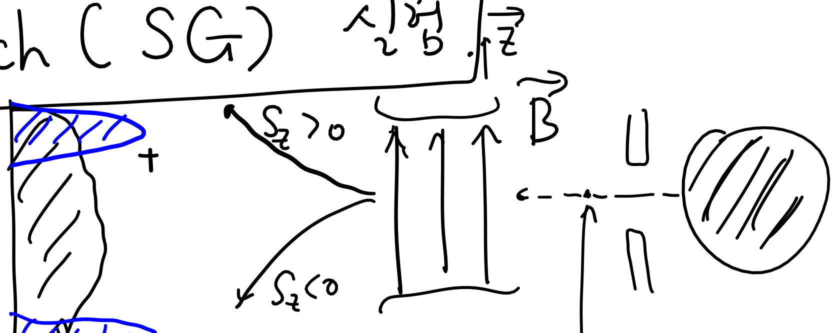
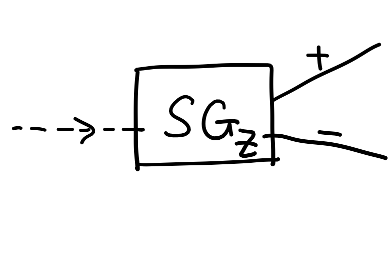
Chap 1. Fundamental Concept.

Stern-Gerlach (SG) 실험



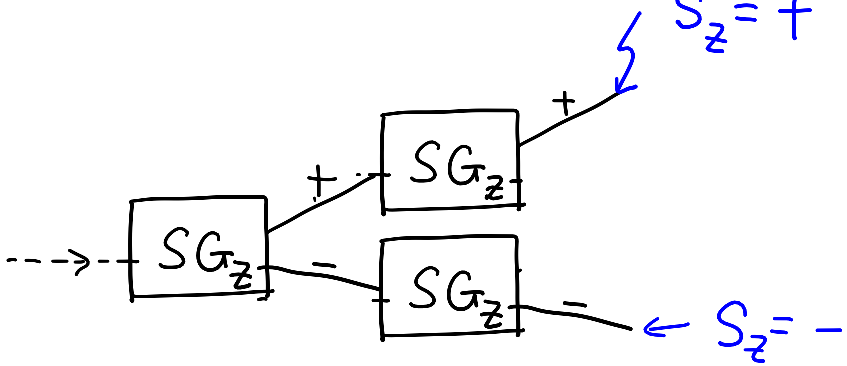
예상

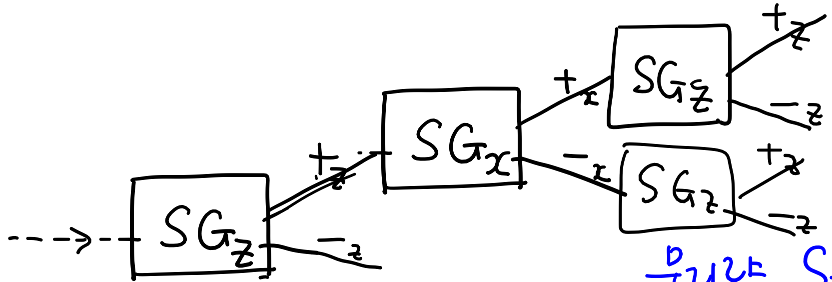


atom: 자석 i.e. mag. moment $\vec{\mu} = g \frac{e}{m_e c} \vec{S} \propto \vec{S}$

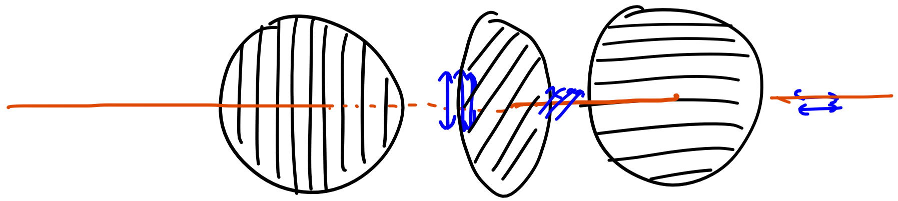
$$V = -\vec{\mu} \cdot \vec{B} \rightarrow \vec{F} = -\vec{\nabla} V = -\vec{\nabla}(-\mu_z B)$$
$$= \mu_z \vec{\nabla} B = \hat{z}\, \mu_z \underbrace{\frac{\partial B}{\partial z}}_{<0}$$

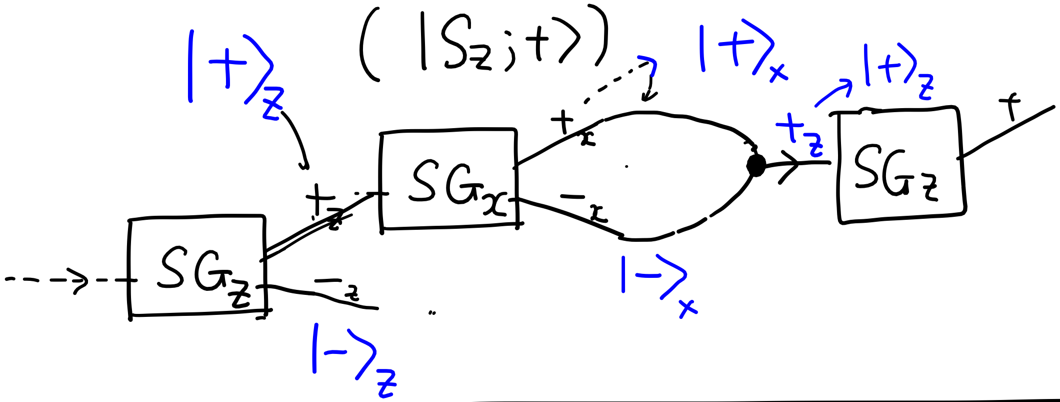
→ spin quantization





물리량 S_z 와 S_x 는 동시에 값을 가질수 없다.





Dirac notation

State vector $\in$ Hilbert space

$\Downarrow$

ket-vector $|\quad\rangle$

$$|\psi\rangle = c\,|\psi\rangle$$

$$|+\rangle_z = \frac{1}{\sqrt{2}}\left(|+\rangle_x + |-\rangle_x\right)$$

$$|c_1|^2 = \frac{1}{2} \to c_{1,2} = \frac{1}{\sqrt{2}} e^{i\alpha}$$

$$|-\rangle_z = \frac{1}{\sqrt{2}}\left(|+\rangle_x - |-\rangle_x\right)$$

$$\alpha = 0$$

$$|+\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle_z + |-\rangle_z\right)$$

$$|-\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle_z - |-\rangle_z\right)$$

