(고전) 역학 (Mechanics)

(양자)

동역학 dynamics

motion (시간에 따라 달라진다)

$$\sum_i \vec{F}_i \neq 0$$

간단한 물리계

갯수가 적은 경우

1개, 2개

3체문제

Newton

공리(Axiom)

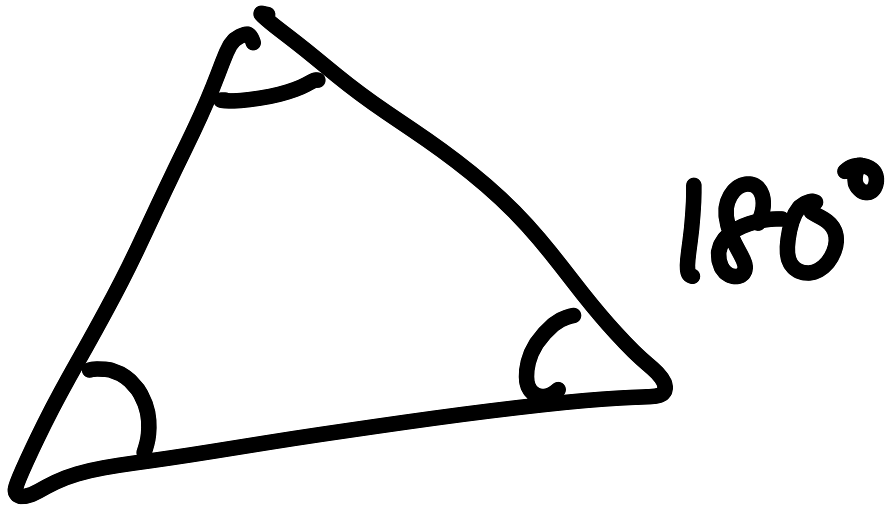
수학: Theorem(정리)

실험법칙.

역학

뉴튼 (Newton)

공리 (axiom)



물리 법칙 (laws of motion)

"계산"

행성

돌

- - - -

사소한 생각 → (데카르트) 공간 (평평한 flat)

