# Quantisation of Conformal Fields in Three-dimensional Anti-de Sitter Black Hole Spacetime

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## Abstract

Utilizing the conformal-flatness nature of 3-dim. Anti-de Sitter ($AdS_3$) black hole solution of Banados, Teitelboim and Zanelli, the quantisation of conformally-coupled scalar and spinor fields in this background spacetime is explicitly carried out. In particular, mode expansion forms and propagators of the fields are obtained in closed forms. The vacuum in this conformally-coupled field theories in $AdS_3$ black hole spacetime, which is conformally-flat, is the conformal vacuum which is unique and has global meaning. This point particularly suggests that now the particle production by $AdS_3$ black hole spacetime should be absent. General argument establishing the absence of real particle creation by $AdS_3$ black hole spacetime for this case of conformal triviality is provided. Then next, using the explicit mode expansion forms for conformally-coupled scalar and spinor fields, the bosonic and fermionic super-radiances are examined and found to be absent confirming the expectation.

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## I. Introduction

We begin with some comments on peculiar features of gravity in (2+1)-dimensions. They are the *local flatness* and the *absence of conformal anomaly.* Firstly on the local flatness ; consider the Einstein field equation in an arbitrary $n(n \geq 3)$ dimensions (our sign convention here is chosen to be that of Misner,Thorne and Wheeler [1])

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}. \tag{1}$$

Now, generally the vanishing of $R_{\mu\nu}$ and $R$ and hence of the Einstein tensor $G_{\mu\nu}$ (due to the absence of the matter, $T_{\mu\nu} = 0$) does not necessarily imply the vanishing Riemann tensor, $R_{\mu\nu\alpha\beta}$ although the converse is true. Namely, the empty space needs not be flat. On the contrary in 3-dimensions, the relation $G^{\mu}_{\nu} = -\frac{1}{4}\epsilon^{\mu\alpha\beta}\epsilon_{\nu\gamma\delta}R^{\gamma\delta}_{\alpha\beta}$, when inverted, yields $R^{\alpha\mu}_{\beta\nu} = -\epsilon^{\alpha\mu\rho}\epsilon_{\beta\nu\sigma}G^{\sigma}_{\rho}$ implying that the Riemann tensor is directly proportional to the Einstein tensor or the matter energy-momentum tensor. As a result, the vanishing of $G_{\mu\nu}$ due to the absense of matter necessarily implies the vanishing of $R_{\mu\nu\alpha\beta}$. This indicates that 3-dimensional empty space is necessarily flat. Several consequenses follow immediately. Since the vacuum spacetime is locally-flat, there are no gravitational waves in the classical theory and upon quantization, there are no quantum gravitons. Matter sources may produce curvature but only locally at the location of the sources. And the forces, if any, between sources are not mediated by graviton exchange since there are no gravitons. This also means that the Newtonian limit of general relativity is lost in 3-dimensions and we are left with an apparently uninteresting theory. In spite of these frustrating observations, the gravity in 3-dimensions is not completely devoid of physical relevances. The vanishing of Riemann tensor means that any point in a spacetime manifold M has a neighborhood that is isometric to the Minkowski spacetime. Thus if M has a trivial topology, a single neighborhood can be extended globally and the geometry is indeed trivial (flat). But if M has a non-trivial topology, say, it contains noncontractable curves, such an extension may not be possible and as a consequence interesting physics may emerge. To name a few, such existence of "global geometry" for spacetimes with nontrivial topologies may arise by coupling point particles (both with and without spin) to gravity or by adding a cosmological constant to the Einstein field equation [3]. Particularly, the elaboration of adding a cosmological constant to the vacuum Einstein theory generated much excitement recently. Namely, due to this peculiar aspect of gravity in 3-dimensions, it had long been thought that black hole solutions cannot exist in 3-dimensions since there is no local gravitational attraction and hence no mechanism to confine large densities of matter. It was, therefore, quite a surprise when Banados, Teitelboim and Zanelli (BTZ) have recently constructed the Anti-de Sitter ($AdS_3$) spacetime solutions to 3-dimensional Einstein equation that can be interpreted as black hole solutions. They included the negative cosmological constant in the 3-dimensional vacuum Einstein theory and then found both rotating and nonrotating black hole solutions. Secondly, we turn to the general features of "conformal flatness" of spacetimes which turns out to be a great advantage that allows us to carry out calculations, in full detail, involved in dealing with the conformally-coupled quantum fields propagating in the background of conformally-flat spacetimes. Note that generally the Riemann tensor $R_{\mu\nu\alpha\beta}$, containing the full information about the curvature of spacetime, may be expressed, for dimensions $n \geq 3$, in terms of its various traces given by the Ricci tensor $R_{\mu\nu}$, the curvature scalar R and the traceless, conformally-invariant piece, the Weyl or conformal tensor, $C_{\mu\nu\alpha\beta}$ as [10]

$$\begin{aligned} R_{\mu\nu\alpha\beta} = & \frac{1}{(n-2)}(g_{\mu\alpha}R_{\nu\beta} + g_{\nu\beta}R_{\mu\alpha} - g_{\mu\beta}R_{\nu\alpha} - g_{\nu\alpha}R_{\mu\beta}) \\ & - \frac{R}{(n-1)(n-2)}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}) + C_{\mu\nu\alpha\beta}. \end{aligned} \quad (2)$$

The Weyl tensor, as is well known, has a dual role; not only is it the traceless part of the Riemann tensor, it also "probes" the conformal properties of a metric. First, $C_{\mu\nu\alpha\beta}$ is invariant under conformal transformation of the metric, $g_{\mu\nu} = \Omega^2(x)\tilde{g}_{\mu\nu}$. Next, but more importantly, it vanishes if and only if the metric is conformally-flat, $g_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu}$ for spacetimes with dimensions $n > 3$. In $n$=3-dimensions, however, the Weyl tensor vanishes "identically" $C_{\mu\nu\alpha\beta} = 0$. In particular, the very consequence of the vanishing Weyl tensor in $n = 3$, i.e.,

$$R_{\mu\nu\alpha\beta} = (g_{\mu\alpha}R_{\nu\beta} + g_{\nu\beta}R_{\mu\alpha} - g_{\mu\beta}R_{\nu\alpha} - g_{\nu\alpha}R_{\mu\beta})$$

$$-\frac{1}{2}R(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}) \quad (3)$$

establishes the fact that (along with the fact that both the Riemann tensor and Ricci tensor have the same number (6) of independent components) the Riemann tensor is linearly proportional to the Ricci and hence to the Einstein tensor, $R^{\alpha\mu}_{\beta\nu} = -\epsilon^{\alpha\mu\rho}\epsilon_{\beta\nu\sigma}G^{\sigma}_{\rho}$ as stated earlier. Since the Weyl tensor $C_{\mu\nu\alpha\beta}$ vanishes identically in $n = 3$, one now needs some other means to probe the conformal flatness of 3-dim. spacetimes. Indeed it is known that in $n = 3$, the "Weyl-Schouten" tensor [11] defined by

$$C_{\lambda\mu\nu} \equiv \nabla_{\nu}R_{\lambda\mu} - \nabla_{\mu}R_{\lambda\nu} - \frac{1}{4}(g_{\lambda\mu}\partial_{\nu}R - g_{\lambda\nu}\partial_{\mu}R)$$

plays the role of Weyl tensor, namely $C_{\lambda\mu\nu} = 0$ if and only if a 3-dim. spacetime is conformally flat. The $AdS_3$ black hole spacetime we mentioned earlier is a solution to the Einstein equation $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0$ which, in turn, implies $R_{\mu\nu} = 2\Lambda g_{\mu\nu}$ and hence $R = 6\Lambda$. Namely, since its scalar curvature is constant and its Ricci tensor is covariantly constant, clearly $C_{\lambda\mu\nu} = 0$ and thus this $AdS_3$ black hole spacetime is conformally-flat. As a matter of fact, this conclusion was expected since we know that generally, (anti) de Sitter spacetimes are spaces of constant curvature and hence are conformally-flat. In this work, we shall first present a general formulation for the quantisation of "conformally-coupled" scalar and spinor fields (but not the vector field which fails to remain conformally-coupled in dimensions other than four) in the background of the conformally-flat $n$-dimensional spacetimes. Then, as a particular and interesting example, we shall take the $AdS_3$ black hole spacetime of BTZ, which is also conformally flat, and perform a quantisation of conformally-coupled matter fields on this background.

Lastly, on the absence of conformal anomaly ; it is natural to wonder if the conformal anomaly necessarily comes into play when considering the quantisation of conformally-coupled scalar and spinor fields on conformally-flat $AdS_3$ black hole spacetime which is of our particular interest. The answer to this question is obviously "no" and one needs not worry about the conformal anomaly in the first place. The reason for this goes as follows

; as is well-known, the conformal anomaly is closely related to the "trace anomaly" [2]. Namely the non-vanishing trace of the renormalized stress tensor leads to the conformal non-invariance of the quantum effective action of the matter field since

$$< T^{\mu}_{ren\ \mu}(x) >= \frac{2}{\sqrt{g}} g_{\mu\nu} \frac{\delta W_{ren}}{\delta g_{\mu\nu}}.$$

Now in the context of zeta function regularization scheme, for example, the renormalized 1-loop effective action is given by [2,15] $W_{ren} = -\frac{1}{2}\xi'(0)$ for real scalar field and $W_{ren} = \xi'(0)$ for spinor field in Euclidean signature. Then the explicit calculation demonstrates that $< T^{\mu}_{ren\ \mu}(x) >$ is non-zero in even dimensions ($d = 2, 4$) while vanishing in odd-dimensions (including $d = 3$). Namely the conformal anomaly emerges only in even dimensions and is absent in odd spacetime dimensions. And this general statement holds true for conformally-flat spacetimes such as $AdS_3$ black hole spacetime as well.

Now, speaking of the motivations of the present work, as already mentioned, partly we wish to present a standard formulation of the general quantization of conformally-coupled scalar and spinor fields in conformally-flat spacetimes. And partly we would like to clarify an unsatisfactory state of affair concerning the existing study of thermodynamics of 3-dimensional $AdS_3$ black hole of BTZ. Let us be more specific on this last point. Essentially, one is bound to be left with the Hawking evaporation and hence the thermodynamics of black holes when he/she considers minimally-coupled (massive or massless) or non-minimally coupled (including the conformal coupling) quantum fields in the background of 4-dimensional black hole spacetimes. Thus, in principle, one may employ any type of matter coupling (to background gravity) to treat the thermodynamics of 4-dimensional black holes although, in practice, general couplings involve highly non-trivial complications. Perhaps because of this state of matter, people do not seem to be cautious in selecting the matter coupling to consider the thermodynamics of black holes in lower dimensions, particularly those in 3-dimensions such as the $AdS_3$ black holes of BTZ. Unlike the 4-dimensional case, however, the $AdS_3$ black hole spacetimes is conformally-flat. Therefore if one negligently employs conformally-coupled quantum fields in the $AdS_3$ black hole spacetimes (which is of our interest here) to discuss its thermodynamics, he/she may be misled and run into a trouble. What happens is, in this case of "conformal triviality" (i.e., when fields are conformally-coupled to the conformally-flat spacetimes), the associated Fock vacuum is the "conformal vacuum" which is unique and has global meaning and hence there is no real particle creation. As a consequence, the Hawking temperature (but not the *local* temperature measured by an accelerating detector) is zero and thus the discussion of black hole thermodynamics becomes irrelevant to start with. Namely, in order to carry out a meaningful examination of $AdS_3$ black holes thermodynamics, one should employ non-conformal matter couplings. Indeed, such confusion regarding the study of thermodynamics of BTZ black holes appeared in the literature [5] and our present work is partly intended to clarify this issue. To be more concrete, for example, the authors of ref. 5 employed conformally-coupled scalar and spinor fields to point out correctly the "statistical inversion" arising in the accelerating detector's response function but to discuss incorrectly the Hawking temperature and black hole thermodynamics which become irrelevant because of the conformally trivial setting. In the present work (particularly in sect.IV), we shall provide a general argument and evidences that explain this seemingly contradictory phenomenon, i.e., the presence of "particle detection" but the absence of real 'particle creation" in the case of conformal triviality.

This paper is organized as follows : In sect. II, we formally review the formalism of quantizing conformally- coupled scalar and spinor fields in general $n$-dimensional conformally flat spacetimes. In sect. III, as an application of the general formalism given in sect. II, the mode expansions of conformally-coupled quantum fields in the $AdS_3$ black hole spacetime of BTZ are obtained in closed forms. Sect. IV will be devoted to the general argument on the absence of particle creation in conformal triviality which is the case at hand. In sect. V, as an explicit demonstration of the absence of real particle creation, the absence of the superradiant scatterings off the $AdS_3$ black hole is shown using the explicit mode expansion forms obtained in sect. III. Finally in sect. VI, we summarize the results of our present work. In addition in the appendix, Green's functions of conformally-coupled quantum fields in the background of $AdS_3$ black hole spacetime are given in closed forms as well.

## II. General Formulation

### 1. Quantization of conformally-coupled scalar field in conformally-flat spacetimes

Although we are particularly interested in the quantisation of conformally-coupled fields in conformally-flat 3-dimensional spacetimes, in what follows we shall formulate the quantisation scheme generally in conformally-flat $n$-dim. spacetimes [2]. The action for a real scalar field coupled conformally to gravity is given by [2] (assuming the gravity as a "background")

$$S = -\int d^n x \sqrt{g}[\frac{1}{2}g^{\mu\nu}\partial_\mu \Phi \partial_\nu \Phi + \frac{1}{2}\xi(n) R \Phi^2] \tag{4}$$

where $\xi(n) = \frac{(n-2)}{4(n-1)}$ in $n$-dimensions. One can readily check that this action is invariant under the Weyl-rescaling

$$g_{\mu\nu} = \Omega^2(x)\tilde{g}_{\mu\nu}, \qquad \Phi = \Omega^{-(\frac{n-2}{2})}\tilde{\Phi} \tag{5}$$

In particular, if the background spacetime is "conformally flat", $g_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu}$, then from $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$, it follows that $\tilde{\Box} = \eta^{\mu\nu}\partial_\mu\partial_\nu$, $\tilde{R}(\tilde{g}) = 0$ and hence, we are left with the action

$$S = -\int d^n x \frac{1}{2}\tilde{\Phi}(-\tilde{\Box})\tilde{\Phi} = -\int d^n x \frac{1}{2}\eta^{\mu\nu}\partial_\mu\tilde{\Phi}\partial_\nu\tilde{\Phi}. \tag{6}$$

This implies that the theory of conformally-coupled scalar field in conformally flat spacetime can be substituted with the usual free, massless scalar field theory in flat spacetime of the Weyl-rescaled field, $\tilde{\Phi}(x) = \Omega^{(\frac{n-2}{2})}(x)\Phi(x)$ with $\Omega(x)$ being the spacetime dependent conformal factor. Thus in order to quantize the original field $\Phi(x)$, we first carry out the familiar quantisation programme for the free, Weyl-rescaled field $\tilde{\Phi}(x)$ and then use the result to eventually recover the quantized theory of the original field $\Phi(x)$ via $\Phi(x) = \Omega^{-(\frac{n-2}{2})}(x)\tilde{\Phi}(x)$ at the end. For instance, if we employ the standard canonical quantisation scheme, we begin by demanding the equal time commutators

$$[\tilde{\Phi}(t,\vec{x}),\tilde{\Pi}(t,\vec{y})] = i\delta^{n-1}(\vec{x}-\vec{y}), \tag{7}$$
$$[\tilde{\Phi}(t,\vec{x}),\tilde{\Phi}(t,\vec{y})] = [\tilde{\Pi}(t,\vec{x}),\tilde{\Pi}(t,\vec{y})] = 0.$$

where $\tilde{\Pi}(x) = \delta S/\delta(\partial_t\tilde{\Phi}) = \partial_t\tilde{\Phi}$ is the momentum conjugate to the Weyl-rescaled scalar field. Then for the particle interpretation, we decompose the scalar field using the mode expansion

$$\tilde{\Phi}(x) = \int \frac{d^{n-1}k}{[(2\pi)^{n-1}2\omega_k]^{1/2}}[a(k)\tilde{u}_k(x) + a^\dagger(k)\tilde{u}_k^*(x)] \qquad (\omega_k \equiv k^0)$$
$$\equiv \sum_{\vec{k}} [a_k\tilde{u}_k(x) + a_k^\dagger\tilde{u}_k^*(x)],$$

where the mode functions, which are the solutions of the Klein-Gordon equation $\partial_\mu\partial^\mu\tilde{\Phi} = \Box\tilde{\Phi} = 0$, are given by

$$\tilde{u}_k(x) = \frac{1}{[(2\pi)^{n-1}2\omega_k]^{\frac{1}{2}}}e^{ik\cdot x} \tag{8}$$

and $a^\dagger(k)$ and $a(k)$ are the creation and the annihilation operators respectively. Now, then the mode expansion of the original field is recovered as

$$\Phi(x) = \Omega^{-(\frac{n-2}{2})}(x)\sum_{\vec{k}}[a_k\tilde{u}_k(x) + a_k^\dagger\tilde{u}_k^*(x)]. \tag{9}$$