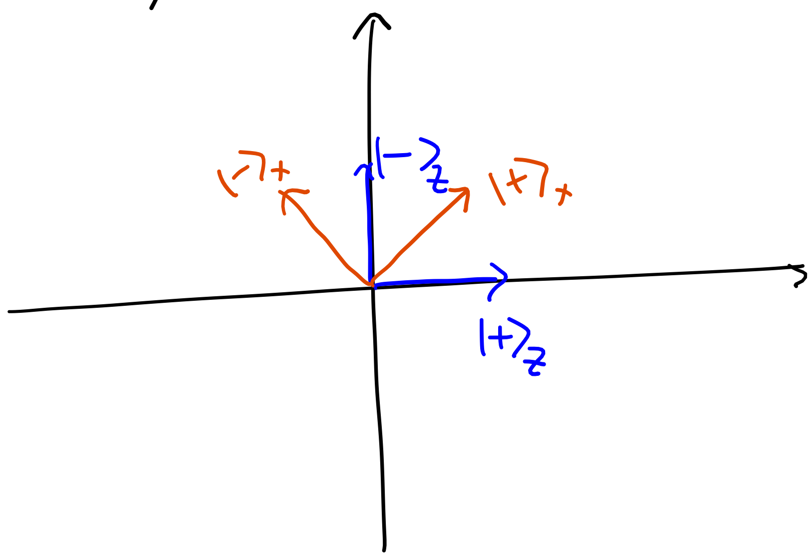
linear combination.

$$\{\vec{v}_i \quad i = 1 \cdots, n\} = \text{basis}$$

$$\vec{v} = \sum_{i=1}^{n} c_i \vec{v}_i \in \mathbb{V}$$

$$c_1 |+\rangle_z + c_2 |-\rangle_z$$ "Superpose"

$$|+\rangle_y = \frac{1}{\sqrt{2}}\left(|+\rangle_z + i|-\rangle_z\right) \qquad |-\rangle_y = \frac{1}{\sqrt{2}}\left(|+\rangle_z - i|-\rangle_z\right)$$



1.2. ket vector, $\boxed{\text{operator}}$ (dynamical variable) $\longleftrightarrow$ 행렬

$(S_x S_z \neq S_z S_x)$

bra vector $\leftarrow$ Hilbert 공간. complex vector space

(ex) $\vec{e}_1^T \cdot \vec{e}_1$

$$(1\ \ 0\ \ 0)\begin{pmatrix}1\\0\\0\end{pmatrix} = 1 \qquad \vec{e}_1\,\vec{e}_1^T = \begin{pmatrix}1\\0\\0\end{pmatrix}(1\ \ 0\ \ 0) = (\text{행렬})$$

column vector = ket vector $\rightarrow$ $|\ \rangle$

row " = bra " $\rightarrow$ $\langle\ |$

$T + * = \dagger$ "Hermitian conjugation"

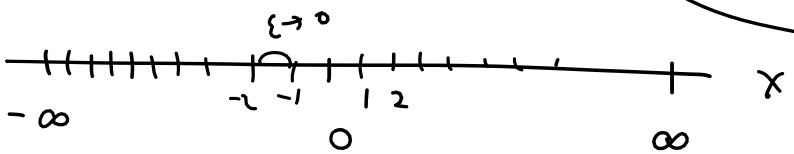
bra"c"ket = $\langle\ |\ \rangle$

$$(|a\rangle)^\dagger = \langle a^*|$$

Schrödinger	Heisenberg
diff. eq	Matrix
미분 연산자	행렬

$$\underbrace{\frac{\partial}{\partial x}}\ \underbrace{\psi(x)}$$

$$-\frac{1}{\epsilon}\begin{pmatrix} \ddots & & & & & & \\ 0 & \cdots & 1 & -1 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 1 & -1 & 0 & \cdots \\ & & & & \ddots & & \end{pmatrix}\begin{pmatrix} \psi(-\infty) \\ \vdots \\ \psi(-1) \\ \psi(0) \\ \psi(1) \\ \vdots \\ \psi(\infty) \end{pmatrix}$$



$$\frac{d}{dx}\psi(x) = \lim_{\epsilon \to 0} \frac{\overset{\psi(x+1)}{\psi(x+\epsilon)} - \overset{\psi(x)}{\psi(x)}}{\epsilon}$$

$|\alpha\rangle \quad , \langle\beta| \quad \Longrightarrow \quad \langle\beta|\alpha\rangle = \text{scalar} \rightarrow \langle\beta|\alpha\rangle^* = \langle\alpha|\beta\rangle$

$|\alpha\rangle\langle\beta| = \text{operator}$

$\langle\alpha|\alpha\rangle^* = \langle\alpha|\alpha\rangle \Rightarrow \langle\alpha|\alpha\rangle$ 실수 $\begin{cases} \text{positive} \begin{cases} 1 \rightarrow \text{"normalized"} \quad (|+\rangle_z)^\dagger = {}_z\langle +| \\ \neq 1 \rightarrow \end{cases} \\ 0 \leftarrow \text{null state} \\ \text{negative} \end{cases}$ ⇒ 제외.

$\downarrow$

"probability"

$c|\alpha\rangle = |\tilde{\alpha}\rangle$

$|\alpha\rangle \equiv |\tilde{\alpha}\rangle$

$\langle\tilde{\alpha}|\tilde{\alpha}\rangle = 1$

$\quad \parallel$

$c^* c \langle\alpha|\alpha\rangle$

$|c|^2 = \frac{1}{\langle\alpha|\alpha\rangle}$

$\therefore c = \frac{1}{\sqrt{\langle\alpha|\alpha\rangle}}$

Operator

$\hat{A}$ 물리량. $\rightarrow \hat{A}^\dagger = \hat{A}$

$\hat{A}|\alpha\rangle = |\alpha'\rangle$

(ex) $\hat{S}_x |+\rangle_z = \begin{Bmatrix} |+\rangle_x \\ |-\rangle_- \end{Bmatrix} = |\psi\rangle$

$\neq$

$|+\rangle_z$

$\hat{A}^\dagger \rightarrow$ (ex) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$

if $\hat{A}^\dagger = \hat{A}$ "Hermitian"

$\downarrow$

eigenvalues 는 "real"

vectors : 내적, superpose

Operators : 덧셈 $(\hat{X} + \hat{Y})|\alpha\rangle = \hat{X}|\alpha\rangle + \hat{Y}|\alpha\rangle$

곱셈. $\hat{X}\hat{Y}|\alpha\rangle = \hat{X}(\underbrace{\hat{Y}|\alpha\rangle})$ "물리량의 연속적 측정"

$\left((\text{ex}) \ \hat{S}_x \cdot \hat{S}_z \right)$

$\hat{X}\hat{Y} \neq \hat{Y}\hat{X}$ "비가환, nonabelian" non commutative

$\hat{X}(\hat{Y}\hat{Z}) = (\hat{X}\hat{Y})\hat{Z}$ "결합법칙, Assosiative"

$X = |\beta\rangle\langle\alpha| \rightarrow X^\dagger = |\alpha\rangle\langle\beta|$

$\longrightarrow (|\beta\rangle\langle\alpha|)|\gamma\rangle = |\beta\rangle\overbrace{\langle\alpha|\gamma\rangle}^{c} \propto |\beta\rangle$

$\begin{pmatrix} \vdots \\ \vdots \end{pmatrix} (\cdots\cdots)$

$\langle\gamma|(|\beta\rangle\langle\alpha|) = \underbrace{\langle\gamma|\beta\rangle}_{c}\langle\alpha| \propto \langle\alpha|$

$\left[(\text{ex}) \quad |\alpha_1\rangle\underbrace{\langle\alpha_2|\alpha_3\rangle\langle\alpha_4|\alpha_5\rangle}\langle\alpha_6| \propto |\alpha_1\rangle\langle\alpha_6| \right]$