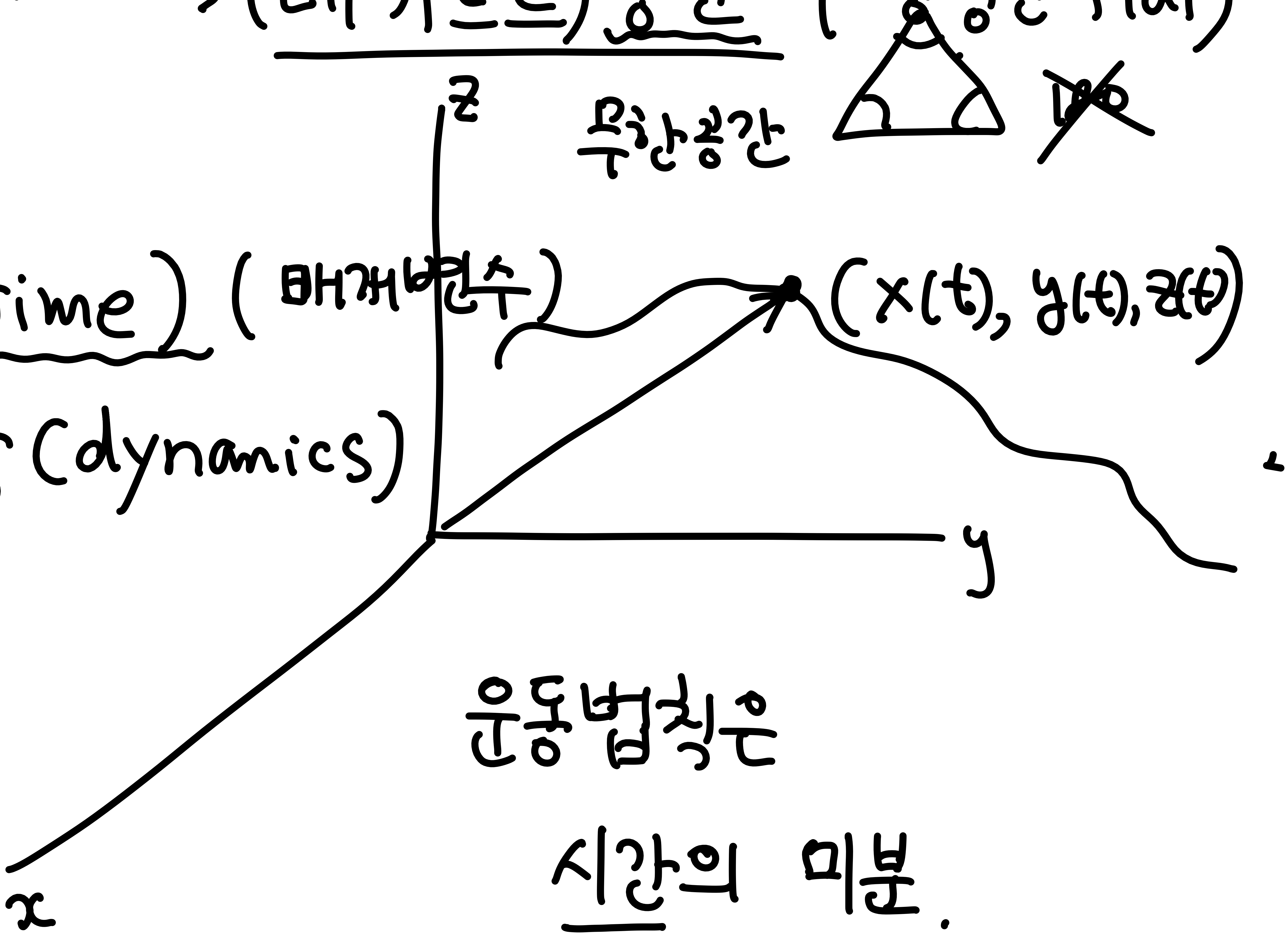
~~180~~

무한공간

시간 (time) (매개변수)

↓

동역학(dynamics)



운동법칙은

시간의 미분.

질량 (mass)

물체의 고유 성질 (ex) (전하)

물리법칙을 정의 $\vec{F} = \underset{\sim}{m}\vec{a}$

$\sin(\overset{\downarrow}{m})$.

$m = 2kg$

$\cancel{m + m^2}$

$F \propto \frac{m_1 m_2}{r^2}$

시간, 공간, 질량 : "차원" (dimension)

$\updownarrow$

숫자 (차원이 없는 양)

$\pi, \sqrt{2}, \cdots$

$\sin(\pi)$

$\pi + \pi^2$

기본

질량 : [M]   kg, g, lb, ...

단위

↓

차원

↑

길이 : [L]   m, cm, ...

시간 : [T]   sec, min, hr, ...

복합

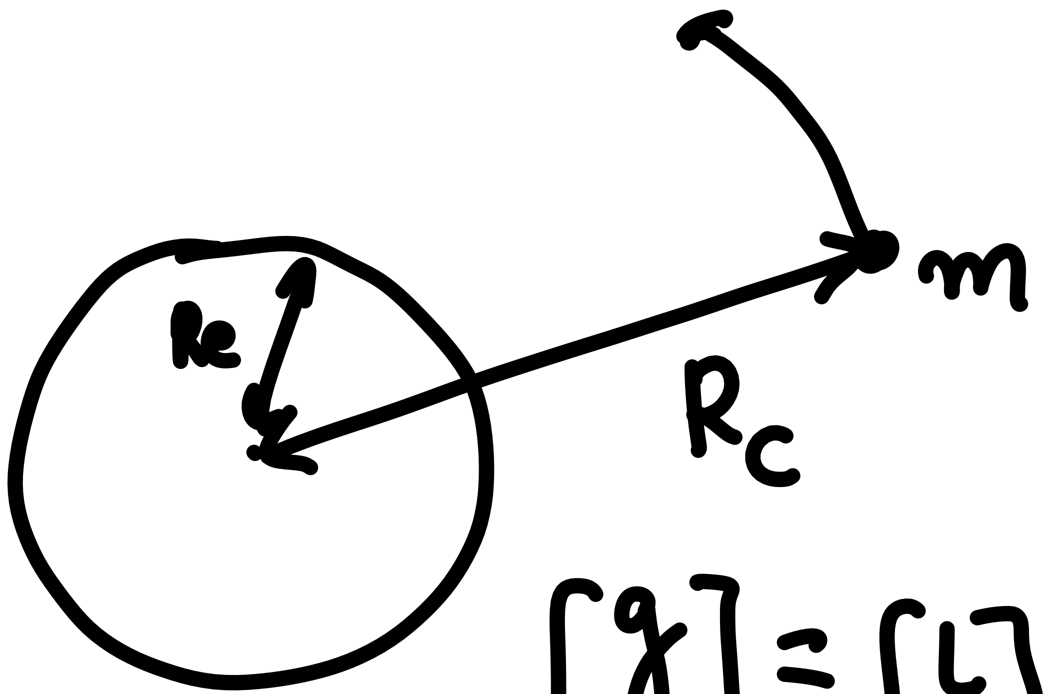
속도 : $[L]/[T] = [L][T]^{-1}$

운동량 : $[M][L][T]^{-1}$ v

에너지 :

$$\cancel{e^{v}} = \overset{\downarrow}{1} + \overset{\downarrow}{v} + \overset{[L]^2[T]^{-2}}{\overset{\downarrow}{\frac{v^2}{2}}} + \frac{v^3}{6} + \cdots$$