Note, here, that the conformal factor $\Omega(x)$ is not a field which is subject to the quantisation nor it participates in the physical particle interpretation. The vacuum state associated with these mode function of the original field above, namely, $a_k|0>= 0$ is called the "conformal vacuum" [2]. The physical meaning of this conformal vacuum may be described as follows; as is well-known, generally in curved spacetimes, a particular set of mode functions of a field equation and the corresponding vacuum and Fock space do not in general have direct physical significance. Or put in plain English, in curved spacetimes, there is in general no meaningful notion of global vacuum state. A vacuum state in a reference frame may be a many-particle state in another. The concept of particle is really observer-dependent. Nevertheless, if there exist geometrical symmetries in the spacetime of interest, it may be that a particular set of modes and corresponding vacuum and Fock space emerge as having natural, physical meaning. The theory of conformally-coupled fields in a conformally-flat spacetime is endowed with such a feature and the associated vacuum state is called "conformal vacuum". Thus the conformal vacuum (just like the unique vacuum in field theory in flat spacetime) remains to be a vacuum with respect to any other reference frame. This characteristic is indeed the essential advantage that greatly simplifies the task of quantizing fields conformally-coupled to a conformally-flat spacetime. Now, in order to discuss the many-particle interpretation of the canonical quantization programme via Hamiltonian and momentum operators, we begin with the energy-momentum tensor (or stress tensor) for the conformally-coupled scalar field

$$T^{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}} \tag{10}$$

which, by a straightforward computation, turns out to be [2]

$$\begin{aligned} T_{\mu\nu} &= (1-2\xi)\nabla_\mu\Phi\nabla_\nu\Phi + (2\xi - \frac{1}{2})g_{\mu\nu}\nabla_\alpha\Phi\nabla^\alpha\Phi \\ &\quad -2\xi\Phi\nabla_\mu\nabla_\nu\Phi + 2\xi g_{\mu\nu}\Phi\Box\Phi + \xi(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\Phi^2. \end{aligned} \tag{11}$$

As is well-known, generally when the classical matter action is invariant under the conformal transformation, the associated classical stress tensor is traceless. This is indeed true for the case at hand since we are dealing with conformally-invariant scalar field theory, i.e.,

$$\begin{aligned} T^\lambda_\lambda &= g^{\mu\nu}T_{\mu\nu} \\ &= [2(n-1)\xi + (1-\frac{n}{2})]\nabla_\mu\Phi\nabla^\mu\Phi + (\frac{2-n}{2})[\Phi(-\Box + \xi R)\Phi] = 0 \end{aligned} \tag{12}$$

where we used $\xi = \frac{(n-2)}{4(n-1)}$ and the Euler-Lagrange's equation of motion $(-\Box + \xi R)\Phi = 0$. Also, note that generally under the conformal transformation in eq.(5), the stress tensor transforms as

$$T_{\mu\nu} = \Omega^{-(n-2)}\tilde{T}_{\mu\nu} \tag{13}$$

where $\tilde{T}_{\mu\nu} = \tilde{T}_{\mu\nu}(\tilde{\Phi}, \tilde{g}_{\mu\nu})$ is the stress tensor of Weyl-rescaled fields, $\tilde{\Phi}$ and $\tilde{g}_{\mu\nu}$. Therefore, for the case of conformally-coupled scalar field in conformally-flat spacetime,

$$\tilde{T}_{\mu\nu} = (\partial_\mu\tilde{\Phi}\partial_\nu\tilde{\Phi} - \frac{1}{2}\eta_{\mu\nu}\partial_\alpha\tilde{\Phi}\partial^\alpha\tilde{\Phi}) - 2\xi[\partial_\mu(\tilde{\Phi}\partial_\nu\tilde{\Phi}) - \eta_{\mu\nu}\partial_\alpha(\tilde{\Phi}\partial^\alpha\tilde{\Phi})] \tag{14}$$

which is the stress tensor for a free, massless scalar field in flat spacetime plus total derivative terms. As a consequence, the Hamiltonian and the total momentum operator becomes generally

$$H = P^0 = \int d^{n-1}x\sqrt{g}\, T^{00} = \int d^{n-1}x\sqrt{\tilde{g}}\, \Omega^{-2}\tilde{T}^{00}, \tag{15}$$

$$P^i = \int d^{n-1}x\sqrt{g}\, T^{i0} = \int d^{n-1}x\sqrt{\tilde{g}}\, \Omega^{-2}\tilde{T}^{i0} \tag{16}$$

where we used $P^\mu = \int d^{n-1}x\sqrt{g}\, T^{\mu 0}$, $T^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}T_{\alpha\beta} = \Omega^{-n-2}\tilde{T}^{\mu\nu}$. Again, for the case of conformally-coupled scalar field in conformally-flat spacetime, using eqs.(14)-(16),

$$H = \int d^{n-1}x\Omega^{-2}[\frac{1}{2}\{\tilde{\Pi}^2 + (\partial_i\tilde{\Phi})^2\} - 2\xi\partial_i(\tilde{\Phi}\partial_i\tilde{\Phi})], \tag{17}$$

$$P^i = \int d^{n-1}x\Omega^{-2}[-\{\tilde{\Pi}(\partial_i\tilde{\Phi}) - 2\xi\partial_i(\tilde{\Phi}\tilde{\Pi})\}].$$

Now, the physical interpretation of the Hamiltonian and the momentum operators should be clear. The Hamiltonian density and the momentum density of a conformally coupled scalar field in a conformally flat spacetime turn out to emerge just as a conformal factor $\Omega^{-2}(x)$ times those of a free scalar field in flat spacetime. Particularly note that terms proportional to $\xi$ in the integrands in eq.(17), which originate from the "total divergence" terms in $\tilde{T}_{\mu\nu}$ in eq.(14), cannot be ignored in this case of conformal scalar field in conformally-coupled spacetime due to the presence of the conformal factor $\Omega^{-2}(x)$. Namely, due to this conformal factor the representation of the Hamiltonian and the momentum operators fails to take a simple form in terms of, say, the number operator.

Next, we provide the Feynman Green's function for this conformally-coupled scalar field in conformally flat spacetime. Generally in curved spacetimes, Green's functions for scalar fields satisfy the wave equation [2]

$$[-\Box_x + \xi R(x)]G_F(x, x') = -\frac{1}{\sqrt{g}(x)}\delta^n(x - x'). \tag{18}$$

And particularly, the Feynman propagator for conformally-coupled scalar field in a conformally-flat spacetime is given by

$$G_F(x, x') = [\Omega^{(\frac{2-n}{2})}(x)G^0_F(x, x')\Omega^{(\frac{2-n}{2})}(x')] \tag{19}$$

where

$$G_F^0(x,x') = \int \frac{d^n k}{(2\pi)^n} e^{ik\cdot(x-x')} \frac{(-1)}{k^2}.$$

is the flat, Minkowski spacetime version of the Feynman propagator for massless scalar fields.

**2. Quantization of conformally-coupled spinor field in conformally-flat spacetimes**

The action for a spinor field conformally coupled to gravity (again assuming the gravity as a "background") is given by [2]

$$\begin{aligned} S &= \int d^n x \sqrt{g} \{ \frac{i}{2} [\bar{\Psi}\gamma^\mu (\nabla_\mu \Psi) - (\nabla_\mu \bar{\Psi})\gamma^\mu \Psi] \} \\ &= \int d^n x \sqrt{g} \{ \frac{i}{2} [\bar{\Psi}\gamma^a e_a^\mu(x) (\nabla_\mu \Psi) - (\nabla_\mu \bar{\Psi})\gamma^a e_a^\mu(x) \Psi] \} \end{aligned} \tag{20}$$

where $e_a^\mu(x)$ is the $n$-bein that can be considered as the square root of the metric $g_{\mu\nu}(x) = \eta_{ab} e_\mu^a(x) e_\nu^b(x)$, with $e_\mu^a e_b^\mu = \delta_b^a$, $e_a^\mu e_\nu^a = \delta_\nu^\mu$ and thus $\gamma^\mu(x) = e_a^\mu(x)\gamma^a$ is the curved spacetime $\gamma$-matrices obeying $\{\gamma^\mu(x), \gamma^\nu(x)\} = -2g^{\mu\nu}(x)$. $\nabla_\mu(x) \equiv [\partial_\mu - \frac{i}{4}\omega_\mu^{ab}(x)\sigma_{ab}]$ is the covariant derivative with $\omega_\mu^{ab}(x)$ being the spin connection and $\sigma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b]$ being the $SO(n-1,1)$ group generator in the spinor representation.

Again, one can readily check that this action is indeed invariant under the Weyl-rescaling

$$\begin{aligned} g_{\mu\nu} &= \Omega^2(x)\tilde{g}_{\mu\nu}, \qquad \text{or} \qquad e_\mu^a = \Omega(x)\tilde{e}_\mu^a, \qquad e_b^\mu = \Omega^{-1}(x)\tilde{e}_b^\mu, \\ \Psi(x) &= \Omega^{-(\frac{n-1}{2})}\tilde{\Psi}(x), \\ \bar{\Psi}(x) &= \Omega^{-(\frac{n-1}{2})}\bar{\tilde{\Psi}}(x) \end{aligned} \tag{21}$$

In particular, if the background spacetime is "conformally-flat", then from $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$, it follows that $\tilde{e}_\mu^a = \delta_\mu^a$, $\tilde{e}_b^\mu = \delta_b^\mu$, $\tilde{\omega}_{\mu b}^a = 0$ and hence $\tilde{\nabla}_\mu = \partial_\mu$ leaving us with the action

$$S = \int d^n x \{ \frac{i}{2} [\bar{\tilde{\Psi}}\gamma^\mu (\partial_\mu \tilde{\Psi}) - (\partial_\mu \bar{\tilde{\Psi}})\gamma^\mu \tilde{\Psi}] \} \tag{22}$$

Like in the case of conformally-coupled scalar field, this implies that the theory of conformally-coupled spinor field in conformally-flat spacetime can be substituted with the usual free, massless spinor field theory in flat spacetime of the Weyl-rescaled field, $\tilde{\Psi}(x) = \Omega^{(\frac{n-1}{2})}(x)\Psi(x)$. Thus in order to eventually quantize the original field $\Psi(x)$, we first carry out the familiar quantisation programme for the free, Weyl-rescaled field $\tilde{\Psi}(x)$ and then use its result to recover the quantized theory of the original field $\Psi(x)$ via $\Psi(x) = \Omega^{-(\frac{n-1}{2})}(x)\tilde{\Psi}(x)$ at the end. Again, if we adopt the canonical quantisation scheme, we begin, for the particle interpretation, by "mode expanding" the spinor field in terms of the mode functions which are the spinors satisfying Dirac equations $i\gamma^{\mu}\vec{\partial}_{\mu}\tilde{\Psi} = 0, \bar{\tilde{\Psi}}i\gamma^{\nu}\overleftarrow{\partial}_{\nu} = 0$,

$$\tilde{\Psi}(x) = \int \frac{d^{n-1}p}{(2\pi)^{n-1}2p^0} \sum_s [b(p,s)\tilde{u}(p,s)e^{ip\cdot x} + d^{\dagger}(p,s)\tilde{v}(p,s)e^{-ip\cdot x}], \tag{23}$$

$$\bar{\tilde{\Psi}}(x) = \int \frac{d^{n-1}p}{(2\pi)^{n-1}2p^0} \sum_s [b^{\dagger}(p,s)\bar{\tilde{u}}(p,s)e^{-ip\cdot x} + d(p,s)\bar{\tilde{v}}(p,s)e^{ip\cdot x}] \tag{24}$$

where $b(d)$ is the positive energy particle (antiparticle) annihilation operator and $b^{\dagger}$ $(d^{\dagger})$ is its creation operator.

Now, slightly differently from the quantisation of bosons (such as the scalar field discussed earlier), we first demand "anticommutation relations" to creation and annihilation operators to arrive eventually at the Pauli exclusion principle and from them, next we determine the anticommutation relations between fields via the mode expansion given above in eqs.(23) and (24)

$$\{\tilde{\Psi}_{\alpha}(t,\vec{x}), \tilde{\Psi}^{\dagger}_{\beta}(t,\vec{y})\} = \delta_{\alpha\beta}\delta^{n-1}(\vec{x}-\vec{y}), \tag{25}$$
$$\{\tilde{\Psi}_{\alpha}(t,\vec{x}), \tilde{\Psi}_{\beta}(t,\vec{y})\} = \{\tilde{\Psi}^{\dagger}_{\alpha}(t,\vec{x}), \tilde{\Psi}^{\dagger}_{\beta}(t,\vec{y})\} = 0$$

where the momentum conjugate to the Weyl-rescaled spinor field is given by $\tilde{\Pi}(x) = i\tilde{\Psi}^{\dagger}(x)$. Here, the essential reason for demanding anticommutation relations among the creation and annihilation operators is to obtain the correct "positive definite" total Hamiltonian operator as will be shown below. Now, the mode expansion of the original field is recovered as

$$\Psi(x) = \Omega^{-(\frac{n-1}{2})}(x) \int \frac{d^{n-1}p}{(2\pi)^{n-1}2p^0} \sum_s [b(p,s)\tilde{u}(p,s)e^{ip\cdot x} + d^{\dagger}(p,s)\tilde{v}(p,s)e^{-ip\cdot x}] \tag{26}$$

and similarly for the adjoint spinor $\bar{\Psi}(x)$. Of course, the vacuum state associated with these mode functions of the original field above, namely, $b(p,s)|0>= 0$, $d(p,s)|0>= 0$ is the "conformal vacuum".

Next, in order to construct eventually the Hamiltonian and momentum operators, we again consider the stress tensor for the conformally-coupled spinor field [2]

$$\begin{aligned} T^{\mu\nu} &= \frac{1}{2(det\ e)}\eta^{ab}[e^{\mu}_{a}\frac{\delta S}{\delta e^{b}_{\nu}} + e^{\nu}_{b}\frac{\delta S}{\delta e^{a}_{\mu}}] \\ &= \frac{i}{4}\{[\bar{\Psi}\gamma^{\mu}(\nabla^{\nu}\Psi) - (\nabla^{\nu}\bar{\Psi})\gamma^{\mu}\Psi] + [\bar{\Psi}\gamma^{\nu}(\nabla^{\mu}\Psi) - (\nabla^{\mu}\bar{\Psi})\gamma^{\nu}\Psi]\}. \end{aligned} \tag{27}$$

Once again, since we are dealing with conformally-invariant spinor field theory, the classical stress tensor above should be traceless as can be easily seen as follows

$$T^{\lambda}_{\lambda} = g_{\mu\nu}T^{\mu\nu} = \frac{i}{2}[\bar{\Psi}\gamma^{\mu}(\nabla_{\mu}\Psi) - (\nabla_{\mu}\bar{\Psi})\gamma^{\mu}\Psi] = 0 \tag{28}$$

where we used the on-shell condition, $i\gamma^{\mu}\overrightarrow{\nabla}_{\mu}\Psi = 0$, $\bar{\Psi}i\gamma^{\mu}\overleftarrow{\nabla}_{\mu} = 0$.

Note also that under the conformal transformation in eq.(21), the stress tensor above transforms as

$$T_{\mu\nu} = \Omega^{-(n-2)}\tilde{T}_{\mu\nu} \tag{29}$$

where $\tilde{T}_{\mu\nu} = \tilde{T}_{\mu\nu}(\tilde{\Psi}, \tilde{g}_{\mu\nu})$ is the stress tensor of Weyl-rescaled fields $\tilde{\Psi}$ and $\tilde{g}_{\mu\nu}$. Therefore, for the case of conformally-coupled spinor field in conformally-flat spacetime

$$\tilde{T}_{\mu\nu} = \frac{i}{4}\{[\bar{\tilde{\Psi}}\gamma_{\mu}(\partial_{\nu}\tilde{\Psi}) - (\partial_{\nu}\bar{\tilde{\Psi}})\gamma_{\mu}\tilde{\Psi}] + [\bar{\tilde{\Psi}}\gamma_{\nu}(\partial_{\mu}\tilde{\Psi}) - (\partial_{\mu}\bar{\tilde{\Psi}})\gamma_{\nu}\tilde{\Psi}]\} \tag{30}$$

which is just the stress tensor for free, massless spinor in flat spacetime.