특히 $\hat{\Lambda}_\alpha = |\alpha\rangle\langle\alpha|$ "Projector"

$$\big(|\alpha\rangle\langle\alpha|\big)\,|\overset{\text{any}}{\psi}\rangle = |\alpha\rangle\underbrace{\langle\alpha|\psi\rangle} \propto |\alpha\rangle$$

$$\underbrace{\langle\beta|\hat{A}}|\alpha\rangle = \langle\beta|\big(\hat{A}|\alpha\rangle\big) = \underbrace{\text{scalar}}^{\dagger} = \text{scalar}^{*}$$

$$= \underbrace{\big(\hat{A}^\dagger|\beta\rangle\big)^\dagger}\,\underbrace{|\alpha\rangle}_{(\langle\alpha|)^\dagger} = \Big(\underbrace{\langle\alpha|}\,\underbrace{\hat{A}^\dagger|\beta\rangle}\Big)^\dagger$$

$$\Big(\big(\hat{A}\hat{B}\big)^\dagger = \hat{B}^\dagger\hat{A}^\dagger\Big)$$

$$\langle\beta|\hat{A}|\alpha\rangle = \Big(\langle\alpha|\hat{A}^\dagger|\beta\rangle\Big)^\dagger = \langle\alpha|\hat{A}^\dagger|\beta\rangle^*$$

1.3. Base kets.

일반적. $\hat{A}\,|a\rangle = |a'\rangle$

특히 $\hat{A}\,\underbrace{|\psi\rangle}_{\uparrow} = a\,|\psi\rangle$ ← $\begin{cases} \text{eigenvector} & |\psi\rangle \to |a\rangle \\ \quad〃 \ \text{value} & a \end{cases}$

only for special vectors

$$\hat{A}\,|a_n\rangle = a_n\,|a_n\rangle \qquad n = 1, \cdots, N$$

$$\boxed{\text{특히} \quad \hat{A}^\dagger = \hat{A} \ \text{라면}, \quad \langle a_n|a_m\rangle = 0 \quad \text{if} \quad a_n \neq a_m}$$

$$\langle a_m|\underbrace{\hat{A}\,|a_n\rangle}_{=\,a_n|a_n\rangle} = a_n\,\langle a_m|a_n\rangle = a_m^*\,\langle a_m|a_n\rangle$$

$$\boxed{\begin{gathered} \therefore\ (a_n - a_m^*)\,\langle a_m|a_n\rangle = 0 \\ \Downarrow \\ a_n \neq a_m^* \\ \Rightarrow \langle a_m|a_n\rangle = 0 \end{gathered}}$$

$$\langle a_m|\hat{A} = \Big(\underbrace{\hat{A}^\dagger}_{=\,\hat{A}}\,|a_m\rangle\Big)^\dagger = \big(a_m|a_m\rangle\big)^\dagger = a_m^*\,\langle a_m|$$

$$\boxed{\langle a_n | a_m \rangle = \delta_{nm}} \rightarrow \begin{array}{l} n \neq m \quad \langle a_n | a_m \rangle = 0 \\ n = m \quad \langle a_n | a_n \rangle = 1 \end{array}$$

$$\sum_{k=1}^{N} \delta_{kn} |a_k\rangle = |a_n\rangle \leftarrow \text{for all } n$$

$$\parallel$$

$$\sum_{k=1}^{N} |a_k\rangle \overbrace{\langle a_k | a_n \rangle}^{\delta_{kn}} = |a_n\rangle$$

$$\parallel$$

$$\underbrace{\left(\sum_{k=1}^{N} |a_k\rangle \langle a_k| \right)}_{\substack{\parallel \\ \mathbb{1}}} |a_n\rangle = |a_n\rangle$$

Completeness.

$$\boxed{\mathbb{1} = \sum_{k=1}^{N} \underbrace{|a_k\rangle \langle a_k|}_{\hat{\Lambda}_{a_k}}}$$

$$\sum_{\alpha} \hat{\Lambda}_{\alpha} = \mathbb{1}$$

for any vector $|\alpha\rangle$

$$\mathbb{1}\,|\alpha\rangle = |\alpha\rangle = \left(\sum_{n=1}^{N} |a_n\rangle\langle a_n|\right)|\alpha\rangle = \sum_{n=1}^{N} |a_n\rangle \underbrace{\langle a_n|\alpha\rangle}_{c_n}$$

$$\mathbb{1} = \sum_{n=1}^{N} |a_n\rangle\langle a_n|$$

$$= \sum_{n=1}^{N} c_n\, |a_n\rangle$$

$$\therefore \quad \{\, |a_n\rangle \,\} \quad \text{basis}$$

$$\langle\alpha|\alpha\rangle = 1 = \sum_{n=1}^{N}\sum_{m=1}^{N} \langle a_m|\, c_m^* c_n \,|a_n\rangle = c_m^* c_n \underbrace{\langle a_m|a_n\rangle}_{\delta_{mn}}$$

$$= \sum_{n=1}^{N} c_n^* c_n = \sum_{n=1}^{N} |c_n|^2$$

$$\hat{A} \longrightarrow \{|a_1\rangle \cdots |a_N\rangle\}$$

$$\downarrow$$

$$\hat{A} = \begin{pmatrix} \langle a_1|\hat{A}|a_1\rangle & \cdots\cdots & \langle a_1|\hat{A}|a_N\rangle \\ \langle a_2|\hat{A}|a_1\rangle & \cdots\cdots & \langle a_2|\hat{A}|a_N\rangle \\ \vdots & & \vdots \\ \langle a_{N-1}|\hat{A}|a_1\rangle & \cdots\cdots & \langle a_{N-1}|\hat{A}|a_N\rangle \\ \langle a_N|\hat{A}|a_1\rangle & \cdots\cdots & \langle a_N|\hat{A}|a_N\rangle \end{pmatrix}$$

(ex) Spin-$\frac{1}{2}$ system $\longrightarrow |+\rangle, |-\rangle \longrightarrow$ $\hat{S}_z|+\rangle = \frac{\hbar}{2}|+\rangle$

$\hat{S}_z|-\rangle = -\frac{\hbar}{2}|-\rangle$

$$|+\rangle\langle+| + |-\rangle\langle-| = \mathbb{1}$$

$$\hat{S}_z = \mathbb{1}\,\hat{S}_z\,\mathbb{1} = |+\rangle \overbrace{\langle+|\hat{S}_z|+\rangle}^{\frac{\hbar}{2}} \langle+| + |+\rangle \overbrace{\langle+|\hat{S}_z|-\rangle}^{0} \langle-|$$

$$+ |-\rangle \underbrace{\langle-|\hat{S}_z|+\rangle}_{0} \langle+| + |-\rangle \underbrace{\langle-|\hat{S}_z|-\rangle}_{-\frac{\hbar}{2}} \langle-|$$

$$\therefore\ = \frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right)$$

$$\hat{S}_+ \equiv \hbar |+\rangle\langle -| \quad , \quad \hat{S}_- \equiv \hbar |-\rangle\langle +|$$

$$\hat{S}_+|+\rangle = 0 \qquad \hat{S}_-|-\rangle = 0$$

$$\hat{S}_+|-\rangle = \hbar|+\rangle \qquad \hat{S}_-|+\rangle = \hbar|-\rangle$$