Dimensional Analysis (차원분석)

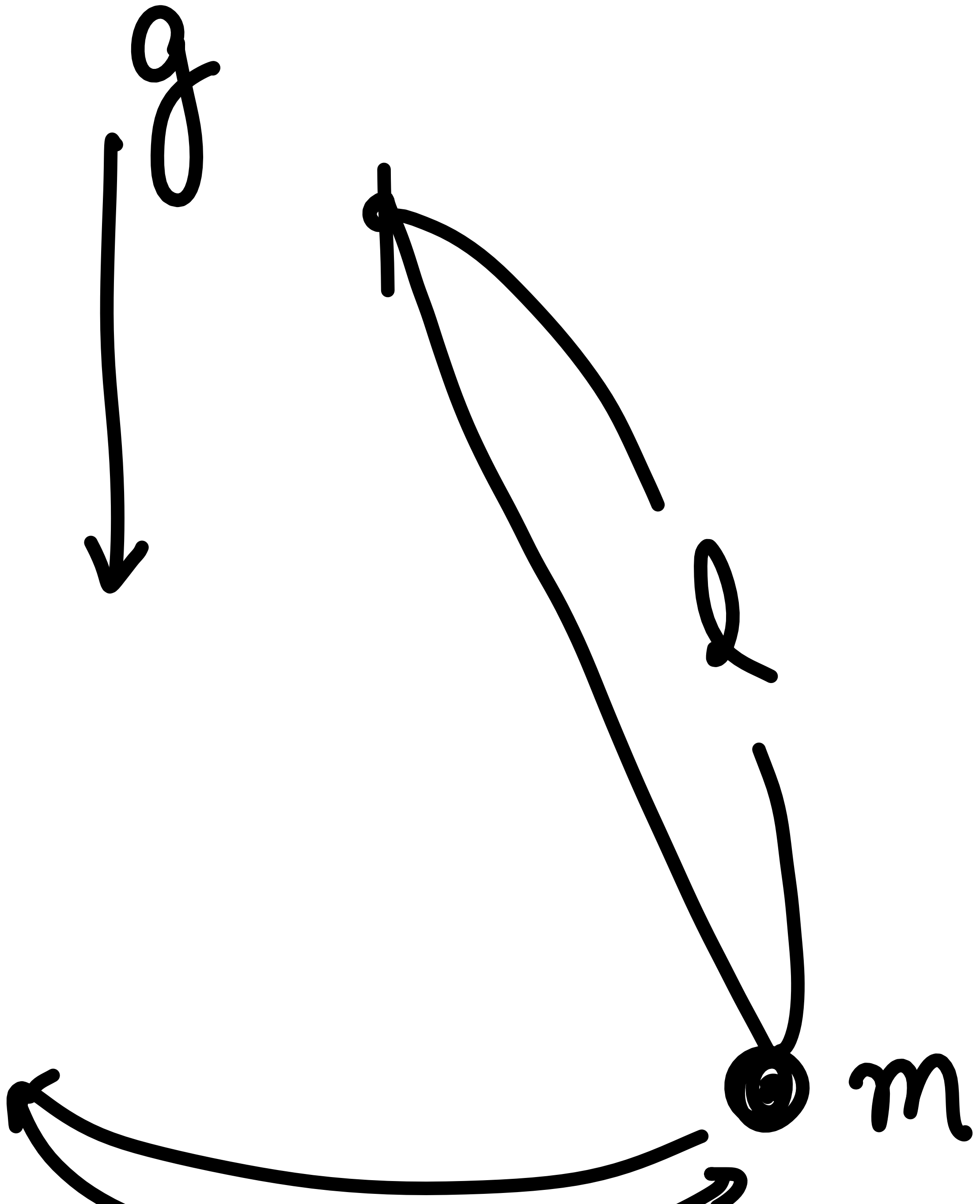


$$v_{Ⓒe} = \left(\frac{g R_e^2}{R_c}\right)^{\frac{1}{2}}$$

$$\downarrow$$

$$[g] = [L][T]^{-2}$$

$$[L][T]^{-1} = \left(\frac{\cancel{[L]}[T]^{-2}[L]^2}{\cancel{[L]}}\right)^{\frac{1}{2}}$$



$$\tau = f(m, l, g) = m^{\overset{0}{\alpha}}\, l^{\overset{\frac{1}{2}}{\beta}}\, g^{\overset{-\frac{1}{2}}{\gamma}}$$