As a consequence, the Hamiltonian and the total momentum operator become generally

$$H = P^{0} = \int d^{n-1}x\sqrt{g}\,T^{00} = \int d^{n-1}x\sqrt{\tilde{g}}\,\Omega^{-2}\tilde{T}^{00}, \tag{31}$$

$$P^{i} = \int d^{n-1}x\sqrt{g}\,T^{i0} = \int d^{n-1}x\sqrt{\tilde{g}}\,\Omega^{-2}\tilde{T}^{i0} \tag{32}$$

where we used $P^{\mu} = \int d^{n-1}x\sqrt{g}\ T^{\mu 0}$ as before and $T^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}T_{\alpha\beta} = \Omega^{-n-2}\tilde{T}^{\mu\nu}$. Again, for the case of conformally-coupled spinor field in conformally-flat spacetime, using eq.(46)

$$H = \int d^{n-1}x\Omega^{-2}\frac{i}{2}[\bar{\tilde{\Psi}}\gamma_{0}(\partial_{t}\tilde{\Psi}) - (\partial_{t}\bar{\tilde{\Psi}})\gamma_{0}\tilde{\Psi}], \tag{33}$$

$$P^{i} = \int d^{n-1}x\Omega^{-2}\frac{-i}{4}\{[\bar{\tilde{\Psi}}\gamma_{i}(\partial_{t}\tilde{\Psi}) - (\partial_{t}\bar{\tilde{\Psi}})\gamma_{i}\tilde{\Psi}] + [\bar{\tilde{\Psi}}\gamma_{0}(\partial_{i}\tilde{\Psi}) - (\partial_{i}\bar{\tilde{\Psi}})\gamma_{0}\tilde{\Psi}]\} \tag{34}$$

Again, the physical interpretation of the Hamiltonian and the momentum operators is clear. The Hamiltonian density and the momentum density of a conformally coupled spinor field in a conformally flat spacetime emerge as a conformal factor $\Omega^{-2}(x)$ times those of a free spinor field in flat spacetime. And as stated earlier, this conformal factor keeps us from representing the Hamiltonian and the momentum operators in a simple form in terms of a number operator.

Next, we provide the Feynman Green's function for this conformally-coupled spinor field in conformally flat spacetime. Generally in curved spacetimes, Green's functions for the massless spinor field satisfies the wave equation [2]

$$[i\gamma^{\mu}\nabla_{\mu}^{x}]S_F(x,x') = \frac{1}{\sqrt{g(x)}}\delta^n(x-x'). \tag{35}$$

And particularly, the Feynman propagator for conformally-coupled spinor field in a conformally-flat spacetime is given by

$$S_F(x,x') = \Omega^{(\frac{1-n}{2})}(x)S_F^0(x,x')\Omega^{(\frac{1-n}{2})}(x') \tag{36}$$

where

$$S_F^0(x,x') = \int \frac{d^n p}{(2\pi)^n} e^{ip\cdot(x-x')} \frac{p_\mu \gamma^\mu}{p^2 + i\epsilon} \tag{37}$$

is the flat, Minkowski spacetime version of the Feynman propagator for massless spinor fields. For future use, we note that the relationship between the Dirac propagator above and the Klein-Gordon propagator (generally in the presence of the mass term)

$$G_F^0(x,x') = \int \frac{d^n p}{(2\pi)^n} e^{ip\cdot(x-x')} \frac{(-1)}{p^2 + m^2}$$

in flat spacetime is given by

$$S_F^0(x,x') = [i\gamma^\alpha \partial_\alpha^x + m]G_F^0(x,x'). \tag{38}$$

Later on, we shall use this relation to obtain the Green's functions for conformally-coupled spinor field from those for conformally-coupled scalar field.

## III. Quantization in the background of $AdS_3$ black hole spacetime

### 1. Introducing "conformal gauge"

Recently, a stationary, axisymmetric solution to the $AdS_3$ Einstein equation $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0$ that can be interpreted as a spinning black hole solution has been discovered by Banados, Teitelboim and Zanelli (BTZ) [3]. The $AdS_3$ black hole solution in Schwarzschild-type coordinates $x^\mu = (t, r, \phi)$ can be given in Arnowitt-Deser-Misner's (ADM) (2+1) space-plus-time split form by [3,4]

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= -(N^\perp)^2 dt^2 + f^{-2}dr^2 + r^2(d\phi + N^\phi dt)^2 \qquad (39) \\ \text{where} \qquad N^\perp &= f = \Big(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\Big)^{\frac{1}{2}}, \qquad N^\phi = -\frac{J}{2r^2} \end{aligned}$$

are the lapse and the angular shift respectively. Of course the $\phi$-coordinate here is taken to be a periodic, angular coordinate satisfying $\phi = \phi + 2\pi n$ $(n \in Z)$. The parameter $l$ is related to the negative cosmological constant by $l^{-2} = -\Lambda$ and $M$ and $a = J/2$ are the ADM mass and the angular momentum per unit mass of the hole respectively. Perhaps it would be appropriate to give a brief description of the structure of this black hole spacetime. The causal structure of this spinning $AdS_3$ black hole has, in many ways, close similarity to that of Kerr black hole spacetime in 4-dim. To begin, since this black hole solution is stationary and axisymmetric, it possesses two Killing fields $\xi^\mu = (\partial/\partial t)^\mu$ and $\psi^\mu = (\partial/\partial\phi)^\mu$ correspondingly. And it is their linear combination $\chi^\mu = \xi^\mu + \Omega_H\psi^\mu$ [10] which is normal to the Killing horizons of this spinning hole. Normally, this is the defining equation of the angular velocity $\Omega_H$ of the rotating holes. Now the spinning $AdS_3$ black hole solution given in eq.(39) has regular Killing horizons at points where the Killing field $\chi^\mu$ becomes null, namely at

$$r_\pm = l\left[\frac{M}{2}\Big\{1 \pm \sqrt{1 - (\frac{J}{Ml})^2}\Big\}\right]^{\frac{1}{2}} \qquad (40)$$

(i.e., $M = (r_+^2 + r_-^2)/l^2$ and $a = r_+r_-/l)$ with $r_+$ being the event horizon and $r_-$ the inner Cauchy horizon respectively provided $| a | \leq Ml/2$. The angular velocity of the event horizon, then, can be evaluated as

$$\Omega_H = -\frac{g_{t\phi}}{g_{\phi\phi}} |_{r_+} = \frac{a}{r_+^2}. \tag{41}$$

This black hole, as one may expects, possesses the "ergoregion" as well. And the "static limit", i.e., the outer boundary of this ergoregion occurs at $r_s = \sqrt{Ml} > r_+$. As we shall discuss later on, the existence of the ergoregion in this black hole spacetime naturally stimulates our curiosity concerning the possible superradiant scattering phenomenon which is known to occur in 4-dim. situation. Having reviewed the structure of the $AdS_3$ black hole spacetime, now consider a coordinate transformation of the $AdS_3$ black hole metric to the "conformal gauge"[4]

$$(t, r, \phi) \rightarrow (z_0, z_1, z_2).$$

Since there are two coordinate singularities at $r = r_+$ and $r_-$, we need two coordinate patches one defined around $r_+$ and the other around $r_-$ (as we usually do when transforming to Kruskal-type coordinates).

(1) Coordinate patch A around the event horizon at $r = r_+$ :

For $r \geq r_+$ (Region I)

$$\begin{aligned}
z_0 &= \left(\frac{r^2 - r_+^2}{r^2 - r_-^2}\right)^{\frac{1}{2}} \sinh\left(\frac{r_+}{l^2}t - \frac{r_-}{l}\phi\right) \exp\left[\frac{r_+}{l}\phi - \frac{r_-}{l^2}t\right], \\
z_1 &= \left(\frac{r^2 - r_+^2}{r^2 - r_-^2}\right)^{\frac{1}{2}} \cosh\left(\frac{r_+}{l^2}t - \frac{r_-}{l}\phi\right) \exp\left[\frac{r_+}{l}\phi - \frac{r_-}{l^2}t\right], \\
z_2 &= \left(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\right)^{\frac{1}{2}} \exp\left[\frac{r_+}{l}\phi - \frac{r_-}{l^2}t\right].
\end{aligned}$$

For $r_- < r \leq r_+$ (Region II)

$$\begin{aligned}
z_0 &= \left(\frac{r_+^2 - r^2}{r^2 - r_-^2}\right)^{\frac{1}{2}} \cosh\left(\frac{r_+}{l^2}t - \frac{r_-}{l}\phi\right) \exp\left[\frac{r_+}{l}\phi - \frac{r_-}{l^2}t\right], \\
z_1 &= \left(\frac{r_+^2 - r^2}{r^2 - r_-^2}\right)^{\frac{1}{2}} \sinh\left(\frac{r_+}{l^2}t - \frac{r_-}{l}\phi\right) \exp\left[\frac{r_+}{l}\phi - \frac{r_-}{l^2}t\right], \\
z_2 &= \left(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\right)^{\frac{1}{2}} \exp\left[\frac{r_+}{l}\phi - \frac{r_-}{l^2}t\right].
\end{aligned} \tag{42}$$

(2) Coordinate patch B around the Cauchy horizon at $r = r_-$ :

For $r_- \leq r < r_+$ (Region II)

$$z_0 = \Big(\frac{r^2 - r_-^2}{r_+^2 - r^2}\Big)^{\frac{1}{2}} \cosh\Big(\frac{r_-}{l^2}t - \frac{r_+}{l}\phi\Big) \exp\Big[\frac{r_-}{l}\phi - \frac{r_+}{l^2}t\Big],$$
$$z_1 = \Big(\frac{r^2 - r_-^2}{r_+^2 - r^2}\Big)^{\frac{1}{2}} \sinh\Big(\frac{r_-}{l^2}t - \frac{r_+}{l}\phi\Big) \exp\Big[\frac{r_-}{l}\phi - \frac{r_+}{l^2}t\Big], \tag{43}$$
$$z_2 = \Big(\frac{r_+^2 - r_-^2}{r_+^2 - r^2}\Big)^{\frac{1}{2}} \exp\Big[\frac{r_-}{l}\phi - \frac{r_+}{l^2}t\Big].$$

For $0 < r \leq r_-$ (Region III)

$$z_0 = \Big(\frac{r_-^2 - r^2}{r_+^2 - r^2}\Big)^{\frac{1}{2}} \sinh\Big(\frac{r_-}{l^2}t - \frac{r_+}{l}\phi\Big) \exp\Big[\frac{r_-}{l}\phi - \frac{r_+}{l^2}t\Big],$$
$$z_1 = \Big(\frac{r_-^2 - r^2}{r_+^2 - r^2}\Big)^{\frac{1}{2}} \cosh\Big(\frac{r_-}{l^2}t - \frac{r_+}{l}\phi\Big) \exp\Big[\frac{r_-}{l}\phi - \frac{r_+}{l^2}t\Big],$$
$$z_2 = \Big(\frac{r_+^2 - r_-^2}{r_+^2 - r^2}\Big)^{\frac{1}{2}} \exp\Big[\frac{r_-}{l}\phi - \frac{r_+}{l^2}t\Big]$$

in terms of which the $AdS_3$ black hole metric takes the "conformally flat" form [4]

$$ds^2 = \frac{l^2}{z_2^2}(-dz_0^2 + dz_1^2 + dz_2^2) = \Omega^2(z)\eta_{\mu\nu}dz^\mu dz^\nu. \tag{44}$$

Note that this type of coordinate transformation is essentially of the same kind as the transformation from Schwarzschild-type to Kruskal coordinates in, say, Reissner-Nordstrom black hole spacetime in 4-dim. Namely, by patching two charts, one well-defined only for all $r > r_-$ and the other again well-defined only for all $r < r_+$, one can go over to a new coordinate system $z^\mu = (z_0, z_1, z_2)$ which is free of coordinate singularities and can cover the whole of black hole spacetime, not just a part of it. And of course in order for these coordinatizations, (both $(t, r, \phi)$ and $(z_0, z_1, z_2)$) to parametrize a black hole spacetime, not just the universal covering space of $AdS_3$ (i.e., $CAdS_3$), an extra condition needs to be imposed. Namely, since the BTZ black hole solution can be obtained from $CAdS_3$ via discrete identifications of points, i.e., the action of discrete subgroup of the $AdS_3$ isometry group $SO(2,2)$, in order for these coordinatizations to represent the BTZ black hole spacetime, we should implicitly assume the identifications

$$\phi = \phi + 2\pi n \quad (n \in Z).$$

As is manifest in this conformal gauge, this BTZ black hole spacetime is indeed conformally-flat as we stressed in the introduction. Thus if one considers quantum fields coupled conformally to this conformally-flat background spacetime, computations of quantities like the mode expansion forms and two-point Green's functions can be carried out exactly. The region II, which lies between the two horizons, is the overlap of two coordinate patches A and B. Thus in order to describe this overlap region, one may select any coordinates belonging to the patch A or to the patch B. Here in this work, we shall take the coordinates belonging to patch A in constructing mode expansion forms (given in the following subsection) and Green's functions (which will be given in the appendix). Note that for the sake of notational simplicity, we shall henceforth use the redefined parameters $a_{\pm} \equiv r_{\pm}/l^2$.

### 2. Mode expansion of conformally-coupled fields in $AdS_3$ black hole spacetime

*i) Real scalar field*

According to the general formulation discussed in the preceding section, the mode expansion for a conformally-coupled real scalar field in 3-dim. conformally-flat spacetime is

$$\Phi(x) = \Omega^{-\frac{1}{2}}(x) \int \frac{d^2k}{(2\pi)^2 2\omega_k} \left[a(k)e^{ik\cdot x} + a^{\dagger}(k)e^{-ik\cdot x}\right]$$

Thus, for the case of our $AdS_3$ black hole background

$$\Phi(z) = (\frac{l}{z_2})^{-\frac{1}{2}} \int \frac{d^2k}{(2\pi)^2 2\omega_k} \left[a(k)e^{ik_\mu z^\mu} + a^{\dagger}(k)e^{-ik_\mu z^\mu}\right]. \tag{45}$$

where $z^\mu = (z_0, z_1, z_2)$ is to be substituted by the original coordinates $(t, r, \phi)$ via the coordinate transformation laws given earlier.

Here, recall that in the coordinatization $(t, r, \phi)$ for the BTZ black hole, the identification $\phi = \phi + 2\pi m$ $(m \in Z)$ was implicitly understood. Thus both the mode expansion forms of the field operators being discussed in this section and the Green's functions that will be provided later in the appendix should exhibit the periodicity in $\phi$-coordinate, i.e., $\Phi(t, r, \phi) = \Phi(t, r, \phi + 2\pi m)$. This can be achieved by taking the infinite linear sum

$$\Phi(t, r, \phi) = \sum_{n=-\infty}^{\infty} \Phi(t, r, \phi_n)$$

where $\phi_n = \phi + 2\pi n$. Then by the superposition principle, this infinite linear sum is automatically a solution to the field equation as well since each $\Phi(t, r, \phi_n)$ is a solution as one can check in a straightforward manner. As we shall see in the appendix, Green's functions exhibiting the same periodicity in $\phi$ can also be constructed via the same infinite sum

$$G^+(\Delta t, \Delta r, \Delta\phi) = \sum_{n=-\infty}^{\infty} G^+(\Delta t, \Delta r, \Delta\phi_n)$$

with $\Delta\phi_n = (\phi - \phi') + 2\pi n$. And this procedure for constructing the Green's function amounts to employing the method of images [15].