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \rightarrow \quad \begin{array}{l} \langle +| = (1 \;,\; 0) \\ \langle -| = (0 \quad 1) \end{array}$$

$$\mathbb{1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1 \;\; 0) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0 \;\; 1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{S}_z = \frac{\hbar}{2}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}(1 \;\; 0) - \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0 \;\; 1)\right) = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left.\begin{array}{l} \hat{S}_+ = \hbar \begin{pmatrix} 1 \\ 0 \end{pmatrix}(0 \;\; 1) = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \hat{S}_- = \hbar \begin{pmatrix} 0 \\ 1 \end{pmatrix}(1 \;\; 0) = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{array}\right\} \begin{array}{l} \text{non Hermitian} \\ \hat{S}_\pm^\dagger = \hat{S}_\mp \end{array}$$

$$\hat{S}_x = \frac{1}{2}\left(\hat{S}_+ + \hat{S}_-\right) \quad , \quad \hat{S}_y = \frac{1}{2i}\left(\underbrace{\hat{S}_+ - \hat{S}_-}\right) \rightarrow \hat{S}_y^\dagger = \hat{S}_y$$

$$\hat{S}_x^\dagger = \hat{S}_x$$

$$\hat{\vec{S}} = \left(\hat{S}_x , \hat{S}_y , \hat{S}_z \right)$$

Copenhagen Interpretation

$$\mathbb{1} \, |\alpha\rangle = \sum_n c_n |a_n\rangle = \begin{pmatrix} \langle a_1|\alpha\rangle \\ \langle a_2|\alpha\rangle \\ \vdots \\ \langle a_N|\alpha\rangle \end{pmatrix} \leftarrow \{|a_n\rangle\}$$

$\underbrace{|\alpha\rangle}_{\text{abstract}}$

↳ 구체적 "Representation" 표현

$\mathbb{1} = \sum_n |a_n\rangle\langle a_n|$ $\hat{A}$ 의 e.v.

$$c_n = \langle a_n|\alpha\rangle$$

$$|c_n|^2 = \text{probability}$$

$\hat{A}$ 를 measure 했을때 그 값이 a_n 일 확률.

$(|\langle a_n|\alpha\rangle|^2$ 의 확률$)$

Measure ; $\hat{A} : |\alpha\rangle \to |a_n\rangle$

$n = 1 \cdots N$

$$\hat{B} \to \mathbb{1} = \sum_{n=1}^{M} |b_n\rangle\langle b_n|$$

$$|a_n\rangle = \mathbb{1}\,|a_n\rangle = \sum_{m=1}^{M} c'_m |b_m\rangle$$

$$\mathbb{1} = \sum_{m=1}^{M} |b_m\rangle\langle b_m|$$

$$c'_m = \langle b_m|a_n\rangle$$

$|b_m\rangle$

$$|a_n\rangle \xrightarrow{=\mathbb{1}|a_n\rangle} |b_m\rangle = \sum_{n=1}^{N} \tilde{C}_n |a_n\rangle$$

$$\vdots$$

(ex) $|+\rangle_z$

$$\begin{cases} \underbrace{|+\rangle_x} \\ |-\rangle_x \end{cases} \longrightarrow \overset{(ex)}{=} \frac{1}{\sqrt{2}}\left(|+\rangle_z + |-\rangle_z\right)$$

$$\sum$$

$$\mathbb{1} = \sum_m |b_m\rangle\langle b_m|$$

$$\hat{S}_z \qquad \hat{B} \equiv \hat{S}_x$$

$$\underbrace{|a_n\rangle = \sum_m \left(\langle b_m|a_n\rangle\right) \underbrace{|b_m\rangle}} \Longrightarrow \overset{\hat{S}_z}{\overset{\shortparallel}{\hat{A}}}|a_n\rangle = a_n|a_n\rangle$$

$$= \sum_m \langle b_m|a_n\rangle \underbrace{\hat{A}|b_m\rangle}$$

Operator $\hat{B}$

$$\mathbb{1}\hat{B}\mathbb{1} = \left(\sum_{n=1}^{N}|a_n\rangle\langle a_n|\right)\hat{B}\left(\sum_{m=1}^{N}|a_m\rangle\langle a_m|\right)$$

$$= \sum_n\sum_m |a_n\rangle \underbrace{\langle a_n|\hat{B}|a_m\rangle}_{\#} \langle a_m|$$

$$= \begin{pmatrix} \langle a_1|\hat{B}|a_1\rangle & \cdots & \langle a_1|\hat{B}|a_N\rangle \\ \vdots & & \vdots \\ \langle a_N|\hat{B}|a_1\rangle & \cdots & \langle a_N|\hat{B}|a_N\rangle \end{pmatrix}$$

특히, $\hat{B} = \hat{A}$

$$\hat{A}|a_m\rangle = a_m|a_m\rangle$$

$$\langle a_n|\hat{A}|a_m\rangle = a_m\,\delta_{nm}$$

$$\hat{A} = \sum_n\sum_m a_m\,\delta_{nm}\,|a_n\rangle\langle a_m|$$

$$= \sum_n a_n\,|a_n\rangle\langle a_n|$$

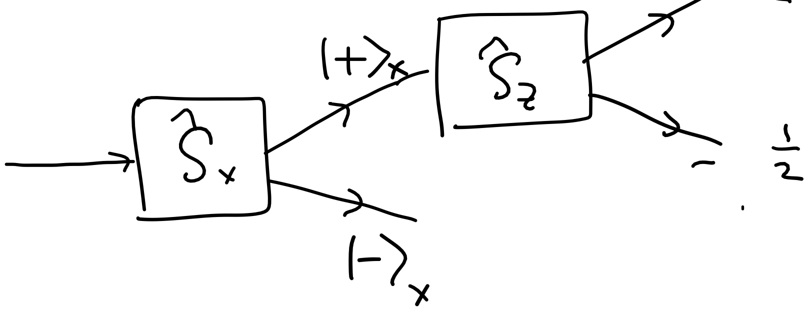
Expectation value $|\alpha\rangle$

$$\langle\alpha|\hat{A}|\alpha\rangle = \sum_{n,m}\langle\alpha|a_n\rangle\underbrace{\langle a_n|\hat{A}|a_m\rangle}_{a_m\delta_{nm}}\langle a_m|\alpha\rangle$$

$$= \sum_n a_n \underbrace{\langle\alpha|a_n\rangle\langle a_n|\alpha\rangle}_{|\langle a_n|\alpha\rangle|^2}$$

(Ex)

$\hat{S}_z \rightarrow |+\rangle, |-\rangle$



$$|\langle +|+\rangle_x|^2 = \frac{1}{2} \qquad |\langle -|+\rangle_x|^2 = \frac{1}{2}$$

$\langle +|$ as $\langle a_1|$, $|+\rangle_x$ as $|\alpha\rangle$

$$\boxed{\begin{aligned} |+\rangle_x &= \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}e^{i\delta_1}|-\rangle \\ |-\rangle_x &= \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}e^{i\delta_1}|-\rangle \end{aligned}}$$

$${}_x\langle +| = \frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}}e^{-i\delta_1}\langle -|$$

$${}_x\langle +|-\rangle_x = 0$$

$$\hat{S}_x = \frac{\hbar}{2}|+\rangle_x\langle+|_x - \frac{\hbar}{2}|-\rangle_x\langle-|_x$$