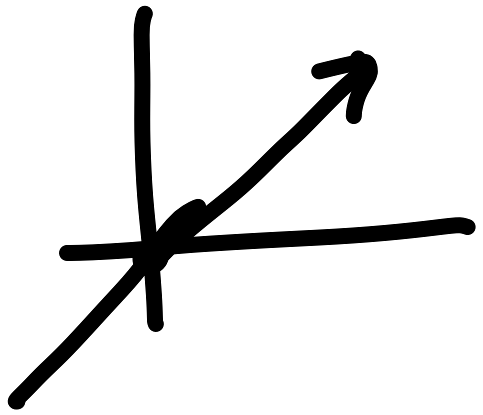
$$= x\sqrt{\frac{l}{g}}$$

$$\downarrow$$

$$[T] = [M]^{\alpha}[L]^{\beta}\left([L][T]^{-2}\right)^{\gamma}$$

$$= [M]^{\alpha}[L]^{\beta+\gamma}[T]^{-2\gamma}$$

$\therefore \alpha = 0,\ \beta+\gamma = 0,\ \gamma = -\frac{1}{2}$



Vector

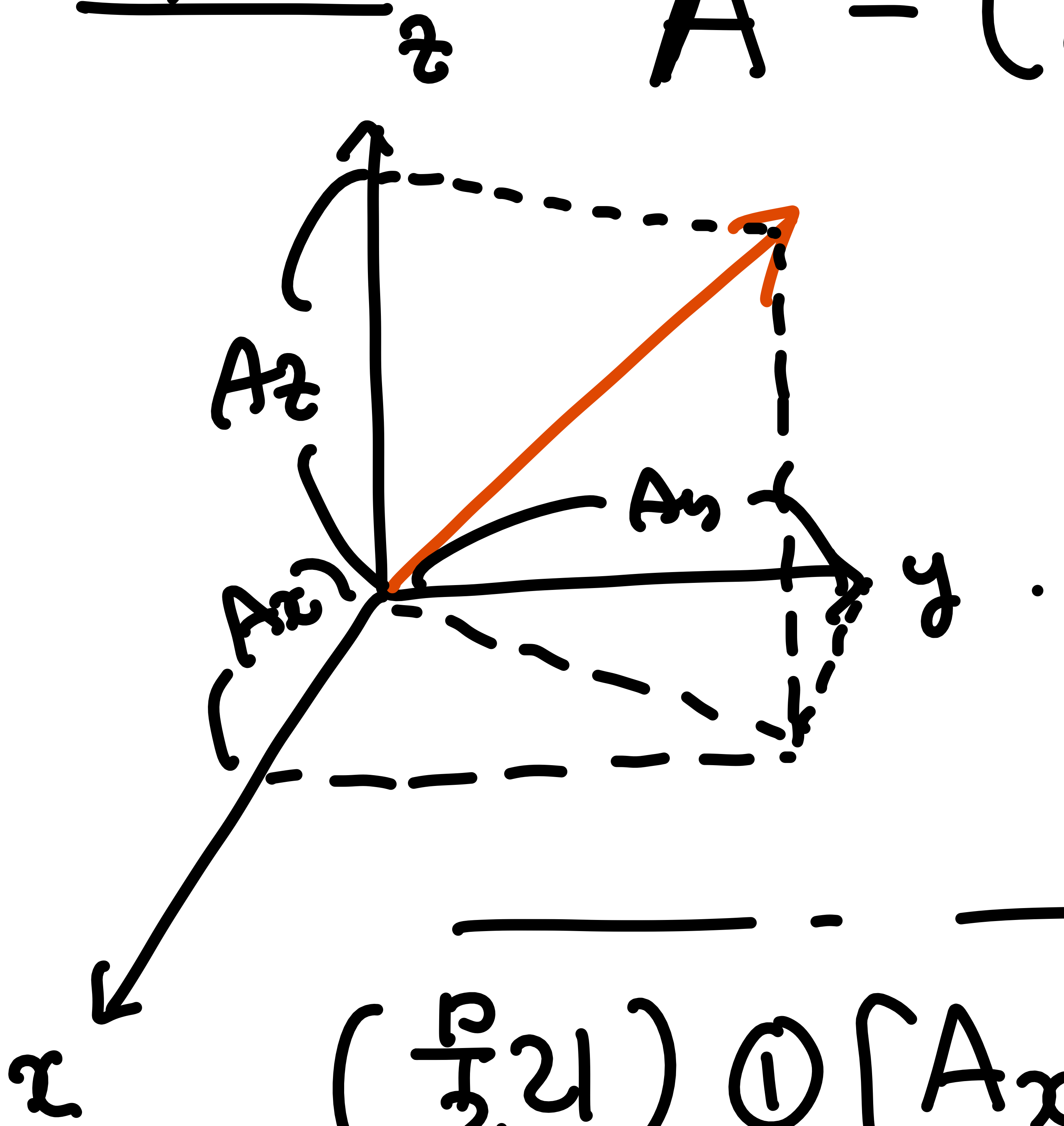
(scalar)