For $r \geq r_+$ (Region I)

$$\Phi(t, r, \phi_n) = l^{-\frac{1}{2}}\Big(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\Big)^{\frac{1}{4}} e^{\frac{1}{2}(la_+\phi_n - a_- t)} \int \frac{d^2k}{(2\pi)^2 2\omega_k}\Big[a(k)e^{iE_I} + a^\dagger(k)e^{-iE_I}\Big]$$

with

$$E_I = e^{(la_+\phi_n - a_- t)}\Big[\big(\frac{r^2 - r_+^2}{r^2 - r_-^2}\big)^{\frac{1}{2}}\{-k^0 \sinh(a_+ t - la_-\phi_n) + k^1 \cosh(a_+ t - la_-\phi_n)\} + \big(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\big)^{\frac{1}{2}} k^2\Big].$$

For $r_- < r \leq r_+$ (Region II)

$$\Phi(t, r, \phi_n) = l^{-\frac{1}{2}}\Big(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\Big)^{\frac{1}{4}} e^{\frac{1}{2}(la_+\phi_n - a_- t)} \int \frac{d^2k}{(2\pi)^2 2\omega_k}\Big[a(k)e^{iE_{II}} + a^\dagger(k)e^{-iE_{II}}\Big]$$

with

$$E_{II} = e^{(la_+\phi_n - a_- t)}\Big[\big(\frac{r_+^2 - r^2}{r^2 - r_-^2}\big)^{\frac{1}{2}}\{-k^0 \cosh(a_+ t - la_-\phi_n) + k^1 \sinh(a_+ t - la_-\phi_n)\} + \big(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\big)^{\frac{1}{2}} k^2\Big].$$

For $0 < r \leq r_-$ (Region III)

$$\Phi(t, r, \phi_n) = l^{-\frac{1}{2}}\Big(\frac{r_+^2 - r_-^2}{r_+^2 - r^2}\Big)^{\frac{1}{4}} e^{\frac{1}{2}(la_-\phi_n - a_+ t)} \int \frac{d^2k}{(2\pi)^2 2\omega_k}\Big[a(k)e^{iE_{III}} + a^\dagger(k)e^{-iE_{III}}\Big]$$

with

$$E_{III} = e^{(la_-\phi_n - a_+ t)}\Big[\big(\frac{r_-^2 - r^2}{r_+^2 - r^2}\big)^{\frac{1}{2}}\{-k^0 \sinh(a_- t - la_+\phi_n) + k^1 \cosh(a_- t - la_+\phi_n)\} + \big(\frac{r_+^2 - r_-^2}{r_+^2 - r^2}\big)^{\frac{1}{2}} k^2\Big]$$

where $k^0 = [(k^1)^2 + (k^2)^2]^{1/2}$. Despite the general formulation we provided in the sect.II, suspicious readers may wonder if the mode expansions obtained in this way are really solutions to the Klein-Gordon equation. Thus we ensure that we checked by a straightforward calculation that the mode expansions above do satisfy the Klein-Gordon equation in $AdS_3$ black hole spacetime written in the Schwarzschild-type coordinates.

*ii) Complex scalar field*

Generally in $n$-dimensional conformally-flat spacetime, the mode expansion of conformally-coupled complex scalar field is given by

$$\Phi(x) = \Omega^{-(\frac{n-2}{2})}(x)\tilde{\Phi}(x), \qquad \Phi^*(x) = \Omega^{-(\frac{n-2}{2})}(x)\tilde{\Phi^*}(x)$$

with

$$\tilde{\Phi}(x) = \int \frac{d^{n-1}k}{(2\pi)^{n-1}2\omega_k}[a(k)e^{ik\cdot x} + b^\dagger(k)e^{-ik\cdot x}],$$
$$\tilde{\Phi}^*(x) = \int \frac{d^{n-1}k}{(2\pi)^{n-1}2\omega_k}[b(k)e^{ik\cdot x} + a^\dagger(k)e^{-ik\cdot x}]$$

where $a(k)$, $a^\dagger(k)$ are annihilation and creation operators for positive energy "particle" respectively, $b(k)$, $b^\dagger(k)$ annihilation and creation operators for positive energy "antiparticle" respectively. Thus, for the case at hand, i.e., in the $AdS_3$ black hole background

$$\Phi(z) = (\frac{l}{z_2})^{-\frac{1}{2}} \int \frac{d^2k}{(2\pi)^2 2\omega_k}\Big[a(k)e^{ik_\mu z^\mu} + b^\dagger(k)e^{-ik_\mu z^\mu}\Big], \tag{46}$$
$$\Phi^*(z) = (\frac{l}{z_2})^{-\frac{1}{2}} \int \frac{d^2k}{(2\pi)^2 2\omega_k}\Big[b(k)e^{ik_\mu z^\mu} + a^\dagger(k)e^{-ik_\mu z^\mu}\Big].$$

Now the correct mode expansion is given by

$$\Phi(t,r,\phi) = \sum_{n=-\infty}^{\infty} \Phi(t,r,\phi_n)$$

and similarly for $\Phi^*(t,r,\phi)$.

For $r \geq r_+$ (Region I)

$$\Phi(t,r,\phi_n) = l^{-\frac{1}{2}}\Big(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\Big)^{\frac{1}{4}} e^{\frac{1}{2}(la_+\phi_n - a_- t)} \int \frac{d^2k}{(2\pi)^2 2\omega_k}\Big[a(k)e^{iE_I} + b^\dagger(k)e^{-iE_I}\Big],$$
$$\Phi^*(t,r,\phi_n) = l^{-\frac{1}{2}}\Big(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\Big)^{\frac{1}{4}} e^{\frac{1}{2}(la_+\phi_n - a_- t)} \int \frac{d^2k}{(2\pi)^2 2\omega_k}\Big[b(k)e^{iE_I} + a^\dagger(k)e^{-iE_I}\Big]$$

with

$$E_I = e^{(la_+\phi_n - a_- t)}\Big[(\frac{r^2 - r_+^2}{r^2 - r_-^2})^{\frac{1}{2}}\{-k^0\sinh(a_+ t - la_-\phi_n) + k^1\cosh(a_+ t - la_-\phi_n)\} + (\frac{r_+^2 - r_-^2}{r^2 - r_-^2})^{\frac{1}{2}}k^2\Big].$$

For $r_- < r \leq r_+$ (Region II)

$$\Phi(t,r,\phi_n) = l^{-\frac{1}{2}}\Big(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\Big)^{\frac{1}{4}} e^{\frac{1}{2}(la_+\phi_n - a_- t)} \int \frac{d^2k}{(2\pi)^2 2\omega_k}\Big[a(k)e^{iE_{II}} + b^\dagger(k)e^{-iE_{II}}\Big],$$

$$\Phi^*(t,r,\phi_n) = l^{-\frac{1}{2}}\Big(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\Big)^{\frac{1}{4}} e^{\frac{1}{2}(la_+\phi_n - a_- t)} \int \frac{d^2k}{(2\pi)^2 2\omega_k}\Big[b(k)e^{iE_{II}} + a^\dagger(k)e^{-iE_{II}}\Big]$$

with

$$E_{II} = e^{(la_+\phi_n - a_- t)}\Big[(\frac{r_+^2 - r^2}{r^2 - r_-^2})^{\frac{1}{2}}\{-k^0\cosh(a_+ t - la_-\phi_n) + k^1\sinh(a_+ t - la_-\phi_n)\} + (\frac{r_+^2 - r_-^2}{r^2 - r_-^2})^{\frac{1}{2}}k^2\Big].$$

For $0 < r \leq r_-$ (Region III)

$$\Phi(t,r,\phi_n) = l^{-\frac{1}{2}}\Big(\frac{r_+^2 - r_-^2}{r_+^2 - r^2}\Big)^{\frac{1}{4}} e^{\frac{1}{2}(la_-\phi_n - a_+ t)} \int \frac{d^2k}{(2\pi)^2 2\omega_k}\Big[a(k)e^{iE_{III}} + b^\dagger(k)e^{-iE_{III}}\Big],$$

$$\Phi^*(t,r,\phi_n) = l^{-\frac{1}{2}}\Big(\frac{r_+^2 - r_-^2}{r_+^2 - r^2}\Big)^{\frac{1}{4}} e^{\frac{1}{2}(la_-\phi_n - a_+ t)} \int \frac{d^2k}{(2\pi)^2 2\omega_k}\Big[b(k)e^{iE_{III}} + a^\dagger(k)e^{-iE_{III}}\Big]$$

with

$$E_{III} = e^{(la_-\phi_n - a_+ t)}\Big[(\frac{r_-^2 - r^2}{r_+^2 - r^2})^{\frac{1}{2}}\{-k^0\sinh(a_- t - la_+\phi_n) + k^1\cosh(a_- t - la_+\phi_n)\} + (\frac{r_+^2 - r_-^2}{r_+^2 - r^2})^{\frac{1}{2}}k^2\Big]$$

where again $k^0 = [(k^1)^2 + (k^2)^2]^{1/2}$. Next, consider the particle number current given by $j^\mu = -i(\Phi^*\nabla^\mu\Phi - \Phi\nabla^\mu\Phi^*)$ and examine the continuity, say, of the particle number density (i.e., the pobability density) flux defined by $n_\mu j^\mu = -\chi^\mu j_\mu = i[\Phi^*(\partial_t + \Omega_H\partial_\phi)\Phi - \Phi(\partial_t + \Omega_H\partial_\phi)\Phi^*]$ where, as introduced earlier, $\chi^\mu = \xi^\mu + \Omega_H\psi^\mu = \delta^\mu_t + \Omega_H\delta^\mu_\phi$ ($\Omega_H$ denotes the angular velocity of the rotating black hole) being the Killing field which is outward normal to the outer event horizon at $r = r_+$ (on which $\Omega_H = \Omega_H(r_+)$) and to the inner null surface at $r = r_-$ (on which $\Omega_H = \Omega_H(r_-)$) and thus $n^\mu$ denotes the inward unit vector which is opposite to the direction of $\chi^\mu$. Then one can readily check that across the event horizon and the Cauchy horizon, $n_\mu j^\mu \mid_{r_\pm + \epsilon} = n_\mu j^\mu \mid_{r_\pm - \epsilon}$, namely the particle number flux is conserved. The particle number conservation across the event horizon at $r = r_+$ and the Cauchy horizon at $r = r_-$ confirms that they are just coordinate singularities.

*iii) Spinor field*

Before we construct the mode expansion of the spinor field in $AdS_3$ black hole spacetime, we should note that there are some subtle properties of fermions in (2+1)-dim. To formulate the algebra of Dirac matrices in (2+1)-dim., we basically need 3-anticommuting Hermitian matrices each of whose square is unity. Clearly the standard $2 \times 2$ Pauli matrices fit the bill. Thus the minimal spinor fields we can work with are 2-component spinors. However, since the two components should describe positive-energy and negative-energy solutions, we are left with just one component to describe the spin state. Actually this is indeed enough in 2-space dimensions, where the rotation group $O(2) \sim U(1)$ is abelian and has 1-dim. irreducible representations. Or in plain English, the spin of a minimal fermion will point in some definite direction - up or down - in the particle's rest frame. Thus in the mode expansion of minimal spinor field in (2+1)-dim. like the AdS$_3$ black hole background, the spin sum is absent. Thus the mode expansion for a conformally-coupled spinor field in the AdS$_3$ black hole background is given by

$$\Psi(z) = (\frac{l}{z_2})^{-1} \int \frac{d^2p}{(2\pi)^2 2p^0} \left[b(p)u(p)e^{ip_\mu z^\mu} + d^\dagger(p)v(p)e^{-ip_\mu z^\mu}\right], \tag{47}$$
$$\bar{\Psi}(z) = (\frac{l}{z_2})^{-1} \int \frac{d^2p}{(2\pi)^2 2p^0} \left[b^\dagger(p)\bar{u}(p)e^{-ip_\mu z^\mu} + d(p)\bar{v}(p)e^{ip_\mu z^\mu}\right].$$

Then again the correct mode expansion is given by

$$\Psi(t,r,\phi) = \sum_{n=-\infty}^{\infty} \Psi(t,r,\phi_n)$$

and similarly for the adjoint spinor $\bar{\Psi}(t,r,\phi)$.

For $r \geq r_+$ (Region I)

$$\Psi(t,r,\phi_n) = l^{-1}\Big(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\Big)^{\frac{1}{2}} e^{(la_+\phi_n - a_- t)} \int \frac{d^2p}{(2\pi)^2 2p^0} \left[b(p)u(p)e^{iE_I} + d^\dagger(p)v(p)e^{-iE_I}\right]$$

with

$$E_I = e^{(la_+\phi_n - a_- t)}\Big[\Big(\frac{r^2 - r_+^2}{r^2 - r_-^2}\Big)^{\frac{1}{2}}\{-p^0 \sinh(a_+t - la_-\phi_n) + p^1 \cosh(a_+t - la_-\phi_n)\} + \Big(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\Big)^{\frac{1}{2}} p^2\Big].$$

For $r_- < r \leq r_+$ (Region II)

$$\Psi(t,r,\phi_n) = l^{-1}\Big(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\Big)^{\frac{1}{2}} e^{(la_+\phi_n - a_- t)} \int \frac{d^2p}{(2\pi)^2 2p^0} \left[b(p)u(p)e^{iE_{II}} + d^\dagger(p)v(p)e^{-iE_{II}}\right]$$

with

$$E_{II} = e^{(la_+\phi_n - a_- t)}\Big[\Big(\frac{r_+^2 - r^2}{r^2 - r_-^2}\Big)^{\frac{1}{2}}\{-p^0 \cosh(a_+t - la_-\phi_n) + p^1 \sinh(a_+t - la_-\phi_n)\} + \Big(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\Big)^{\frac{1}{2}} p^2\Big].$$

For $0 < r \le r_-$ (Region III)

$$\Psi(t,r,\phi_n) = l^{-1}\Big(\frac{r_+^2 - r_-^2}{r_+^2 - r^2}\Big)^{\frac{1}{2}} e^{(la_-\phi_n - a_+t)} \int \frac{d^2p}{(2\pi)^2 2p^0}\Big[b(p)u(p)e^{iE_{III}} + d^\dagger(p)v(p)e^{-iE_{III}}\Big]$$

with

$$E_{III} = e^{(la_-\phi_n - a_+t)}\Big[\Big(\frac{r_-^2 - r^2}{r_+^2 - r^2}\Big)^{\frac{1}{2}}\{-p^0\sinh(a_-t - la_+\phi_n) + p^1\cosh(a_-t - la_+\phi_n)\} + \Big(\frac{r_+^2 - r_-^2}{r_+^2 - r^2}\Big)^{\frac{1}{2}} p^2\Big]$$

and similarly for $\bar{\Psi}(t,r,\phi_n)$.

We now end this subsection with some comments accounting for the relation between the results given above and those appeared in the recent works. Recently, there have been attempts to determine explicit form of "grey body factor (or absorption coefficient)" $\sigma_{abs}$ in the modified Hawking radiation spectrum (or decay rate)

$$\Gamma = \frac{\sigma_{abs}}{e^{\frac{\omega}{T_H}} - 1}$$

of BTZ black hole. (Here, the grey body factor is defined as the ratio of the particle flux entering the horizon to the incoming flux at spatial infinity.) And they include the works by Birmingham, Sachs, and Sen [17], by Lee, Kim, and Myung [18], by Teo [17], and by Dasgupta [19]. And to this end, these authors computed the mode functions of quantum scalar or spinor fields in the BTZ black hole spacetime. The mode expansion forms obtained there, however, appear to be different from those given here in the present work. As a matter of fact, this is not surprising for the following reason. In the present work, we considered only the conformally- coupled scalar and spinor fields whereas in [17], they considered minimally-coupled massless scalar fields and in [18], the authors considered non-minimally coupled dilaton field in the context of low energy string theory in 3-dimensions. Besides in [19], the author considered conformally-coupled spinor field but in a rather non-standard manner. Moreover, in all of these works, they *assumed* the mode functions to take the $(t,\phi)$-dependence of the form $e^{-i\omega t + im\phi}$ (with $\omega$ and $m$ being the frequency and the azimuthal number respectively) which, as we shall stress later on, can be regarded as an *ad hoc*. Namely, this type of $(t,\phi)$-dependence of the mode functions is by no means rigorous in that it simply has been chosen so based on the fact that the BTZ black hole spacetime has time-translational and rotational isometries generated by the Killing fields $(\partial/\partial t)^{\mu}$ and $(\partial/\partial\phi)^{\mu}$ respectively. In the present work, as we considered only the conformally-coupled fields (in the conformally-flat BTZ black hole spacetime), we could *construct* the mode functions in a straightforward manner without having to assume any *ad hoc* $(t,\phi)$-dependence and this is why the mode functions in the present work and those in [17-19] came out to be different.