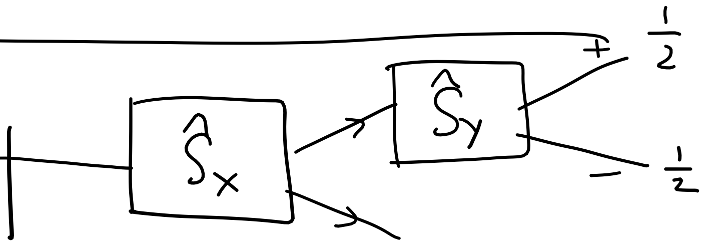
$$\underbrace{\hat{A} = \sum a_n |a_n\rangle\langle a_n|}$$

$$= \frac{\hbar}{2}\left[e^{-i\delta_1}|+\rangle\langle-| + e^{i\delta_1}|-\rangle\langle+|\right]$$

$$\hat{S}_y \longrightarrow |\pm\rangle_y \longrightarrow \underbrace{|\pm\rangle_y = \frac{1}{\sqrt{2}}\left(|+\rangle \pm e^{i\delta_2}|-\rangle\right)}$$

$$\hat{S}_y = \frac{\hbar}{2}\left[e^{-i\delta_2}|+\rangle\langle-| + e^{i\delta_2}|-\rangle\langle+|\right]$$

$$\left|{}_x\langle+|+\rangle_y\right| = \frac{1}{\sqrt{2}} = \frac{1}{2}\left|\underbrace{\left(1+e^{i(\delta_2-\delta_1)}\right)}_{\sqrt{2}}\right|$$



$$\left|{}_y\langle-|+\rangle_x\right| = \frac{1}{\sqrt{2}} = \frac{1}{2}\left|\underbrace{\left(1-e^{i(\delta_1-\delta_2)}\right)}_{\sqrt{2}}\right|$$

$$e^{i(\delta_1-\delta_2)} = \pm i \rightarrow \underbrace{\delta_1-\delta_2 = \pm\frac{\pi}{2}}$$

$$(\,|1\pm i| = \sqrt{2}\,)$$

$${}_x\langle+| = \frac{1}{\sqrt{2}}\left(\langle+| + e^{-i\delta_1}\langle-|\right)$$

$${}_y\langle-| = \frac{1}{\sqrt{2}}\left(\langle+| - e^{-i\delta_2}\langle-|\right)$$

$$|+\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\delta_1}|-\rangle\right)$$

$$\Rightarrow \quad \delta_1 - \delta_2 = \pm \frac{\pi}{2} \longrightarrow \begin{cases} \delta_1 = 0 \\ \delta_2 = \frac{\pi}{2} \end{cases}$$

$$\frac{\hbar}{2}\left(|+\rangle_y \langle +|_y - |-\rangle_y \langle -|_y\right)$$

$$|\pm\rangle_y = \frac{1}{\sqrt{2}}\left(|+\rangle \pm i\,|-\rangle\right) \longrightarrow \hat{S}_y = \frac{\hbar}{2}\left(-i\,|+\rangle\langle -| + i\,|-\rangle\langle +|\right)$$

$$|\pm\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle \pm |-\rangle\right) \longrightarrow \hat{S}_x = \frac{\hbar}{2}\left(|+\rangle\langle -| + |-\rangle\langle +|\right)$$

$\{|+\rangle_z, |-\rangle_z\}$ basis

$$\hat{S}_z = \frac{\hbar}{2}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\parallel} \qquad \hat{S}_y = \frac{\hbar}{2}\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_y} \qquad \hat{S}_x = \frac{\hbar}{2}\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_x}$$

$|+\rangle\langle -|$ $|-\rangle\langle +|$

Pauli matrix ; σ_z

$$\sigma_i \sigma_j = i\varepsilon_{ijk}\sigma_k \quad (i \neq j)$$

$$\underbrace{\sigma_i^2 = \mathbb{1}}\qquad \underbrace{\sigma_i \sigma_j = -\sigma_j \sigma_i}$$

$$\{A, B\} \equiv AB + BA \quad \Rightarrow \quad \underline{\{\sigma_i, \sigma_j\} = 2\,\mathbb{1}\,\delta_{ij}}, \quad \underline{[\sigma_i, \sigma_j] = 2\sigma_i\sigma_j = 2i\varepsilon_{ijk}\sigma_k}_{\,i \neq j}$$

$$\hat{\vec{S}} = \frac{\hbar}{2}\vec{\sigma} \longrightarrow [\hat{S}_i, \hat{S}_j] = \hbar i \varepsilon_{ijk} \hat{S}_k \neq 0 \quad (i \neq j)$$

$$\{|\pm\rangle\}$$

$$\{\hat{S}_i, \hat{S}_j\} = \frac{\hbar^2}{2} \mathbb{1} \delta_{ij}$$

$$\hat{S}_i^2 = \frac{\hbar^2}{4} \mathbb{1}$$

$$\hat{\vec{S}}^2 = \sum_{i=x}^{z} \hat{S}_i^2 = \frac{3}{4}\hbar^2 \mathbb{1}$$

$$[\hat{\vec{S}}^2, \hat{S}_i] = 0 \quad , \quad [\hat{S}_i, \hat{S}_j] \neq 0 \ (i \neq j)$$

$$\{\hat{\vec{S}}^2, \hat{S}_z\}$$

complete set of commuting operators (CSCO)

Compatible Observables

$[\hat{A}, \hat{B}] = 0$ $\quad$ $\hat{A}, \hat{B}$

[Theorem] if $[\hat{A}, \hat{B}] = 0 \rightarrow$ common eigenstates

$$\hat{A}|a_n\rangle = a_n|a_n\rangle$$

$$\hat{B}\hat{A}|a_n\rangle = \underline{a_n \hat{B}|a_n\rangle = \hat{A}(\hat{B}|a_n\rangle)}$$

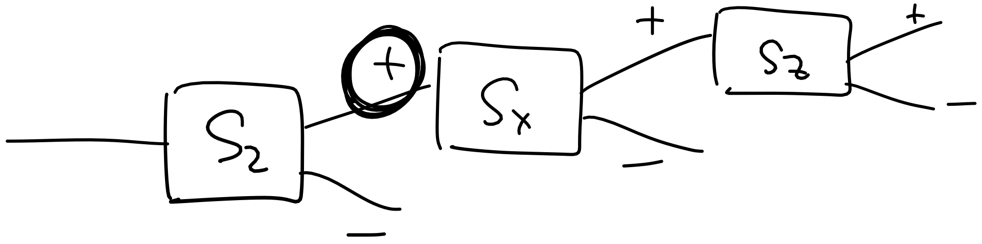
$\therefore$ $\hat{B}|a_n\rangle$는 $\hat{A}$의 e.v. (evalue $= a_n$)

("degeneracy"

같은 e.v. a_n을 주는 e.v. $|a_n\rangle$이 여러개 일 때.)