물리량	벡타	스칼라
변수 (variable)	위치, 변위, 속도, 가속도 힘, 운동량, 전기장, 자기장	에너지, 속력, 거리, 길이,
상수 (constant)	×	질량, 전하, 뉴턴상수 G 광속 c, ...

벡타

$$\vec{A} = (A_x, A_y, A_z)$$

성분(component)

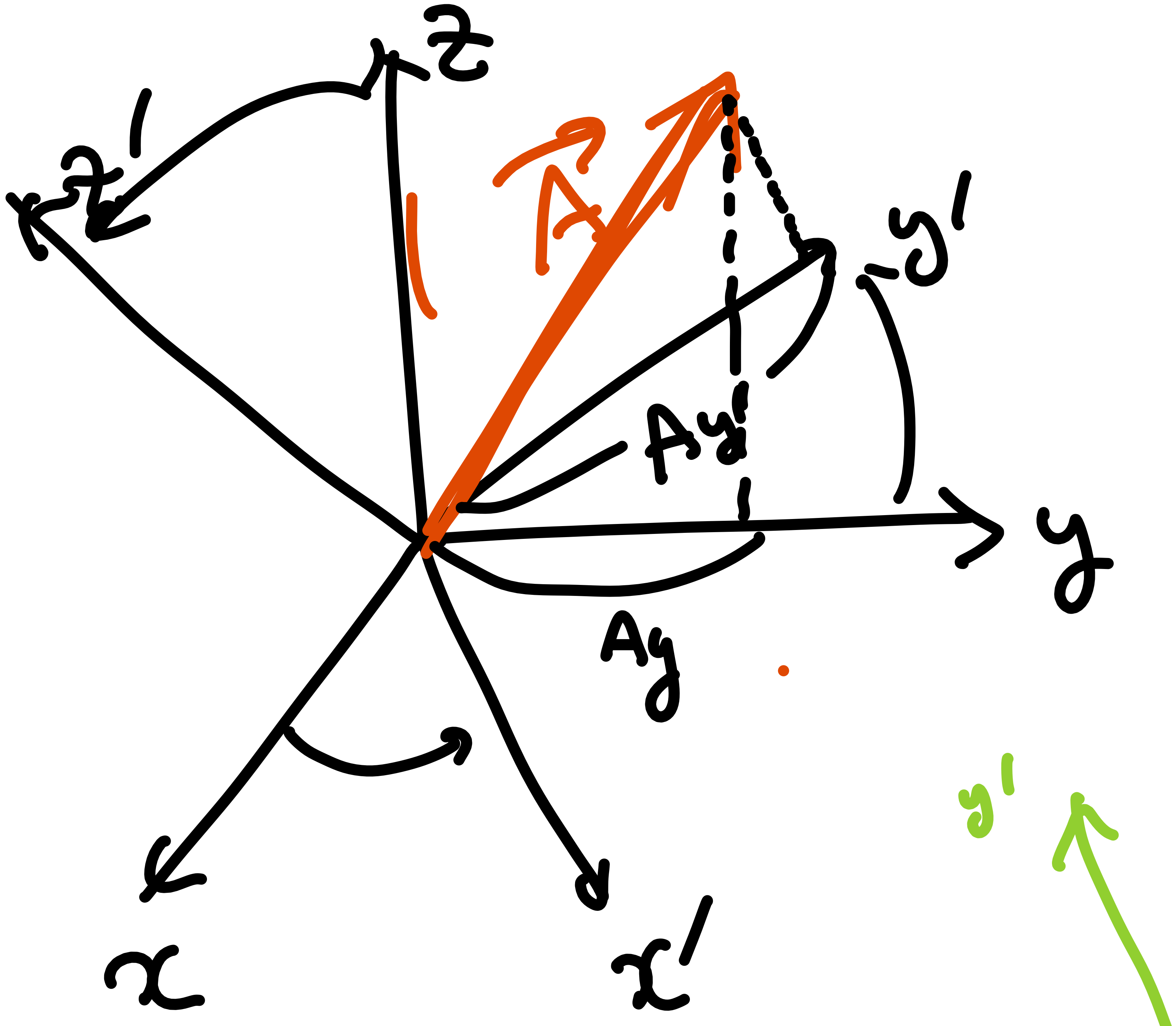