### IV. Absence of particle creation in conformal triviality

In the preceding sections, we discussed the quantisation of conformally- coupled scalar and spinor fields in $AdS_3$ black hole spacetime which is conformally-flat. One crucial point in this case of "conformal triviality" is the absence of real particle creation. Here in this section, we shall provide a formal argument establishing this statement. In "conformally-trivial" situations [2] when conformally- invariant fields are propagating in a conformally flat background spacetime, due to the high degree of symmetry possessed by the spacetime, a unique, global physical vacuum state, called conformal vacuum exists as mentioned earlier and as a consequence no real particles are created from the conformal vacuum. Note that the essence in Hawking's derivation of particle creation by black holes [13] is the non-trivial "Bogoliubov transformation" [2] relation between mode functions (or the associated creation and annihilation operators) of the field operator at early and at late times, namely the "mode-mixing". Perhaps, therefore, the most straightforward way of displaying the absence of particle production by the present $AdS_3$ black hole spacetime is to exhibit that the mixing of modes is absent or equivalently that the Bogoliubov coefficient "$\beta$" associated with the mode-mixing is zero. Thus to do so, we write the $AdS_3$ black hole metric again in "conformal gauge"

$$ds^2 = (\frac{l}{z_2})^2(-dz_0^2 + dz_1^2 + dz_2^2) = \Omega^2(z)\eta_{\mu\nu}dz^{\mu}dz^{\nu}$$

then in terms of which the mode expansion form of a real scalar field coupled conformally to this spacetime is given by

$$\Phi(z) = (\frac{l}{z_2})^{-1/2} \int \frac{d^2k}{(2\pi)^2 2\omega_k} [a(k) e^{ik^\mu z_\mu} + a^\dagger(k) e^{-ik^\mu z_\mu}]$$

with the conformal vacuum being defined by $a(k) \mid 0 >= 0$. Consider now the mode functions

$$u_k(z) = (\frac{l}{z_2})^{-1/2} \frac{1}{[(2\pi)^2 2\omega_k]^{1/2}} e^{ik^\mu z_\mu}$$

which are "positive frequency" modes with respect to the timelike Killing field $\partial/\partial z_0 \equiv \partial_0$, i.e.,

$$\pounds_{\partial_0} u_k(z) = -i\omega_k u_k(z). \quad (\omega_k =\mid \vec{k} \mid > 0)$$

Obviously, these modes $u_k(z)$, which are positive frequency with respect to the conformal vacuum $\mid 0 >$ at one time, remain so for all time, $z_0$. Namely the positive frequency mode function $u_k(z)$ will remain identical both at early ($z_0 \to -\infty$) and at late ($z_0 \to \infty$) times and hence the associated conformal vacuum which was annihilated by $a(k)$ at early times will be so at late times, too, $\mid 0, \text{in} >=\mid 0, \text{out} >$. Also note that since $z_0$ and $t$ are related in the region $r > r_+$ by the coordinate transformation

$$\begin{aligned} z_0 &= (\frac{r^2 - r_+^2}{r^2 - r_-^2})^{1/2} \sinh(\frac{r_+}{l^2} t - \frac{r_-}{l} \phi) \exp[\frac{r_+}{l} \phi - \frac{r_-}{l^2} t] \\ &= (\frac{r^2 - r_+^2}{r^2 - r_-^2})^{1/2} \frac{1}{2} [\exp\{(r_+ - r_-)(\frac{1}{l^2} t + \frac{1}{l} \phi)\} - \exp\{-(r_+ + r_-)(\frac{1}{l^2} t - \frac{1}{l} \phi)\}] \end{aligned} \tag{48}$$

it is clear that $t \to -\infty$ corresponds to $z_0 \to -\infty$ and $t \to \infty$ corresponds to $z_0 \to \infty$. Therefore, in the Bogoliubov transformation

$$\begin{aligned} a_{out}(k) &= \sum_{k'} [\alpha_{k'k} a_{in}(k') + \beta^*_{k'k} a^\dagger_{in}(k')], \\ a^\dagger_{out}(k) &= \sum_{k'} [\alpha^*_{kk'} a^\dagger_{in}(k') + \beta_{kk'} a_{in}(k')] \end{aligned} \tag{49}$$

the Bogoliubov coefficient $\beta_{kk'}$ associated with the mode mixing is zero and hence there will be no particle creation, i.e.,

$$\begin{aligned} N_k &= < \text{in}, 0 \mid a^\dagger_{out}(k) a_{out}(k) \mid 0, \text{in} > \\ &= \sum_{k'} \mid \beta_{kk'} \mid^2 = 0. \end{aligned} \tag{50}$$

And of course the same argument holds for the exhibition of the absence of fermionic real particle creation. Now that we have convinced ourselves of the absence of particle creation (i.e., both the superradiance, which is the stimulated emission and the Hawking radiation, which is the spontaneous emission) by the $AdS_3$ black hole when the fields are conformally-coupled. However, one may still be puzzled since it has been known in the literature that accelerating detectors (of Unruh and DeWitt) outside the event horizon of the $AdS_3$ black hole do detect particles. Recall that according to the theory of particle detection formulated by Unruh [2,14] and by DeWitt [2,15], the transition probability to all possible excited states when the (accelerating) detector registered quanta is given by [2]

$$\Gamma = c^2 \sum_E |< E \mid m(0) \mid E_0 >|^2 \, F(\omega) \tag{51}$$

with $c$ being the small coupling constant, $m(\tau)$ detector's monopole moment operator and $F(\omega)$ the "detector response function" given by

$$F(\omega) = \int_{-\infty}^{\infty} d(\Delta\tau) e^{-i\omega\Delta\tau} g(\Delta\tau) \tag{52}$$

where $\omega = (E - E_0)$ and $g(\Delta\tau) = G^+(x(\tau), x(\tau'))$ with $G^+(x, x')$ being the positive-frequency Wightman function. The detector response functions have been calculated, in the case of scalar and spinor fields coupled conformally to the nonrotating $AdS_3$ black hole, by [4,5]

$$F_{boson}(\omega) \sim \frac{1}{2}\frac{1}{e^{\omega/T} + 1}, \qquad F_{spinor}(\omega) \sim \frac{i\omega}{2}\frac{1}{e^{\omega/T} - 1} \tag{53}$$

with the "local temperature" $T$ being given by $T = r_+/2\pi l(r^2 - r_+^2)^{1/2}$ *which vanishes asymptotically (i.e., as* $r \to \infty$*) incidentally confirming the absence of real particle creation.* Apparently, there also exists an issue concerning the "wrong statistics" between scalar and spinor case detector response functions known as "statistical inversion". We believe that the wrong statistics can be attributed to the odd-dimensionality ($d = 3$) of the $AdS_3$ black hole spacetime. We are, thus, in an uncomfortable situation where on one hand, there is no real particle creation and on the other, there is particle detection. In order to reconcile these two seemingly conflicting results, we need to note the distinction between the two concepts, "particle creation" and "particle detection". As a matter of fact, the former, particle creation by curved spacetime can be best understood by "Hawking effect" [2,13], while the latter, particle detection (which is possible even in flat Minkowski spacetime) can be best understood by "Unruh effect" [2]. It is clear enough that there is no particle creation in flat Minkowski spacetime. This, however, does not necessarily mean that particle detectors never click in this spacetime. The detection by particle detectors depends on the state of motion of the detector. Indeed, an accelerated detector in 2-dim. Minkowski spacetime, namely the Rindler observer will see a non-zero particle spectrum. This is the phenomenon known as Unruh effect [2]. And it can be attributed to the fact that the accelerating Rindler observers exist only in part of the Minkowski spacetime, i.e., the "Rindler wedge" separated from other regions by a horizon. Unruh effect is the well-known example of the fact that even without particle creation, there can be particle detection. Meanwhile in the Hawking effect [2,13], real particles are actually created by a black hole and thus can be detected as well.

To conclude, our situation associated with the conformally-coupled fields in conformally-flat $AdS_3$ black hole spacetime can be thought of as an analogue of the Unruh effect or of the case of conformally- coupled fields in the spatially-flat Friedmann-Robertson-Walker spacetimes [2]. Namely, although there is no real particle creation, particle spectrum is still detected because of the accelerating motion of the particle detector. And as we have seen, the absence of particle creation most clearly manifests itself in the conformal coordinates $z^\mu = (z_0, z_1, z_2)$ and the illustration of particle detection has been done in the Schwarzschild-type coordinates $x^\mu = (t, r, \phi)$ which is an accelerating coordinate system. In view of this, it seems meaningless to consider thermodynamics of $AdS_3$ black hole as long as quantum fields of interest are conformally-coupled since it never radiates real particles and hence it never possesses even a temperature (defined in the asymptotic region, $r \to \infty$) with which to start investigating its thermodynamic behaviors. In the presence of non-conformal coupling such as the minimal coupling, however, the $AdS_3$ black hole may evaporate and therefore one may consider its thermodynamics. In this regard, it seems appropriate to check, at this point, whether the main conclusion of the present work given above (on the absence of real particle creation by BTZ black hole in the case of "conformal triviality") can indeed be consistent with the results of recent works [17-19] aiming at the determination of grey body factor in the decay rate of BTZ black hole that we referred earlier. Firstly in [17,18], the authors are considering the case when minimally-coupled scalar field or non-minimally coupled dilaton field are employed to study the decay (or evaporation) of BTZ black hole. As we just pointed out, in these cases the BTZ black hole may be shown to evaporate and thus there is no contradiction between the conclusions of the present work and the results of [17,18]. Secondly in [19], the author considers the case when conformally-coupled (but in a non-standard manner) spinor field is employed again to study the decay of the BTZ black hole, particularly to compute the grey body factor. Since this author also employs the conformally-coupled spinor field, it may seem that there is a contradiction between the conclusion of this work and that of our present work. There is, however, no contradiction. As the author of [19] mentioned carefully in his work, "since the BTZ metric is asymptotically anti-de Sitter, the local temperature measured by any timelike observer decreases with distance and becomes zero at spatial infinity". Thus he chooses "a BTZ-observer, sitting at finite radial distance, detecting radiation". Namely, he studies the radiation from BTZ black hole measured by an accelerated observer placed at a "finite" distance from the hole. As we stressed, an accelerated observer can detect particle spectrum ("acceleration radiation") and thus the computation in [19] does not contradict to the absence of real particle creation by BTZ black hole in this conformally-trivial setting.

In the following section, we shall demonstrate, as a concrete evidence of the absence of real particle creation in this case of conformal triviality, the absence of both bosonic and fermionic superradiances, i.e., the absence of the stimulated emission. And to do so, the explicit mode expansion forms of scalar and spinor fields obtained in sect. III will play a central role.

## V. Demonstration of the absence superradiant scatterings

### 1. Introduction

A black hole is, by definition, a "region of no escape". No massive object or even the massless light ray, therefore, can ever be extracted from a black hole. When it comes to rotating black holes such as the Kerr family of solutions, however, things are not so simple and indeed energy can be extracted from black holes as was first noted by Penrose [6]. Briefly, this energy extraction mechanism proposed by Penrose and hence is called "Penrose process" [6] can be understood as follows. In Kerr geometry, the surface on which $g_{tt}$ vanishes does not coincide with the event horizon except at the poles. The toroidal space inbetween the two surfaces is called "ergosphere" and in particular the outer boundary of this ergosphere on which $g_{tt}$ vanishes is dubbed "static limit" because it can be seen that inside of which no observer can possibly remain static. Namely the time translational Killing field $\xi^{\mu} = (\partial/\partial t)^{\mu}$ becomes spacelike inside the ergosphere and so does the conserved component $p_t$ of the four momentum. As a consequence, the energy of a particle in this ergoregion, as perceived by an observer at infinity, can be negative. This last fact leads to a peculiar possibility that, in principle, one can devise a physical process which extracts energy and angular momentum from the black hole. The Penrose process, however, requires a precisely timed breakup of the incident particle at the relativistic velocities and thus is not a very practical energy extraction scheme. Perhaps because of this reason, an alternative study of energy extraction mechanism, known as "superradiant scattering" [7] was considered. In a sense, it can be thought of as a wave analogue of the Penrose process. If a wave is incident upon a black hole, the part of the wave ("transmitted" wave) will be absorbed by the black hole and the part of the wave ("reflected" wave) will escape back to infinity. Normally, the transmitted wave will carry positive energy into the black hole and the reflected wave will have less energy than the incident wave. However, for a scalar wave with the time ($t$) and azimuthal angle ($\phi$) dependence given by $e^{i(m\phi-\omega t)}$ (with $m$ and $\omega$ being the azimuthal number and the frequency respectively), the transmitted wave will carry negative energy into the black hole and the reflected wave will escape to infinity with greater amplitude and energy than the originally incident one provided the scalar wave has the frequency in the range [10]

$$0 < \omega < m\Omega_H$$

where again $\Omega_H$ denotes the angular velocity of the rotating hole at the event horizon. The "scalar waves" such as electromagnetic and gravitational waves exhibit this superradiance [8] when they have frequency in the range given above. Curiously enough, it is known that fermion fields do not display superradiance [9].

The spinning $AdS_3$ black hole spacetime that we are considering also possesses very similar structure to that of the Kerr black hole in 4-dim. including particularly the existence of ergoregion in which the time translational Killing field $\xi^\mu$ becomes spacelike. Consequently, the natural question one comes to ask is whether or not there are superradiant scatterings off this $AdS_3$ black hole. Moreover, we are in a perfect position to quantitatively check the possibility of superradiance since the precise mode expansions of the scalar (both real and complex) and the spinor fields are available now. In other words, unlike the case of scalar or spinor field in the Kerr black hole spacetime in 4-dim., now we need not assume the *ad hoc* time and azimuthal angle dependences of the fields to be $e^{i(m\phi-\omega t)}$ but we can pick a scalar or spinor wave with definite frequency to check the occurrence of superradiance. As mentioned earlier in the introduction and as we shall see shortly, as long as fields are conformally-coupled, the superradiant scattering off the $AdS_3$ black hole spacetime is absent. Therefore in order to demonstrate this absence of superradiance, in the following subsection, a succinct superradiance-checking algorithm employing the particle number or energy current will be formally reviewed and then applied to our $AdS_3$ black hole case.

**2. A simple superradiance-checking algorithm**

In this subsection, we would like to first set up a general formalism [10] that allows us to determine whether or not the superradiance is actually present in the case of scalar or fermion field. To this end, we introduce two quantities of central importance, "energy current" and "particle number current". First, we begin with the energy current. Generally, the "energy current" of a field in curved background spacetime is defined by [10]

$$J_\mu \equiv -T_{\mu\nu}\xi^\nu \tag{54}$$

with $\xi^\mu = (\partial/\partial t)^\mu$ being the time translational Killing field of a stationary, axisymmetric spacetime (which is the Kerr black hole spacetime for our case). This quantity is obviously conserved owing to the energy-momentum conservation and the Killing equation $\nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu = 0$ satisfied by the Killing field $\xi^\mu$, i.e.,

$$\nabla^\mu J_\mu = -(\nabla^\mu T_{\mu\nu})\xi^\nu - T_{\mu\nu}(\nabla^\mu\xi^\nu) = 0.$$

Next, we turn to the particle number current. Generally speaking, for field theories with action possessing the global U(1) transformation (i.e., phase transformation) symmetry, (e.g. complex scalar field theory and fermion field theory) the associated Noether current can be identified with the particle number current. Namely the Noether current of the typical form

$$j^\mu = \frac{\delta\mathcal{L}}{\delta(\nabla_\mu\phi^i)}\delta\phi^i \tag{55}$$

(where $\mathcal{L}$ denotes the Lagrangian density) is defined to be the particle number density. Then this particle number density is covariantly conserved as well due to the Euler-Lagrange's equation of motion and the invariance of the action, $\nabla_\mu j^\mu = 0$.

Now in order eventually to determine the presence or absence of the superradiant scattering, we consider a region $K$ of spacetime of which the boundary consists of two spacelike hypersurfaces $\Sigma_1$ at $(t)$ and $\Sigma_2$ at $(t+\delta t)$ (the constant time slice $\Sigma_2$ is a time translate of $\Sigma_1$ by $\delta t$) and two timelike hypersurfaces $H$ (black hole horizon at $r = r_+$) and $S_\infty$ (large sphere at spatial infinity $r \to \infty$). The appropriate directions of the hypersurface normal vector $n^\mu$ on each part of the boundary are ; $n^\mu$ is future-directed on $\Sigma_1$, past-directed on $\Sigma_2$. It is pointing inward the black hole on the event horizon $H$ and pointing outward to infinity on $S_\infty$. Then next, consider integrating the quantity $\nabla^\mu J_\mu$ (which leads to the "energy flux" crossing each part of the boundary upon utilizing the Gauss's theorem) or $\nabla_\mu j^\mu$ (which leads to the "particle number current") over the region $K$ of spacetime. By using Gauss's theorem we have

$$
\begin{aligned}
0 &= \int_K d^4x\sqrt{g}\nabla_\mu j^\mu \\
&= \int_{\partial K} d^3x\sqrt{h}n_\mu j^\mu \\
&= \int_{\Sigma_1(t)} n_\mu j^\mu + \int_{\Sigma_2(t+\delta t)} n_\mu j^\mu + \int_{H(r_+)} n_\mu j^\mu + \int_{S_\infty} n_\mu j^\mu
\end{aligned}
\tag{56}
$$

where $h_{\mu\nu}$ denotes the 3-metric induced on the boundary $\partial K$ of the region $K$. Now the terms in the last line of eq.(89) need some explanations. For boson or fermion field with time dependence of the form $e^{-i\omega t}$ (which we shall assume throughout) the first two terms cancel with each other by time translation symmetry. The third term represents the net particle number flow into the rotating black hole while the last term stands for the net particle number flow out of $K$ to infinity, i.e., the outgoing minus incoming particle number through $S_\infty$ during the time $\delta t$. Thus we end up with the result

$$
\int_{S_\infty} n_\mu j^\mu = -\int_{H(r_+)} n_\mu j^\mu \tag{57}
$$

which states that the *net particle number flow out of $K$ or the "outgoing minus incoming particle number" equals "minus" of the net particle number flow into the rotating black hole.* Therefore, now we can establish the criterion for the occurrence of superradiant scattering ; If the quantity on the right hand side $\int_{H(r_+)} n_\mu j^\mu$, namely the net particle number flowing down the hole, is negative (zero or positive), it means that the outgoing particle number flux is greater (smaller) than the incident one and hence the superradiance is present (absent). Thus far we have established the criterion for the occurrence of superradiance in terms of the "particle number current" $j^\mu$. An equivalent criterion can be derived in terms of the "energy current" $J_\mu$ if we replace $n_\mu j^\mu$ by $< n^\mu J_\mu >$ (where $< ... >$ denotes time averaged quantity) and replace "particle number current" with "energy current" respectively in the above formalism.