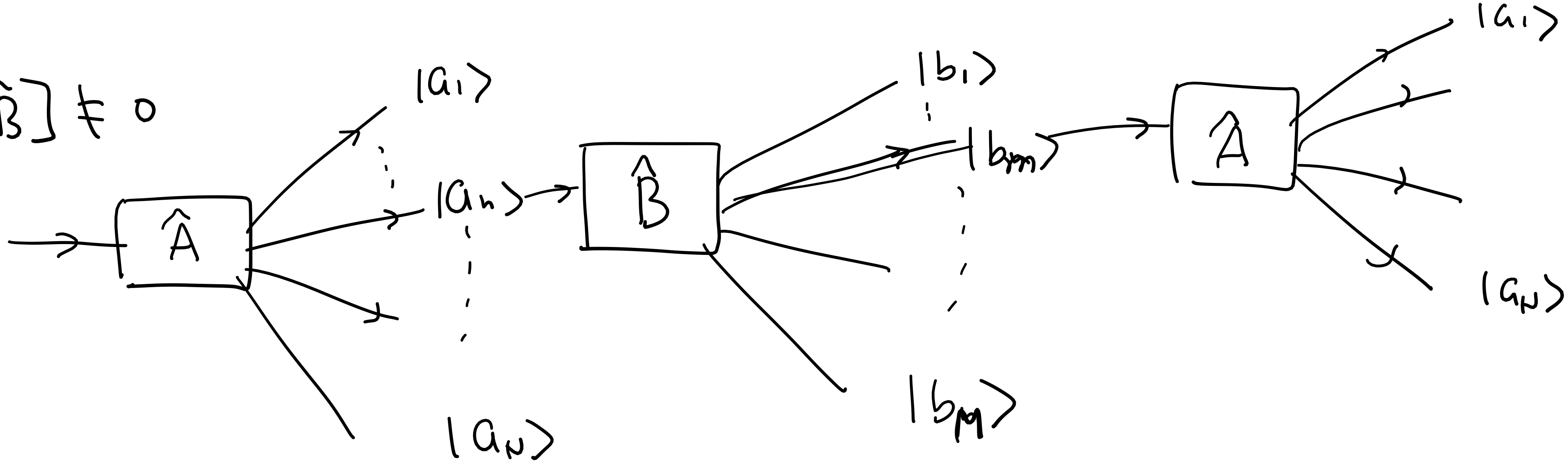
if non degenerate ; $\hat{B}|a_n\rangle \propto |a_n\rangle \Rightarrow \underline{\hat{B}|a_n\rangle = b_n|a_n\rangle}$

b_n: 비례상수

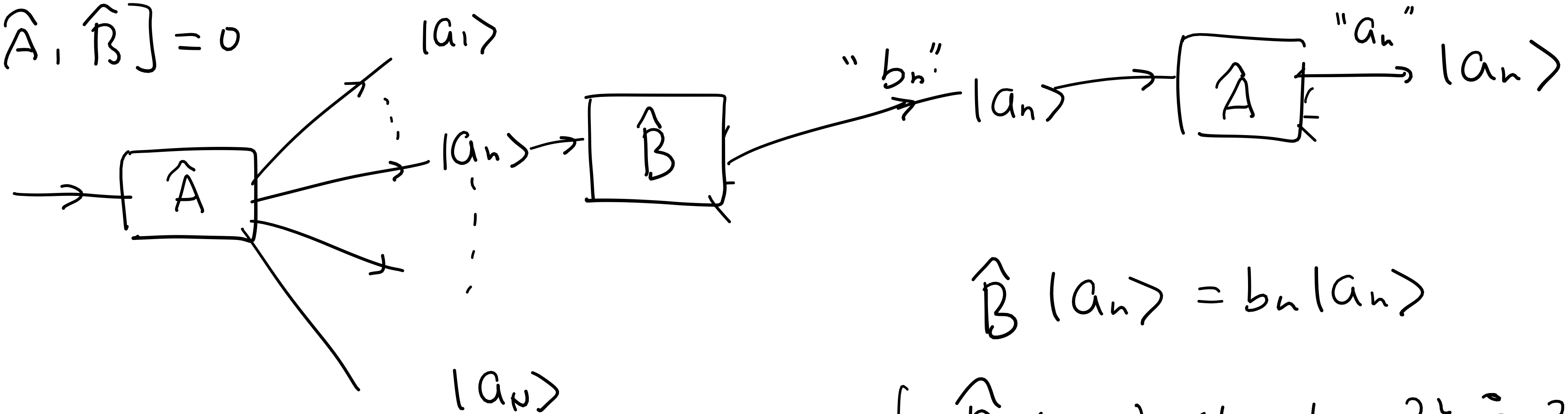


$\rightarrow \quad [\hat{S}_x, \hat{S}_z] \neq 0$

$[\hat{A}, \hat{B}] \neq 0$



$[\hat{A}, \hat{B}] = 0$



$$\hat{B}|a_n\rangle = b_n|a_n\rangle$$

$\hat{B}$는 항상 b_n 값을 갖고
$\hat{A}$는 〃 a_n 〃

$$\hat{A}\,|\underset{\sim}{a_n, b_n}\rangle = a_n |a_n, b_n\rangle$$

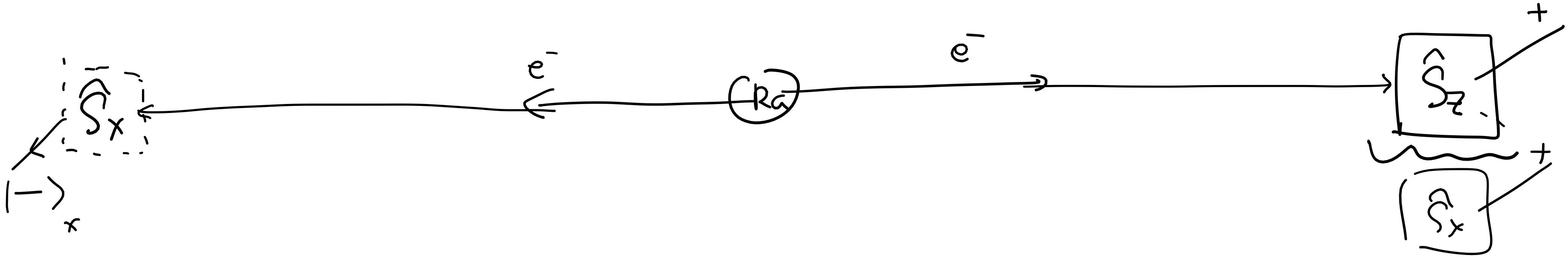
$$\hat{B}\,|\underset{\sim}{a_n, b_n}\rangle = b_n |a_n, b_n\rangle$$

Bell Inequality	EPR (Einstein-Podolsky-Rosen)