3차원, 2차원, ··· n 차원

(물리) ① $[A_x] = [A_y] = [A_z]$

[···] → 차원 ~~(1kg, 1m, 1초)~~

② 공간의 회전에 따라 같이 회전

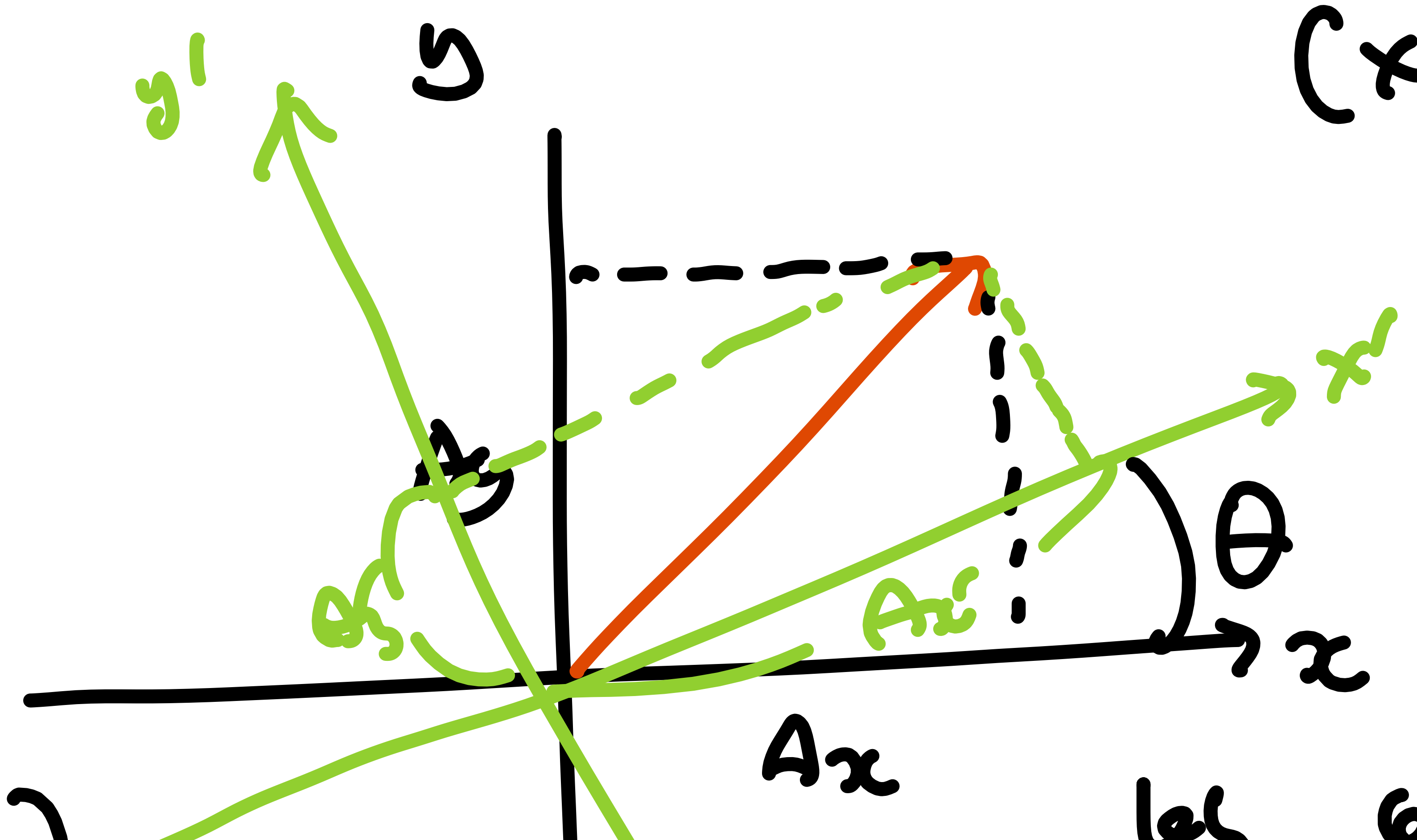


$\vec{A} = (A_x, A_y, A_z)$

(x-y-z)

$= (A_{x'}, A_{y'}, A_{z'})$

(x'-y'-z')