Also note that the hypersurface normal $n^\mu$ on the black hole event horizon $H$ is pointing inward the hole and hence is opposite to the direction of the Killing field [10]

$$
\chi^\mu = \xi^\mu + \Omega_H \psi^\mu \tag{58}
$$

which is outer normal to the rotating hole's event horizon. Thus our task of checking the presence or absence of superradiance reduces to the computation of the net particle number (or energy) current flowing into the rotating hole through its event horizon

$$\int_{H(r_+)} n_\mu j^\mu = -\int_{H(r_+)} \chi_\mu j^\mu. \tag{59}$$

Now, before we demonstrate the absence of the superradiance in the case of $AdS_3$ black hole spacetime, it will be comparative to illustrate the presence of the superradiance in a boson field case in the background of Kerr black hole in 4-dim. using the superradiance-checking formalism introduced above.

Thus consider a complex scalar field theory in a stationary, axisymmetric background spacetime (which we take to be the Kerr black hole geometry) described by the action (remember that here in this work, we employ the Misner-Thorne-Wheeler sign convention [1] in which the metric has the sign of $g_{\mu\nu} = diag(-+++)$)

$$S = -\int d^4x\sqrt{g}[\nabla_\mu\Phi^*\nabla^\mu\Phi + (M^2 + \xi R)\Phi^*\Phi] \tag{60}$$

and the classical field equations

$$\nabla_\mu\nabla^\mu\Phi - (M^2 + \xi R)\Phi = 0, \tag{61}$$
$$\nabla_\mu\nabla^\mu\Phi^* - (M^2 + \xi R)\Phi^* = 0$$

where $\xi$, $M$ and $R$ denotes some constant (for example, $\xi = 1/6$ with $M = 0$ corresponds to "conformal coupling"), the mass of the scalar field and the scalar curvature of the background spacetime respectively. Then we consider a situation when a complex scalar wave with particular frequency

$$\Phi(x) = \Phi_0(r,\theta)e^{i(m\phi-\omega t)} \tag{62}$$

is incident on and reflected by the Kerr black hole. Since the Lagrangian density of this complex scalar field in eq.(60) is invariant under the global U(1) (or phase) transformation

$$\Phi(x) \to e^{-i\alpha}\Phi(x),$$
$$\Phi^*(x) \to e^{i\alpha}\Phi^*(x)$$

corresponding Noether current exists and it is

$$j^{\mu} = -i(\Phi^{*}\nabla^{\mu}\Phi - \Phi\nabla^{\mu}\Phi^{*}). \tag{63}$$

This Noether current is the particle number current and it can be seen to be conserved owing to the classical field equations in eq.(61)

$$\nabla_{\mu}j^{\mu} = 0.$$

According to the criterion for the occurrence of the superradiance stated earlier, all we have to do now is to evaluate the net particle number flowing into the black hole, $\int_{H(r_+)} n_{\mu}j^{\mu}$ and see if it can be negative. Thus on the horizon $r = r_+$, we compute the particle number flux and it is

$$\begin{aligned} n_{\mu}j^{\mu} &= -\chi^{\mu}j_{\mu} \\ &= i(\Phi^{*}\chi^{\mu}\nabla_{\mu}\Phi - \Phi\chi^{\mu}\nabla_{\mu}\Phi^{*}) \\ &= i[\Phi^{*}(\frac{\partial}{\partial t} + \Omega_H\frac{\partial}{\partial\phi})\Phi - \Phi(\frac{\partial}{\partial t} + \Omega_H\frac{\partial}{\partial\phi})\Phi^{*}] \\ &= 2(\omega - m\Omega_H) \mid \Phi_0 \mid^2 . \end{aligned} \tag{64}$$

Thus for a complex scalar field with frequency in the range

$$0 < \omega < m\Omega_H \tag{65}$$

the net particle number flowing down the hole is negative and hence

$$\int_{S_\infty} n_{\mu}j^{\mu} = -\int_{H(r_+)} n_{\mu}j^{\mu} > 0 \tag{66}$$

namely the outgoing minus incident particle number flux through the large sphere $S_\infty$ is positive indicating the occurrence of superradiance in the case of a scalar field.

**3. Absence of superradiant scattering off $AdS_3$ black hole**

As usual, we would like to check the occurrence of superradiance with both boson and fermion fields. Thus we shall consider the three cases when the real, complex scalar and spinor fields are scattered off the rotating $AdS_3$ black hole respectively. There is, however, a crucial point that we would like to stress again. In 4-dim., we *assumed* that both the boson and the fermion fields possess the time and azimuthal angle dependences given by $e^{i(m\phi-\omega t)}$. This choice of the form of scalar and spinor waves is by no means rigorous in that we never really confirmed that solutions to the Klein-Gordon equation for the scalar field and solutions to the Dirac equation for the spinor field in the background of Kerr black hole do take this time and azimuthal angle dependences. We simply assumed this type of dependence based on the fact that Kerr black hole spacetime has time-translational and rotational isometries generated by Killing fields $\xi^\mu = (\partial/\partial t)^\mu$ and $\psi^\mu = (\partial/\partial\phi)^\mu$ respectively. For the present case of rotating $AdS_3$ black hole, however, as we have seen in the preceding section, exact mode expansions for both scalar and spinor fields are available. Namely, we do know the precise solution forms of the Klein-Gordon and Dirac equations in the background of the rotating $AdS_3$ black hole spacetime. Therefore we shall naturally use these exact solutions as test scalar and spinor waves to be scattered off the rotating $AdS_3$ black hole. We believe that this use of actual solutions will provide the correct answer to our question on the occurrence of superradiance in the case of $AdS_3$ black hole.

First we start with the case with real scalar field. As we have seen, the mode expansion of real scalar field coupled conformally to the background of spinning $AdS_3$ black hole is given in eq.(45). Since this mode expansion is the superposition of plane waves, we pick one plane wave solution with particular frequency

$$\Phi(z) = \mathrm{Re}[(\frac{l}{z_2})^{-1/2}e^{ik_\mu z^\mu}] = (\frac{l}{z_2})^{-1/2}\cos(k_\mu z^\mu) \tag{67}$$

and consider its scattering off the spinning $AdS_3$ black hole. In order to check the existence of superradiant scattering, we need to evaluate the energy flux flowing into the black hole across the horizon and see if it can be negative under certain circumstances. Thus on the horizon, $r = r_+$, using

$$\Phi(t, r_+, \phi) = \sum_{n=-\infty}^{\infty} l^{-1/2} e^{\frac{1}{2}(la_+\phi_n - a_- t)} \cos[k_2 e^{(la_+\phi_n - a_- t)}]$$

the time-averaged energy flux is

$$
\begin{aligned}
< n^\mu J_\mu > &= - < \chi^\mu J_\mu > \\
&= < T_{\mu\nu}\chi^\mu \xi^\nu > = < (\chi^\mu \nabla_\mu \Phi)(\xi^\nu \nabla_\nu \Phi) > = < (\partial_t + \Omega_H \partial_\phi)\Phi \partial_t \Phi > \qquad (68) \\
&= a_-(a_- - \Omega_H l a_+) < \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} A(t, r_+, \phi_n) A(t, r_+, \phi_m) >
\end{aligned}
$$

where

$$A(t, r_+, \phi_n) = l^{-1/2} e^{\frac{1}{2}(la_+ \phi_n - a_- t)} \{ \frac{1}{2} \cos[k_2 e^{(la_+ \phi_n - a_- t)}] - k_2 e^{(la_+ \phi_n - a_- t)} \sin[k_2 e^{(la_+ \phi_n - a_- t)}] \}.$$

Now, if the quantity in the last line could be negative, then the superradiance occurs. Obviously, however, this cannot really happen since $\Omega_H = a/r_+^2 = r_-/lr_+$ and $a_\pm = r_\pm/l^2$,

$$(a_- - \Omega_H l a_+) = 0. \qquad (69)$$

Therefore, regardless of the value of its frequency, (real) scalar field does not display superradiance. And this is in contrast to what happens in the case of Kerr black holes in 4-dim. spacetime.

Secondly, we consider the case of complex scalar field. The mode expansion of complex scalar field coupled conformally to the background of spinning $AdS_3$ black hole is given in eq.(46). Again, since these mode expansions are the superpositions of plane waves, we pick one plane wave solution with particular frequency

$$
\begin{aligned}
\Phi(z) &= (\frac{l}{z_2})^{-1/2} e^{ik_\mu z^\mu}, \qquad (70) \\
\Phi^*(z) &= (\frac{l}{z_2})^{-1/2} e^{-ik_\mu z^\mu}.
\end{aligned}
$$

and consider its scattering off the spinning $AdS_3$ black hole. In order to see if there is a superradiant scattering, we need to compute the particle number flux flowing into the black hole across the horizon and check whether it can be negative under certain circumstances. Thus on the horizon, $r = r_+$, using

$$
\begin{aligned}
\Phi(t, r_+, \phi) &= \sum_{n=-\infty}^{\infty} l^{-1/2} e^{\frac{1}{2}(la_+ \phi_n - a_- t)} \exp[ik_2 e^{(la_+ \phi_n - a_- t)}], \\
\Phi^*(t, r_+, \phi) &= \sum_{n=-\infty}^{\infty} l^{-1/2} e^{\frac{1}{2}(la_+ \phi_n - a_- t)} \exp[-ik_2 e^{(la_+ \phi_n - a_- t)}].
\end{aligned}
$$

the particle number flux is

$$
\begin{aligned}
n_\mu j^\mu &= -\chi^\mu j_\mu \\
&= i(\Phi^* \chi^\mu \nabla_\mu \Phi - \Phi \chi^\mu \nabla_\mu \Phi^*) \\
&= i[\Phi^*(\partial_t + \Omega_H \partial_\phi)\Phi - \Phi(\partial_t + \Omega_H \partial_\phi)\Phi^*] \\
&= (a_- - \Omega_H l a_+) k_2 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} B(t, r_+, \phi_n) B^*(t, r_+, \phi_m) [e^{(la_+ + \phi_n - a_- t)} + e^{(la_+ + \phi_m - a_- t)}]
\end{aligned} \tag{71}
$$

where $B(t, r_+, \phi_n) = l^{-1/2} e^{\frac{1}{2}(la_+ + \phi_n - a_- t)} \exp[i k_2 e^{(la_+ + \phi_n - a_- t)}]$. Again, if the quantity in the last line could be negative, then the superradiance occurs which, as we have realized earlier in the case of real scalar field, cannot really happen since $\Omega_H = a/r_+^2 = r_-/lr_+$ and $a_\pm = r_\pm/l^2$, thus

$$
(a_- - \Omega_H l a_+) = 0. \tag{72}
$$

Again, irrespective of the value of its frequency, complex scalar field does not display superradiance either.

Finally, we turn to the case of spinor field. As already mentioned and as we shall see shortly as well, in order to check the occurrence of superradiance in the fermion field case, one needs the concrete geometry structure of the background $AdS_3$ black hole. Besides, the standard formulation of spinor field theory in curved background spacetime is associated with the Riemann-Cartan formulation of general relativity in which one of the basic computational tools is the use of the non-holonomic basis 1-form (i.e., "soldering form"). Thus here we begin with the $AdS_3$ black hole metric (given in Schwarzschild-like coordinates) written in the ADM's (2+1) space-plus-time split form which proves to be suitable to be converted to the one in non-coordinate basis

$$
\begin{aligned}
ds^2 &= -N^2 dt^2 + h_{rr} dr^2 + h_{\phi\phi}[d\phi + N^\phi dt]^2 \\
&= g_{\mu\nu} dx^\mu dx^\nu = \eta_{ab} e^a e^b
\end{aligned} \tag{73}
$$

where Greek indices refer to the accelerated frame of reference (i.e., coordinate basis, $\mu = t,\ r,\ \phi$) and the Roman indices refer to the locally inertial reference frame (i.e., non-coordinate basis, $a = 0,\ 1,\ 2$). Also we used the definitions for the soldering form ("dreibein") $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$ and the non-coordinate basis 1-form $e^a = e^a_\mu dx^\mu$. In the ADM's (2+1) split form above, the lapse, shift functions and the spatial metric components are given respectively by

$$
\begin{aligned}
N^2(r) &= (-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}), \\
N^\phi(r) &= -\frac{J}{2r^2}, \quad N^r(r) = 0, \\
h_{rr}(r) &= N^{-2}(r), \quad h_{\phi\phi}(r) = r^2, \\
h_{r\phi} &= h_{\phi r} = 0.
\end{aligned}
\tag{74}
$$

where $M$ and $J = 2a$ denote the ADM mass and the angular momentum per unit mass of the $AdS_3$ black hole respectively and $l^{-2} = (-\Lambda)$ as indicated earlier. As is well known the event horizon developes at the larger zero of $N^2(r_+) = 0$.

Actually, the virtue of writing the $AdS_3$ black hole metric in the ADM's (2+1) split form as in eq.(73) above is that from which now one can read off the non-coordinate basis 1-form easily as follows

$$
\begin{aligned}
e^0 &= e^0_\mu dx^\mu = N dt, \\
e^1 &= e^1_\mu dx^\mu = \sqrt{h_{rr}} dr = N^{-1} dr, \\
e^2 &= e^2_\mu dx^\mu = \sqrt{h_{\phi\phi}} (d\phi + N^\phi dt) = r(d\phi + N^\phi dt).
\end{aligned}
\tag{75}
$$

Equvalently, the dreibein and the inverse dreibein can be read off as

$$
e^a_\mu = \begin{pmatrix} N & 0 & 0 \\ 0 & N^{-1} & 0 \\ rN^\phi & 0 & r \end{pmatrix}, \quad e^\mu_a = \begin{pmatrix} N^{-1} & 0 & 0 \\ 0 & N & 0 \\ -N^{-1}N^\phi & 0 & r^{-1} \end{pmatrix}
\tag{76}
$$

Further, the spin connection 1-form $\omega^{ab} = \omega^{ab}_\mu dx^\mu$ can be obtained from the Cartan's 1st structure equation, $de^a + \omega^a_b \wedge e^b = 0$ using the non-coodinate basis 1-form given in eq.(75). Here, however, we do not look for the spin connection since we shall not really need its explicit form in the discussion below leading to the conclusion on the absence of fermionic superradiance.