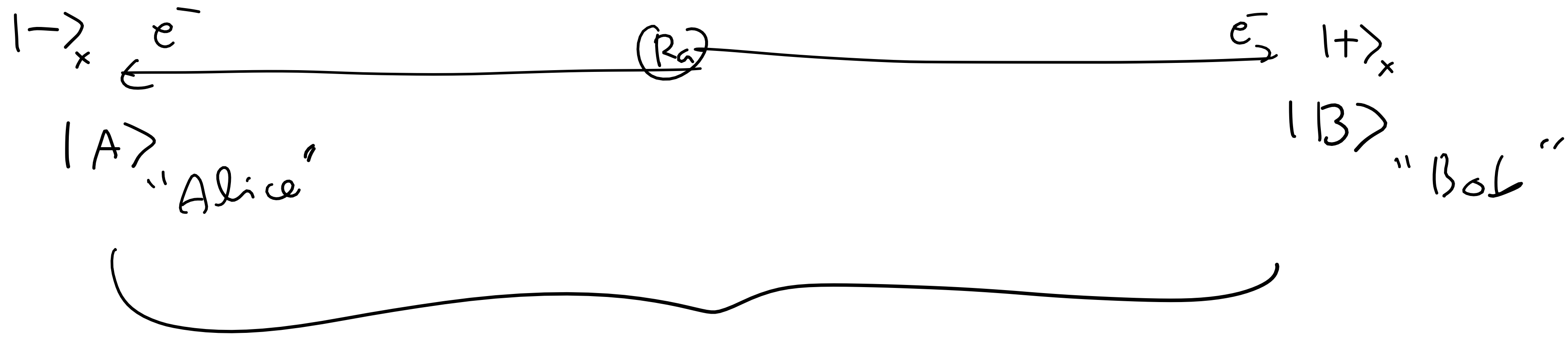
$$|+\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$

superposition

EPR



(entanglement 꼬임 상태)



$$|A\rangle \otimes |B\rangle$$

"non-locality"

tensor product

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} & b\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ c\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} & d\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \end{pmatrix}$$

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

$$(\hat{S}_z)_B \, |-\rangle^A |+\rangle^B = \frac{\hbar}{2} |-\rangle |+\rangle$$

$$(\hat{S}_x)_A \left(|-\rangle^A |+\rangle^B \right) = \left(\hat{S}_x |-\rangle \right)^A |+\rangle^B =$$

$\downarrow$

$$|-\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle_x + e^{i\delta} |-\rangle_x \right)$$

$$|+\rangle_x \to n(|+\rangle_z) + n(|-\rangle_z) = N$$

$$n(|+\rangle_z) \leq N$$

$$n|+_x +_z\rangle \subseteq n|+_x\rangle$$

$$n|+_x +_y +_z\rangle \subseteq n|+^A_x +^A_y\rangle$$

Bell's inequality

$$= n|+^A_x -^B_y\rangle$$

Experiments (Aspect ...)

~~Bell inequality~~

$$|+\rangle_x^A \, |-\rangle_x^B = \frac{1}{\sqrt{2}} \underbrace{\left(|+\rangle_A |-\rangle_B + |-\rangle_A |+\rangle_B \right)}_{\text{entangled state.}}$$

Quantum Computy (Cryptography)

Pshor's algorithm.

2 1 7 8 9 2 5 7 2 3 5 6

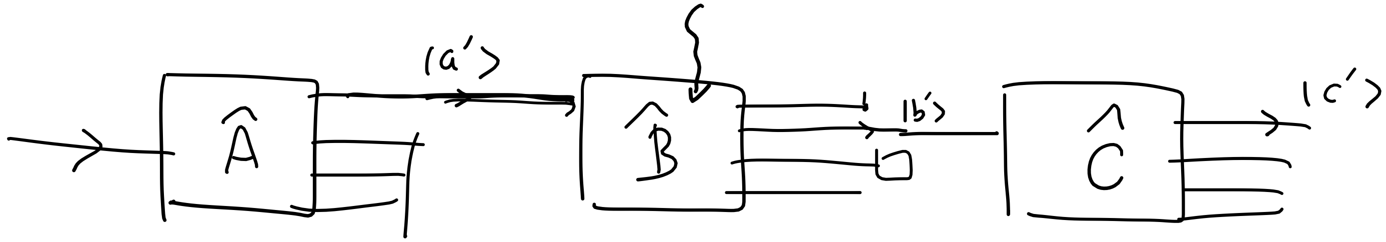
$\underset{\sim}{P}$ $\underset{\sim}{nP}$ problem

$$\left(\sum_{n_1 \cdots n_{N_1} = 0,1} |n_1\rangle \cdots |n_{N_1}\rangle \right) \otimes \left(\sum_{n_{N_1+1} \cdots n_{N_2}} |n\rangle \cdots \right) \otimes \left(\quad \right)$$

$= 1 \; 0 \; 1 \; 1 \; 0 \; 0 \; \cdots \; - \; - \; - \; -$

$$|n_1\rangle |n_2\rangle \cdots\cdots\cdots \, |n_N\rangle$$

$\hat{A}, \hat{B}, \hat{C}, \left([\hat{A}, \hat{B}] \neq 0 \;\ldots\right)$



$$\text{prob.}(a' \to c') = \sum_{b'} |\langle b' | a' \rangle|^2 \, |\langle c' | b' \rangle|^2 \quad \leftarrow$$

$$|\langle c' | a' \rangle|^2 = |\langle c' | \underbrace{\mathbb{1}}_{= \sum_{b'} |b'\rangle\langle b'|} | a' \rangle|^2 = \left| \sum_{b'} \langle c' | b' \rangle \langle b' | a' \rangle \right|^2$$

$$\underbrace{\left| \sum_{b'} \langle c' | b' \rangle \langle b' | a' \rangle \right|^2}_{\text{no } \hat{B} \text{ measure}} \neq \underbrace{\sum_{b'} \left| \langle c' | b' \rangle \langle b' | a' \rangle \right|^2}_{\hat{B} \text{ measure}}$$

Uncertainty relation. $[\hat{A}, \hat{B}] \neq 0$

$$\Delta \hat{A} = \hat{A} - \langle \hat{A} \rangle$$

$$\langle (\Delta \hat{A})^2 \rangle = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$$

$$\boxed{\langle (\Delta \hat{A})^2 \rangle \langle (\Delta \hat{B})^2 \rangle \geqq \frac{1}{4} |\langle [\hat{A}, \hat{B}] \rangle|^2}$$

Lemma 1. Schwarz inequality, $|\alpha\rangle, |\beta\rangle$

$$\underline{\langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \geqq |\langle \alpha | \beta \rangle|^2}$$

$$\left((\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) \geqq (\vec{a} \cdot \vec{b})^2 \right)$$

(ex) $(a_x^2 + a_y^2)(b_x^2 + b_y^2) - (a_x b_x + a_y b_y)^2$

$$\quad = (a_y b_x - a_x b_y)^2 \geqq 0$$

$|\psi\rangle = |\alpha\rangle + \lambda |\beta\rangle$

$a\lambda^2 + 2b\lambda + c$, $a \geqq 0$

$\langle \psi | \psi \rangle \geqq 0$ $\quad (\langle \alpha | + \lambda^* \langle \beta |)(|\alpha\rangle + \lambda |\beta\rangle) = \langle \alpha | \alpha \rangle + |\lambda|^2 \langle \beta | \beta \rangle$

$$+ \underline{\lambda \langle \alpha | \beta \rangle + \lambda^* \langle \beta | \alpha \rangle}$$

$$\Rightarrow |\langle \beta | \alpha \rangle|^2 - \langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \leqq 0$$

$$|\alpha\rangle = \Delta\hat{A}\,|\psi\rangle$$
$$|\beta\rangle = \Delta\hat{B}\,|\psi\rangle$$
} Lemma 1.