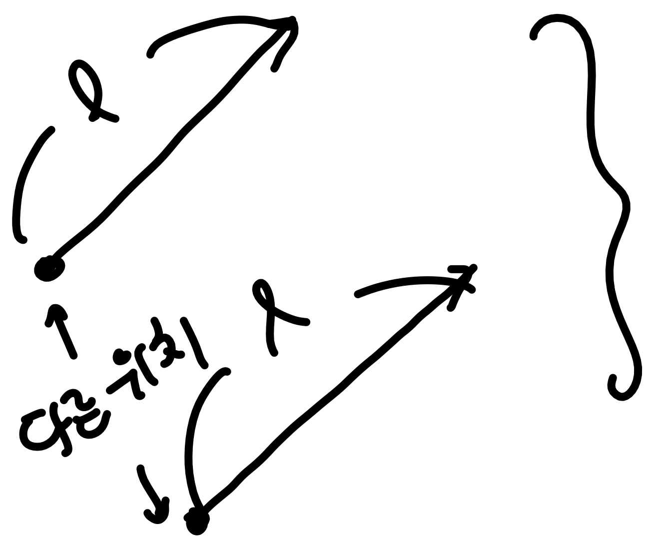


$(A_x, A_y) \leftrightarrow (A_{x'}, A_{y'})$

$\vec{m} \propto (\overset{kg}{40}, \overset{kg}{41}, \overset{kg}{42})$

Vector의 연산

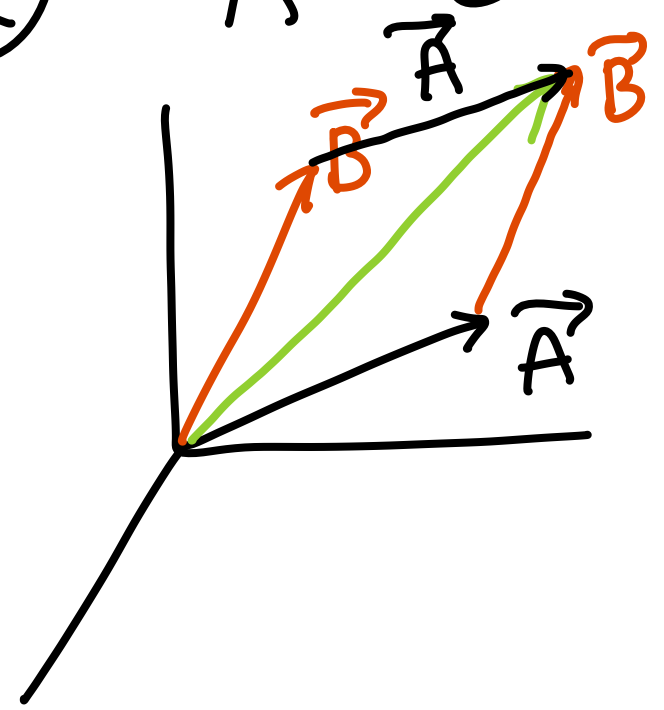
① $\vec{A} = \vec{B} \Rightarrow A_x = B_x, A_y = B_y, \cdots$



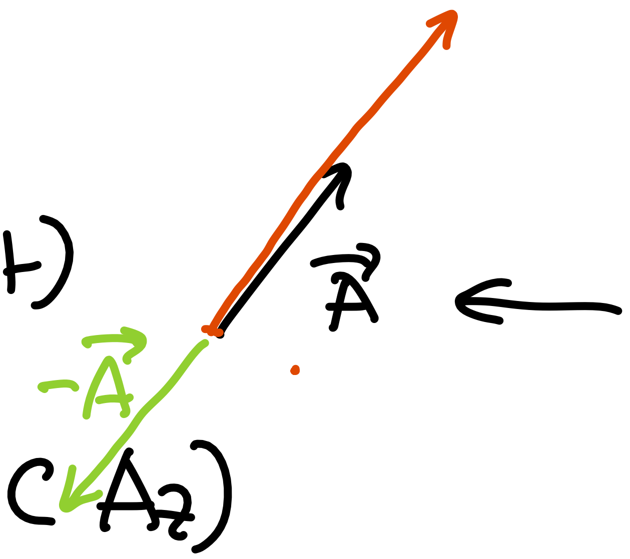
동일.