Now, for later use we write the $\gamma$-matrices with coordinate basis indices in accelerated frame (i.e., in Schwarzschild-type coordinates) in terms of those with non-coordinate basis indices in locally-inertial frame using the soldering form (inverse dreibein) given above, i.e., $\gamma^{\mu}(x) = e^{\mu}_{a}(x)\gamma^{a}$

$$\begin{aligned}
\gamma^{t} &= e^{t}_{a}\gamma^{a} = N^{-1}\gamma^{0}, \\
\gamma^{r} &= e^{r}_{a}\gamma^{a} = N\gamma^{1}, \\
\gamma^{\phi} &= e^{\phi}_{a}\gamma^{a} = -N^{-1}N^{\phi}\gamma^{0} + r^{-1}\gamma^{2}.
\end{aligned} \tag{77}$$

With this preparation, now we consider the conformally-coupled spinor field theory in the background of this $AdS_3$ black hole spacetime described by the action [2]

$$S = \int d^{3}x\sqrt{g}\{\frac{i}{2}[\bar{\psi}\gamma^{\mu}\overrightarrow{\nabla}_{\mu}\psi - \bar{\psi}\gamma^{\mu}\overleftarrow{\nabla}_{\mu}\psi]\} \tag{78}$$

and the classical field equations, i.e., curved spacetime Dirac equations

$$i\gamma^{\mu}\overrightarrow{\nabla}_{\mu}\psi = 0, \quad \bar{\psi}i\gamma^{\nu}\overleftarrow{\nabla}_{\nu} = 0. \tag{79}$$

The mode expansion of spinor field coupled conformally to the background of spinning $AdS_3$ black hole was given in eq.(47). Again, since this mode expansion is the superposition of plane waves, we consider a situation when a spinor wave with particular frequency and spin

$$\psi(z) = (\frac{l}{z_2})^{-1}u(p)e^{ip_{\mu}z^{\mu}} \tag{80}$$

is incident on and reflected by the $AdS_3$ black hole. Here the 2-component Dirac spinor $u(p)$ satisfies the Dirac equation in curved spacetime given above. As usual, in order to check the occurrence of superradiant scattering, we need to evaluate the particle number flux across the event horizon and see if it can be negative under certain circumstances. To this end, we begin by defining the particle number current for spinor field. Since the Lagrangian density of this spinor field given in eq.(78) is also invariant under the global U(1) (or phase) transformation

$$\psi(x) \to e^{-i\alpha}\psi(x), \quad \bar{\psi}(x) \to \bar{\psi}(x)e^{i\alpha}$$

corresponding Noether current exists and it is

$$j^\mu = \frac{\delta \mathcal{L}}{\delta(\overrightarrow{\nabla}_\mu \psi)}\delta\psi + \delta\bar{\psi}\frac{\delta \mathcal{L}}{\delta(\bar{\psi}\overleftarrow{\nabla}_\mu)} = \bar{\psi}\gamma^\mu\psi. \tag{81}$$

This Noether current is identified with the particle number current and it can be seen to be conserved due to the Dirac equations given above

$$\nabla_\mu j^\mu = \bar{\psi}\gamma^\mu \overleftarrow{\nabla}_\mu \psi + \bar{\psi}\gamma^\mu \overrightarrow{\nabla}_\mu \psi = 0$$

Thus on the horizon, $r = r_+$, using

$$\psi(t, r_+, \phi) = \sum_{n=-\infty}^{\infty} l^{-1} e^{(la_+\phi_n - a_- t)} u(p) \exp[ip_2 e^{(la_+\phi_n - a_- t)}],$$
$$\bar{\psi}(t, r_+, \phi) = \sum_{n=-\infty}^{\infty} l^{-1} e^{(la_+\phi_n - a_- t)} \bar{u}(p) \exp[-ip_2 e^{(la_+\phi_n - a_- t)}]$$

the particle number flux is

$$\begin{aligned} n_\mu j^\mu &= -\chi^\mu j_\mu = -\bar{\psi} g_{\alpha\beta}\chi^\alpha\gamma^\beta\psi \\ &= -\bar{\psi} g_{\alpha\beta}(\delta^\alpha_t + \Omega_H \delta^\alpha_\phi)\gamma^\beta\psi \\ &= -\bar{\psi}[(g_{tt} + \Omega_H g_{t\phi})\gamma^t + (g_{t\phi} + \Omega_H g_{\phi\phi})\gamma^\phi]\psi \\ &= -\bar{\psi}[-N\gamma^0 + r(N^\phi + \Omega_H)\gamma^2]\psi = 0. \end{aligned} \tag{82}$$

where we used the relation between $\gamma$-matrices with coordinate basis indices in accelerated frame (i.e., in Schwarzschild-type coordinates) and those with non-coordinate basis indices in locally-inertial frame, $\gamma^\mu(x) = e^\mu_a(x)\gamma^a$ derived earlier in eq.(77) and $g_{tt} = -[N^2 - r^2(N^\phi)^2]$, $g_{t\phi} = r^2 N^\phi$ and $g_{\phi\phi} = r^2$. And to get the last equality to zero we used

$$N^2(r_+) = 0, \quad N^\phi(r_+) = -(\frac{a}{r_+^2}) = -\Omega_H.$$

This result indicates that the net fermionic particle number flux flowing down the hole through its event horizon is zero *irrespective of the frequency of the fermion field.* Therefore the outgoing minus incident fermionic particle number flux through the large sphere $S_\infty$ is zero

$$\int_{S_\infty} n_\mu j^\mu = -\int_{H(r_+)} n_\mu j^\mu = 0$$

establishing again the absence of superradiance in the case of fermion field as well.

Thus far we have illustrated the absence of fermionic superradiance in terms of the particle number flux. One can draw the same conclusion in terms of the energy flux we introduced earlier by showing that it also is zero through the event horizon of the $AdS_3$ black hole. Since it is rather straightforward to demonstrate this, we shall not get into the detail further.

We now wish to provide physical interpretation of the absence of both bosonic and fermionic superradiances observed above. To this end, we should go back and cite our earlier statement on the physical meaning of the conformal vacuum. In curved spacetimes, generally there is no meaningful notion of global vacuum and global Fock space. The vacuum and hence the concept of particle is really observer-dependent. Nevertheless, if there exist geometrical symmetries in the background spacetime, it may be that a particular set of modes and the corresponding vacuum and Fock space emerge as having natural physical meaning. The conformally-coupled scalar and spinor fields in conformally-flat spacetimes like $AdS_3$ black hole spacetime are endowed with such a feature and the associated vacuum state is the "conformal vacuum" which remains to be a vacuum with respect to any other reference frame. This means that particle production is absent since $\mid 0, \text{in} >$ and $\mid 0, \text{out} >$ are identical. In view of this, the absence of the superradiance which is a stimulated emission phenomenon from the ergoregion seems to be a natural consequence. Furthermore, this observation implies that if one sticks to consider solely the conformally-coupled fields in this $AdS_3$ black hole spacetime, the Hawking radiation [13], namely the spontaneous emission of particles should be absent as well.

## VI. Discussions

Now we summarize what has been done in this work. Noticing that the $AdS_3$ black hole solution discovered recently by BTZ [3] has attracted a lot of interest currently, we attempted quantisation of scalar and spinor fields in the background of this $AdS_3$ black hole spacetime. And to do so we particularly employed the "conformal gauge" in which the metric takes the manifestly conformally-flat form. Mode expansions (given in sect.III) and two point Green's functions (which will be given in the appendix below) for the scalar and spinor fields have been obtained in closed forms. To be more concrete, first the construction of two-point Green's functions for conformally-coupled fields performed in the present work can be thought of as employing the "transparent boundary conditions" introduced originally by Avis, Isham and Storey [12]. Besides the construction of the Green's functions for conformally-coupled spinor field given in the present work has not yet been attempted in the literature. It *is* a new contribution. Next, the construction of mode expansions for quantum fields in explicit, closed forms allowed several important observations. Firstly, the particle number conservation across the event horizon and the Cauchy horizon of the $AdS_3$ black hole confirms that indeed the inner and outer horizons are just coordinate singularities on which nothing special happens. Secondly, but more importantly, on the absence of superradiant scattering off the spinning $AdS_3$ black hole. Since the spinning $AdS_3$ black hole possesses very similar causal structure to that of Kerr black hole in 4-dim. including the existence of ergoregion, one naturally gets interested in the possibility of superradiance phenomenon. However, unlike the conventional study of superradiance for the Kerr black hole in 4-dim., now one needs not assume the *ad hoc* time and azimuthal angle dependences of the fields to be $e^{i(m\phi-\omega t)}$ since the precise mode expansion forms for the scalar and spinor fields are known. Thus using these exact solutions to the Klein-Gordon and Dirac equations in the background of the $AdS_3$ black hole spacetime as test scalar and spinor waves to be scattered off the rotating $AdS_3$ black hole, both the bosonic and fermionic superradiances have been displayed to be absent. The physical interpretation of this absence of superradiance (i.e., the stimulated emission) and the absence of Hawking evaporation (i.e., the spontaneous emission) of $AdS_3$ black hole for the case of "conformal triviality" (namely, the case of conformally-coupled fields in conformally-flat spacetime), then, has been provided in terms of the conformal vacuum, which is unique and has global meaning. Finally, it is our hope that our general formulation of the quantisation of conformally-coupled fields in conformally flat spacetimes in arbitrary dimensions and our explicit demonstration particularly in $AdS_3$ black hole spacetime may find interesting applications in various other situations.

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## Appendix : The Green's functions of conformally-coupled fields

Here, we shall provide the 2-point Green's functions, particularly the positive-frequency Wightman functions [2] of the scalar and spinor fields coupled conformally to the spinning $AdS_3$ black hole spacetime in closed forms.

### 1. Scalar field

For any 2-point Green's function of scalar field conformally coupled to a conformally flat spacetime is given by (in $n$-dimension) [2]

$$G(x,x') = \Omega^{-(\frac{n-2}{2})}(x) G_0(x,x') \Omega^{-(\frac{n-2}{2})}(x') \tag{83}$$

with $G_0$ being the free field propagator in flat, Minkowski spacetime. Consider that we are interested in the positive-frequency Wightman function for the real scalar field in the background of $AdS_3$ black hole spacetime. Then it is given by

$$G^+(z,z') = \Omega^{-\frac{1}{2}}(z) G_0^+(z,z') \Omega^{-\frac{1}{2}}(z') \tag{84}$$

We first have to evaluate, in flat 3-dimensional spacetime

$$G_0^+(z,z') = \int \frac{d^3k}{(2\pi)^3} \frac{-1}{k^2} e^{ik\cdot(z-z')} \tag{85}$$

where a particularly prescribed integration contour in the complex $k^0$ plane is assumed. However, if we analytically continue to the Euclidean space via the "Wick rotation"

$$t \to t_E = it, \quad k^0 \to k_E^0 = ik^0 \tag{86}$$

then the evaluation of the integral can be done rather easily to yield

$$\begin{aligned} G_0^+(z,z') &= i \int \frac{d^3k_E}{(2\pi)^3} \frac{1}{k_E^2} e^{ik_\mu^E (z-z')^\mu} \\ &= \frac{i}{4\pi} \frac{1}{|z-z'|} = (\frac{i}{4\pi}) \Big[ -(z_0 - {z'}_0)^2 + (z_1 - {z'}_1)^2 + (z_2 - {z'}_2)^2 \Big]^{-\frac{1}{2}}. \end{aligned} \tag{87}$$

Therefore

$$\begin{aligned} G^+(z,z') &= \Omega^{-\frac{1}{2}}(z) G_0^+(z,z') \Omega^{-\frac{1}{2}}(z') \\ &= \frac{i}{4\pi l} \Big[ \frac{-(z_0 - {z'}_0)^2 + (z_1 - {z'}_1)^2 + (z_2 - {z'}_2)^2}{z_2 {z'}_2} \Big]^{-\frac{1}{2}}. \end{aligned} \tag{88}$$

As was stated when obtaining the correct mode expansions of the scalar and spinor fields conformally coupled to this BTZ black hole spacetime, the correct Green functions exhibiting the periodicity in $\phi$-coordinate can be constructed via the infinite linear sum

$$G^{+}(z,z') = \sum_{n=-\infty}^{\infty} G^{+}(z_n, z'_n)$$

(where $z_n = z(t,r,\phi_n)$, $z'_n = z(t,r,\phi'_n)$ with again $\phi_n = \phi + 2\pi n$ and $\phi'_n = \phi' + 2\pi n$) which amounts to employing the method of images [15].

(1) For $z, z' \in$ Region (I); $r, r' \geq r_+$

$$G^{+}(z_n, z'_n) = \frac{i}{4\sqrt{2}\pi l}\left\{\frac{(r^2-r_-^2)^{\frac{1}{2}}(r'^2-r_-^2)^{\frac{1}{2}}}{r_+^2-r_-^2}\cosh F_-(\Delta t, \Delta\phi_n) - \frac{(r^2-r_+^2)^{\frac{1}{2}}(r'^2-r_+^2)^{\frac{1}{2}}}{r_+^2-r_-^2}\cosh F_+(\Delta t, \Delta\phi_n) - 1\right\}^{-\frac{1}{2}}.$$

(2) For $z, z' \in$ Region (II); $r_- < r, r' \leq r_+$

$$G^{+}(z_n, z'_n) = \frac{i}{4\sqrt{2}\pi l}\left\{\frac{(r^2-r_-^2)^{\frac{1}{2}}(r'^2-r_-^2)^{\frac{1}{2}}}{r_+^2-r_-^2}\cosh F_-(\Delta t, \Delta\phi_n) + \frac{(r_+^2-r^2)^{\frac{1}{2}}(r_+^2-r'^2)^{\frac{1}{2}}}{r_+^2-r_-^2}\cosh F_+(\Delta t, \Delta\phi_n) - 1\right\}^{-\frac{1}{2}}.$$

(3) For $z, z' \in$ Region (III); $0 < r, r' \leq r_-$

$$G^{+}(z_n, z'_n) = \frac{i}{4\sqrt{2}\pi l}\left\{\frac{(r_+^2-r^2)^{\frac{1}{2}}(r_+^2-r'^2)^{\frac{1}{2}}}{r_+^2-r_-^2}\cosh F_+(\Delta t, \Delta\phi_n) - \frac{(r_-^2-r^2)^{\frac{1}{2}}(r_-^2-r'^2)^{\frac{1}{2}}}{r_+^2-r_-^2}\cosh F_-(\Delta t, \Delta\phi_n) - 1\right\}^{-\frac{1}{2}}.$$

(4) For $z \in$ Region(I); $(r \geq r_+)$ $\quad z' \in$ Region (II); $(r_- < r' \leq r_+)$

$$G^{+}(z_n, z'_n) = \frac{i}{4\sqrt{2}\pi l}\left\{\frac{(r^2-r_-^2)^{\frac{1}{2}}(r'^2-r_-^2)^{\frac{1}{2}}}{r_+^2-r_-^2}\cosh F_-(\Delta t, \Delta\phi_n) + \frac{(r^2-r_+^2)^{\frac{1}{2}}(r_+^2-r'^2)^{\frac{1}{2}}}{r_+^2-r_-^2}\sinh F_+(\Delta t, \Delta\phi_n) - 1\right\}^{-\frac{1}{2}}.$$

(5) For $z \in$ Region (II); $(r_- < r \leq r_+)$ $\quad z' \in$ Region (III); $(0 < r' \leq r_-)$

$$G^{+}(z_n, z'_n) = \frac{i}{4\sqrt{2}\pi l}\left\{\frac{(r^2-r_-^2)^{\frac{1}{2}}(r_+^2-r'^2)^{\frac{1}{2}}}{r_+^2-r_-^2}\cosh[(a_-t - a_+t') - l(a_+\phi_n - a_-\phi'_n)] - \frac{(r_+^2-r^2)^{\frac{1}{2}}(r_-^2-r'^2)^{\frac{1}{2}}}{r_+^2-r_-^2}\sinh[(a_+t - a_-t') - l(a_-\phi_n - a_+\phi'_n)] - 1\right\}^{-\frac{1}{2}}.$$

where we used the short-hand notation $F_{\pm}(\Delta t, \Delta\phi_n) \equiv [a_{\pm}(t-t') - la_{\mp}(\phi - \phi' + 2\pi n)]$ with $a_{\pm} = r_{\pm}/l^2$ and $\phi_n = \phi + 2\pi n$, $\Delta\phi_n = (\phi - \phi') + 2\pi n$ as defined earlier.