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle = \langle\psi|(\Delta\hat{A})^2|\psi\rangle\,\langle\psi|(\Delta\hat{B})^2|\psi\rangle$$
$$\geq \left|\langle\psi|\Delta\hat{A}|\psi\rangle\langle\psi|\Delta\hat{B}|\psi\rangle\right|^2$$

for any $|\psi\rangle$

$$\Rightarrow \langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geq |\langle\Delta\hat{A}\,\Delta\hat{B}\rangle|^2$$

$[\hat{A},\hat{B}] \neq 0 \rightarrow$ no common e. vector $\rightarrow$ no values

$\rightarrow$ uncertainty

$[\hat{A},\hat{B}] = 0 \rightarrow$ eigenvalues. $\rightarrow$ no uncertainty

"classical"

(ex) H-atom $\quad CSCO = \{ \hat{H}, \vec{L}^2, L_z, \vec{S}^2, S_z \}$

$$| n \quad l \quad m \quad \tfrac{1}{2} \quad m_s \rangle \xrightarrow{|\vec{x}\rangle} \langle \vec{x} | n l m m_s \rangle = \psi_{nlm,m_s}(r,\theta,\phi)$$

representation

$$\psi_{nlm}(r,\theta,\phi) \to R_{nl}(r) Y_{lm}(\theta,\phi)$$

1.5. Change basis $([\hat{A},\hat{B}] \neq 0)$

$n = 1 \cdots N$

$\hat{A} \to \{ |a_n\rangle \}$ $\quad \hookrightarrow 1, \cdots, N$

$\hat{B} \to \{ |b_n\rangle \}$

$$\hat{U} | a_n \rangle = | b_n \rangle \qquad n = 1 \cdots N$$

$$\hat{U} |a_1\rangle = |b_1\rangle \quad \vdots$$

$$\hat{U} = \sum_n | b_n \rangle \langle a_n |$$

$$\hat{U} | a_k \rangle = \sum_n | b_n \rangle \underbrace{\langle a_n | a_k \rangle}_{\delta_{nk}} = | b_k \rangle$$

$$\hat{U}^\dagger = \sum_n | a_n \rangle \langle b_n |$$

$\underline{\underline{\hat{U} \text{ : unitary}}}$

$$\hat{U}^\dagger \hat{U} = \left(\sum_n | a_n \rangle \langle b_n | \right) \left(\sum_m | b_m \rangle \langle a_m | \right)$$

$$= \sum_{n,m} | a_n \rangle \underbrace{\langle b_n | b_m \rangle}_{\delta_{nm}} \langle a_m | = \sum_n | a_n \rangle \langle a_n | = \underline{\mathbb{1}}$$

$$|\alpha\rangle = \sum_n \langle a_n|\alpha\rangle |a_n\rangle$$

$$= \sum_m \langle b_m|\alpha\rangle \underbrace{|b_m\rangle}_{\hat{U}|a_m\rangle}$$

$$\underbrace{\langle a_k|\alpha\rangle} = \sum_m \langle a_k|\hat{U}|a_m\rangle \underbrace{\langle b_m|\alpha\rangle}$$

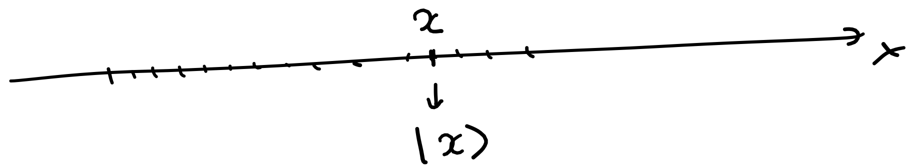
$$\begin{pmatrix} \langle a_1|\alpha\rangle \\ \vdots \\ \langle a_N|\alpha\rangle \end{pmatrix} = \begin{pmatrix} \langle a_1|\hat{U}|a_1\rangle & \cdots & \langle a_1|\hat{U}|a_N\rangle \\ \vdots & & \vdots \\ \langle a_N|\hat{U}|a_1\rangle & \cdots & \langle a_N|\hat{U}|a_N\rangle \end{pmatrix} \begin{pmatrix} \langle b_1|\alpha\rangle \\ \vdots \\ \langle b_N|\alpha\rangle \end{pmatrix}$$

$$\underset{\uparrow \atop \{|b\rangle\}}{X'} = U^\dagger \underset{\uparrow \atop \{|a\rangle\}}{X}\, U$$

Special Representation (continuous)

① $\hat{X}|x\rangle = x|x\rangle$ $\quad x =$ 위치 $\quad x = -\infty \;—\; \infty$

연속값



$$\mathbb{1} = \underbrace{\sum_x}_{\overset{\shortparallel}{\int dx}} |x\rangle\langle x| = \int dx\, |x\rangle\langle x| \Rightarrow \int d^3r\, |\vec{r}\rangle\langle\vec{r}| = \mathbb{1}$$

$$\hat{\vec{r}} = (\hat{x}, \hat{y}, \hat{z})$$

$$|\alpha\rangle = \mathbb{1}|\alpha\rangle = \int dx\, |x\rangle \underbrace{\langle x|\alpha\rangle}_{\equiv \psi_\alpha(x)} = \int dx\, \psi_\alpha(x)\, |x\rangle$$

$$\Big(\quad |\alpha\rangle = \big(|+\rangle\langle+| + |-\rangle\langle-|\big)|\alpha\rangle = \langle+|\alpha\rangle|+\rangle + \langle-|\alpha\rangle|-\rangle$$

$$[\hat{x}, \hat{y}] = 0 \quad \cdots$$

$$\vec{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3) \rightarrow [\hat{x}_i, \hat{x}_j] = 0$$

② momentum representation

$$\hat{\vec{p}}\,|\vec{p}\rangle = \vec{p}\,|\vec{p}\rangle \qquad \{|\vec{p}\rangle\} \text{ basis}$$

(if $\vec{x}$ is not restricted)

$$\mathbb{1} = \int d^3\vec{p}\;\; |\vec{p}\rangle\langle\vec{p}|$$

x is restricted

$$0 \leq x \leq L$$

$$\downarrow$$

$$\left\{ p_n = \frac{2\pi n}{L} \quad n = 0, \pm 1, \pm 2, \cdots \right\}$$

$$\mathbb{1} = \sum_n |p_n\rangle\langle p_n|$$

$$|x\rangle \longleftrightarrow |p\rangle$$

$$\mathbb{1}\,|x\rangle = \int dp\, |p\rangle\langle p|x\rangle$$

$$\langle x|\psi\rangle = \psi(x) = \int dp\, \langle x|p\rangle \underbrace{\langle p|\psi\rangle}_{\tilde{\psi}(p)}$$

$$\psi(x) = \int dp\, \langle x|p\rangle\, \tilde{\psi}(p)$$

$$\langle p|\psi\rangle = \tilde{\psi}(p) = \int dx\, \langle p|x\rangle\langle x|\psi\rangle = \int dx\, \underbrace{\langle p|x\rangle}_{\frac{e^{i\frac{px}{\hbar}}}{\sqrt{2\pi\hbar}}}\, \psi(x)$$

Fourier transform

$$\langle x|p\rangle = \frac{e^{-i\frac{px}{\hbar}}}{\sqrt{2\pi\hbar}}$$

$$[\hat{x}, \hat{p}] = i\hbar \longrightarrow (\Delta x)^2 (\Delta p)^2 \geq \frac{\hbar^2}{4}$$

3차원 $$[\hat{x}_i, \hat{p}_j] = i\hbar\,\delta_{ij}$$

1장 끝.