**2. Spinor field**

Generally, any 2-point Green's function of spinor field conformally-coupled to a conformally-flat spacetime is given by [2]

$$S(x,x') = \Omega^{-(\frac{n-1}{2})}(x) S_0(x,x') \Omega^{-(\frac{n-1}{2})}(x') \tag{89}$$

with $S_0$ being the free, massless field Green's function in flat, Minkowski spacetime as mentioned earlier. Besides, as pointed out earlier, the spinor field Green's function in flat spacetime is related to that of the scalar field by (generally in the presence of the mass)

$$S_0(x,x') = (i\gamma^{\alpha}\partial^x_{\alpha} + m) G_0(x,x'). \tag{90}$$

Again, consider that we are interested in the positive-frequency Wightman function for the spinor field in the background of the $AdS_3$ black hole spacetime. Then it is given by

$$S^+(z,z') = \Omega^{-1}(z) S^+_0(z,z') \Omega^{-1}(z') \tag{91}$$

where $S^+_0$ can be obtained from $G^+_0$ evaluated earlier as

$$\begin{aligned} S^+_0(z,z') &= (i\gamma^{\alpha}\partial^z_{\alpha}) G^+_0(z,z') \\ &= -(\gamma^{\alpha}\partial^z_{\alpha})\Big[\frac{1}{4\pi}\frac{1}{|z-z'|}\Big] = (\frac{1}{4\pi})\frac{\gamma^{\alpha}(z-z')_{\alpha}}{|z-z'|^3}. \end{aligned} \tag{92}$$

Finally, therefore

$$\begin{aligned} S^+(z,z') &= (\frac{l}{z_2})^{-1}\left\{(\frac{1}{4\pi})\frac{-\gamma^0(z_0-z'_0)+\gamma^1(z_1-z'_1)+\gamma^2(z_2-z'_2)}{[-(z_0-z'_0)^2+(z_1-z'_1)^2+(z_2-z'_2)^2]^{\frac{3}{2}}}\right\}(\frac{l}{z'_2})^{-1} \\ &= \frac{1}{4\pi l^2}\left\{\frac{-(z_0-z'_0)^2+(z_1-z'_1)^2+(z_2-z'_2)^2}{z_2 z'_2}\right\}^{-\frac{3}{2}} \\ &\quad \times (z_2 z'_2)^{-\frac{1}{2}}[-\gamma^0(z_0-z'_0)+\gamma^1(z_1-z'_1)+\gamma^2(z_2-z'_2)]. \end{aligned} \tag{93}$$

And again the correct Green function is given by

$$S^{+}(z,z') = \sum_{n=-\infty}^{\infty} S^{+}(z_n, z'_n)$$

(1) For $z, z' \in$ Region (I); $r, r' \geq r_+$

$$(z_0 - {z'}_0) = \left\{ \Big(\frac{r^2 - r_+^2}{r^2 - r_-^2}\Big)^{\frac{1}{2}} \sinh F_+(t,\phi_n) e^{[-F_-(t,\phi_n)]} - \Big(\frac{r'^2 - r_+^2}{r'^2 - r_-^2}\Big)^{\frac{1}{2}} \sinh F_+(t',\phi'_n) e^{[-F_-(t',\phi'_n)]} \right\},$$

$$(z_1 - {z'}_1) = \left\{ \Big(\frac{r^2 - r_+^2}{r^2 - r_-^2}\Big)^{\frac{1}{2}} \cosh F_+(t,\phi_n) e^{[-F_-(t,\phi_n)]} - \Big(\frac{r'^2 - r_+^2}{r'^2 - r_-^2}\Big)^{\frac{1}{2}} \cosh F_+(t',\phi'_n) e^{[-F_-(t',\phi'_n)]} \right\},$$

$$(z_2 - {z'}_2) = \left\{ \Big(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\Big)^{\frac{1}{2}} e^{[-F_-(t,\phi_n)]} - \Big(\frac{r_+^2 - r_-^2}{r'^2 - r_-^2}\Big)^{\frac{1}{2}} e^{[-F_-(t',\phi'_n)]} \right\}.$$

$$\begin{aligned} S^{+}(z_n, z'_n) = \frac{1}{8\sqrt{2}\pi l^2} & \left\{ \frac{(r^2 - r_-^2)^{\frac{1}{2}}(r'^2 - r_-^2)^{\frac{1}{2}}}{r_+^2 - r_-^2} \cosh F_-(\Delta t, \Delta\phi_n) \right. \\ & \left. - \frac{(r^2 - r_+^2)^{\frac{1}{2}}(r'^2 - r_+^2)^{\frac{1}{2}}}{r_+^2 - r_-^2} \cosh F_+(\Delta t, \Delta\phi_n) - 1 \right\}^{-\frac{3}{2}} \\ & \times \frac{(r^2 - r_-^2)^{\frac{1}{4}}(r'^2 - r_-^2)^{\frac{1}{4}}}{(r_+^2 - r_-^2)^{\frac{1}{2}}} \exp\frac{1}{2}\Big[F_-(t,\phi_n) + F_-(t',\phi'_n)\Big] \\ & \times \Big[-\gamma^0(z_0 - {z'}_0) + \gamma^1(z_1 - {z'}_1) + \gamma^2(z_2 - {z'}_2)\Big]. \end{aligned}$$

(2) For $z, z' \in$ Region (II); $r_- < r, r' \leq r_+$

$$(z_0 - {z'}_0) = \left\{ \Big(\frac{r_+^2 - r^2}{r^2 - r_-^2}\Big)^{\frac{1}{2}} \cosh F_+(t,\phi_n) e^{[-F_-(t,\phi_n)]} - \Big(\frac{r_+^2 - r'^2}{r'^2 - r_-^2}\Big)^{\frac{1}{2}} \cosh F_+(t',\phi'_n) e^{[-F_-(t',\phi'_n)]} \right\},$$

$$(z_1 - {z'}_1) = \left\{ \Big(\frac{r_+^2 - r^2}{r^2 - r_-^2}\Big)^{\frac{1}{2}} \sinh F_+(t,\phi_n) e^{[-F_-(t,\phi_n)]} - \Big(\frac{r_+^2 - r'^2}{r'^2 - r_-^2}\Big)^{\frac{1}{2}} \sinh F_+(t',\phi'_n) e^{[-F_-(t',\phi'_n)]} \right\},$$

$$(z_2 - {z'}_2) = \left\{ \Big(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\Big)^{\frac{1}{2}} e^{[-F_-(t,\phi_n)]} - \Big(\frac{r_+^2 - r_-^2}{r'^2 - r_-^2}\Big)^{\frac{1}{2}} e^{[-F_-(t',\phi'_n)]} \right\}.$$

$$\begin{aligned} S^{+}(z_n, z'_n) = \frac{1}{8\sqrt{2}\pi l^2} & \left\{ \frac{(r^2 - r_-^2)^{\frac{1}{2}}(r'^2 - r_-^2)^{\frac{1}{2}}}{r_+^2 - r_-^2} \cosh F_-(\Delta t, \Delta\phi_n) \right. \\ & \left. + \frac{(r_+^2 - r^2)^{\frac{1}{2}}(r_+^2 - r'^2)^{\frac{1}{2}}}{r_+^2 - r_-^2} \cosh F_+(\Delta t, \Delta\phi_n) - 1 \right\}^{-\frac{3}{2}} \\ & \times \frac{(r^2 - r_-^2)^{\frac{1}{4}}(r'^2 - r_-^2)^{\frac{1}{4}}}{(r_+^2 - r_-^2)^{\frac{1}{2}}} \exp\frac{1}{2}\Big[F_-(t,\phi_n) + F_-(t',\phi'_n)\Big] \\ & \times \Big[-\gamma^0(z_0 - {z'}_0) + \gamma^1(z_1 - {z'}_1) + \gamma^2(z_2 - {z'}_2)\Big]. \end{aligned}$$

(3) For $z, z' \in$ Region (III); $0 < r, r' \leq r_-$

$$(z_0 - {z'}_0) = \left\{ \Big(\frac{r_-^2 - r^2}{r_+^2 - r^2}\Big)^{\frac{1}{2}} \sinh F_-(t,\phi_n) e^{[-F_+(t,\phi_n)]} - \Big(\frac{r_-^2 - r'^2}{r_+^2 - r'^2}\Big)^{\frac{1}{2}} \sinh F_-(t',\phi'_n) e^{[-F_+(t',\phi'_n)]} \right\},$$

$$(z_1 - {z'}_1) = \left\{ \Big(\frac{r_-^2 - r^2}{r_+^2 - r^2}\Big)^{\frac{1}{2}} \cosh F_-(t,\phi_n) e^{[-F_+(t,\phi_n)]} - \Big(\frac{r_-^2 - r'^2}{r_+^2 - r'^2}\Big)^{\frac{1}{2}} \cosh F_-(t',\phi'_n) e^{[-F_+(t',\phi'_n)]} \right\},$$

$$(z_2 - {z'}_2) = \left\{ \Big(\frac{r_+^2 - r_-^2}{r_+^2 - r^2}\Big)^{\frac{1}{2}} e^{[-F_+(t,\phi_n)]} - \Big(\frac{r_+^2 - r_-^2}{r_+^2 - r'^2}\Big)^{\frac{1}{2}} e^{[-F_+(t',\phi'_n)]} \right\}.$$

$$\begin{aligned} S^+(z_n, z'_n) = \frac{1}{8\sqrt{2}\pi l^2} \Big\{ & \frac{(r_+^2 - r^2)^{\frac{1}{2}} (r_+^2 - r'^2)^{\frac{1}{2}}}{r_+^2 - r_-^2} \cosh F_+(\Delta t, \Delta\phi_n) \\ & - \frac{(r_-^2 - r^2)^{\frac{1}{2}} (r_-^2 - r'^2)^{\frac{1}{2}}}{r_+^2 - r_-^2} \cosh F_-(\Delta t, \Delta\phi_n) - 1 \Big\}^{-\frac{3}{2}} \\ & \times \frac{(r_+^2 - r^2)^{\frac{1}{4}} (r_+^2 - r'^2)^{\frac{1}{4}}}{(r_+^2 - r_-^2)^{\frac{1}{2}}} \exp \frac{1}{2} \Big[ F_+(t,\phi_n) + F_+(t',\phi'_n) \Big] \\ & \times \Big[ -\gamma^0 (z_0 - {z'}_0) + \gamma^1 (z_1 - {z'}_1) + \gamma^2 (z_2 - {z'}_2) \Big]. \end{aligned}$$

(4) For $z \in$ Region(I); $(r \geq r_+)$ $\quad z' \in$ Region (II); $(r_- < r' \leq r_+)$

$$(z_0 - {z'}_0) = \left\{ \Big(\frac{r^2 - r_+^2}{r^2 - r_-^2}\Big)^{\frac{1}{2}} \sinh F_+(t,\phi_n) e^{[-F_-(t,\phi_n)]} - \Big(\frac{r_+^2 - r'^2}{r'^2 - r_-^2}\Big)^{\frac{1}{2}} \cosh F_+(t',\phi'_n) e^{[-F_-(t',\phi'_n)]} \right\},$$

$$(z_1 - {z'}_1) = \left\{ \Big(\frac{r^2 - r_+^2}{r^2 - r_-^2}\Big)^{\frac{1}{2}} \cosh F_+(t,\phi_n) e^{[-F_-(t,\phi_n)]} - \Big(\frac{r_+^2 - r'^2}{r'^2 - r_-^2}\Big)^{\frac{1}{2}} \sinh F_+(t',\phi'_n) e^{[-F_-(t',\phi'_n)]} \right\},$$

$$(z_2 - {z'}_2) = \left\{ \Big(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\Big)^{\frac{1}{2}} e^{[-F_-(t,\phi_n)]} - \Big(\frac{r_+^2 - r_-^2}{r'^2 - r_-^2}\Big)^{\frac{1}{2}} e^{[-F_-(t',\phi'_n)]} \right\}.$$

$$\begin{aligned} S^+(z_n, z'_n) = \frac{1}{8\sqrt{2}\pi l^2} \Big\{ & \frac{(r^2 - r_-^2)^{\frac{1}{2}} (r'^2 - r_-^2)^{\frac{1}{2}}}{r_+^2 - r_-^2} \cosh F_-(\Delta t, \Delta\phi_n) \\ & + \frac{(r^2 - r_+^2)^{\frac{1}{2}} (r_+^2 - r'^2)^{\frac{1}{2}}}{r_+^2 - r_-^2} \sinh F_+(\Delta t, \Delta\phi_n) - 1 \Big\}^{-\frac{3}{2}} \\ & \times \frac{(r^2 - r_-^2)^{\frac{1}{4}} (r'^2 - r_-^2)^{\frac{1}{4}}}{(r_+^2 - r_-^2)^{\frac{1}{2}}} \exp \frac{1}{2} \Big[ F_-(t,\phi_n) + F_-(t',\phi'_n) \Big] \\ & \times \Big[ -\gamma^0 (z_0 - {z'}_0) + \gamma^1 (z_1 - {z'}_1) + \gamma^2 (z_2 - {z'}_2) \Big]. \end{aligned}$$

(5) For $z \in$ Region(II); $(r_- < r \leq r_+)$ $\quad z' \in$ Region (III); $(0 < r' \leq r_-)$

$$(z_0 - {z'}_0) = \left\{ \Big(\frac{r_+^2 - r^2}{r^2 - r_-^2}\Big)^{\frac{1}{2}} \cosh F_+(t,\phi_n) e^{[-F_-(t,\phi_n)]} - \Big(\frac{r_-^2 - r'^2}{r_+^2 - r'^2}\Big)^{\frac{1}{2}} \sinh F_-(t',\phi'_n) e^{[-F_+(t',\phi'_n)]} \right\},$$

$$(z_1 - {z'}_1) = \left\{ \Big(\frac{r_+^2 - r^2}{r^2 - r_-^2}\Big)^{\frac{1}{2}} \sinh F_+(t,\phi_n) e^{[-F_-(t,\phi_n)]} - \Big(\frac{r_-^2 - r'^2}{r_+^2 - r'^2}\Big)^{\frac{1}{2}} \cosh F_-(t',\phi'_n) e^{[-F_+(t',\phi'_n)]} \right\},$$

$$(z_2 - {z'}_2) = \left\{ \Big(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\Big)^{\frac{1}{2}} e^{[-F_-(t,\phi_n)]} - \Big(\frac{r_+^2 - r_-^2}{r_+^2 - r'^2}\Big)^{\frac{1}{2}} e^{[-F_+(t',\phi'_n)]} \right\}.$$

$$
\begin{aligned}
S^{+}(z_n, z'_n) = \frac{1}{8\sqrt{2}\pi l^2} & \left\{ \frac{(r^2-r_-^2)^{\frac{1}{2}}(r_+^2-r'^2)^{\frac{1}{2}}}{r_+^2-r_-^2} \cosh[(a_- t - a_+ t') - l(a_+\phi_n - a_-\phi'_n)] \right. \\
& \left. - \frac{(r_+^2-r^2)^{\frac{1}{2}}(r_-^2-r'^2)^{\frac{1}{2}}}{r_+^2-r_-^2} \sinh[(a_+ t - a_- t') - l(a_-\phi_n - a_+\phi'_n)] - 1 \right\}^{-\frac{3}{2}} \\
& \times \frac{(r^2-r_-^2)^{\frac{1}{4}}(r_+^2-r'^2)^{\frac{1}{4}}}{(r_+^2-r_-^2)^{\frac{1}{2}}} \exp\frac{1}{2}\Big[F_-(t,\phi_n) + F_+(t',\phi'_n)\Big] \\
& \times \Big[-\gamma^0(z_0 - {z'}_0) + \gamma^1(z_1 - {z'}_1) + \gamma^2(z_2 - {z'}_2)\Big].
\end{aligned}
$$

where again we used the short-hand notation, $F_\pm(t,\phi_n) \equiv [a_\pm t - la_\mp\phi_n]$, $F_\pm(\Delta t, \Delta\phi_n) \equiv [a_\pm(t-t') - la_\mp(\phi - \phi' + 2\pi n)]$ and where in (2+1)-dimension, Dirac $\gamma$-matrices $\gamma^a = (\gamma^0, \gamma^1, \gamma^2)$ obeying

$$
\begin{aligned}
(\gamma^0)^\dagger = \gamma^0, (\gamma^i)^\dagger = -\gamma^i, \\
(\gamma^0)^2 = I, (\gamma^i)^2 = -I, \\
\{\gamma^a, \gamma^b\} = -2\eta^{ab}
\end{aligned}
$$

are given, for example, in standard representation, by

$$
\begin{aligned}
\gamma^0 &= \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\gamma^1 &= i\sigma_1 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \\
\gamma^2 &= i\sigma_2 = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\end{aligned}
$$

This completes the explicit evaluation of Green's functions for the scalar and spinor field in the background of $AdS_3$ black hole spacetime.

Perhaps it would be appropriate to mention the issue of boundary conditions at infinity on any Green's function for fields propagating on general $AdS_3$ spacetimes. As is the case with all $AdS_3$ spacetimes, the $AdS_3$ black hole spacetimes of BTZ [3] we are dealing with is *not* globally hyperbolic. And this global non-hyperbolicity invites some delicacy in constructing the two-point Green's functions which is under the consideration. To be more concrete, the spatial infinity $i^0$ of this $AdS_3$ black hole spacetime is "timelike". This has been demonstrated in the Carter-Penrose conformal diagram provided by BTZ [3]. And physically, it means that information can leak in or out through the spatial infinity in a finite coordinate time. In order to deal with this fact, boundary conditions may be imposed on the two-point Green's functions. As has been pointed out by Avis, Isham and Storey [12], it is still possible to define a quantization scheme on globally non-hyperbolic spacetimes like the present $AdS_3$ black hole spacetime without using boundary conditions, which is usually referred to as "transparent" boundary conditions. In view of this, the construction of two-point Green's functions for conformally-coupled scalar and spinor field performed in the present work can be thought of as corresponding to this option. The two-point Green's functions for quantum fields in $AdS_3$ black hole spacetime have been provided in the literature [16]. But only the Green's functions for conformally-coupled scalar field with the choice of Neumann or Dirichlet boundary conditions are given. Thus the Green's functions for conformally-coupled spinor field given in the present work can be regarded as a new contribution.

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