② $\vec{A} + \vec{B} = (A_x + B_x, A_y + B_y, A_z + B_z)$

$= \vec{B} + \vec{A}$ ⑥ (교환)



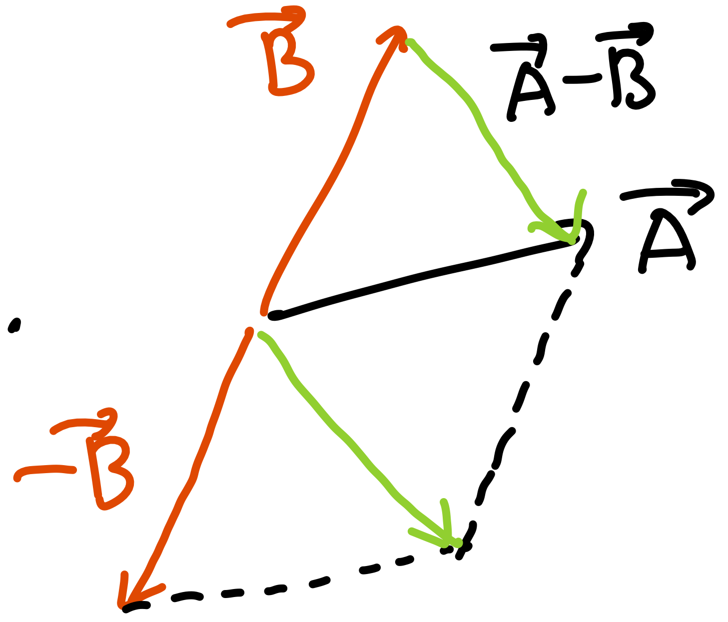
③ 곱셈 (스칼라 × 벡타)



$c\vec{A} = (cA_x, cA_y, cA_z)$

$(-1)\vec{A} = -\vec{A}$

④ 뺄셈 $\underbrace{\vec{A}-\vec{B}}_{\vec{A}+(-\vec{B})} = (A_x - B_x, A_y - B_y, A_z - B_z)$



⑤ null vector $\vec{0} = (0, 0, 0)$

$\vec{A} - \vec{A} = \vec{0}$

⑦ Associative Law (결합) for addition

$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$

⑧ 분배법칙

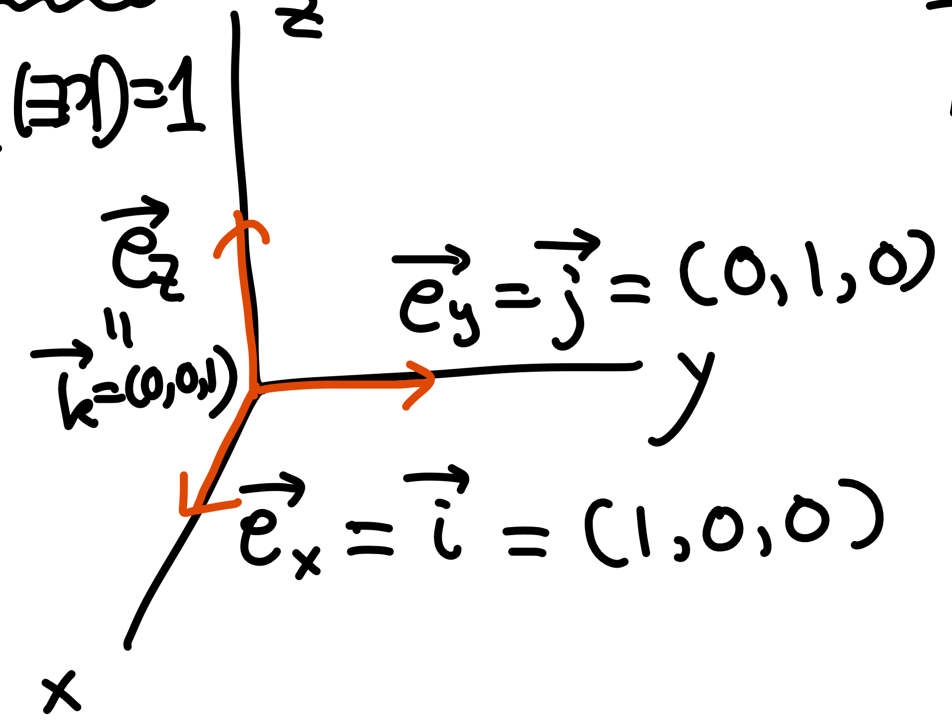
$c(\vec{A} + \vec{B}) = c\vec{A} + c\vec{B}$

⑨ 크기 (magnitude)

$|\vec{A}| = A = \sqrt{A_x^2 + A_y^2 + A_z^2}$

⑩ 단위 벨타 (좌표축)

길이(크기)=1



$$\vec{A}=(A_x, A_y, A_z)$$
$$=A_x(1,0,0)$$
$$+A_y(0,1,0)$$
$$+A_z(0,0,1)$$
$$=A_x\vec{e_x}+A_y\vec{e_y}+A_z\vec{e_z}$$
$$=A_x\vec{i}+A_y\vec{j}+A_z\vec{k}$$

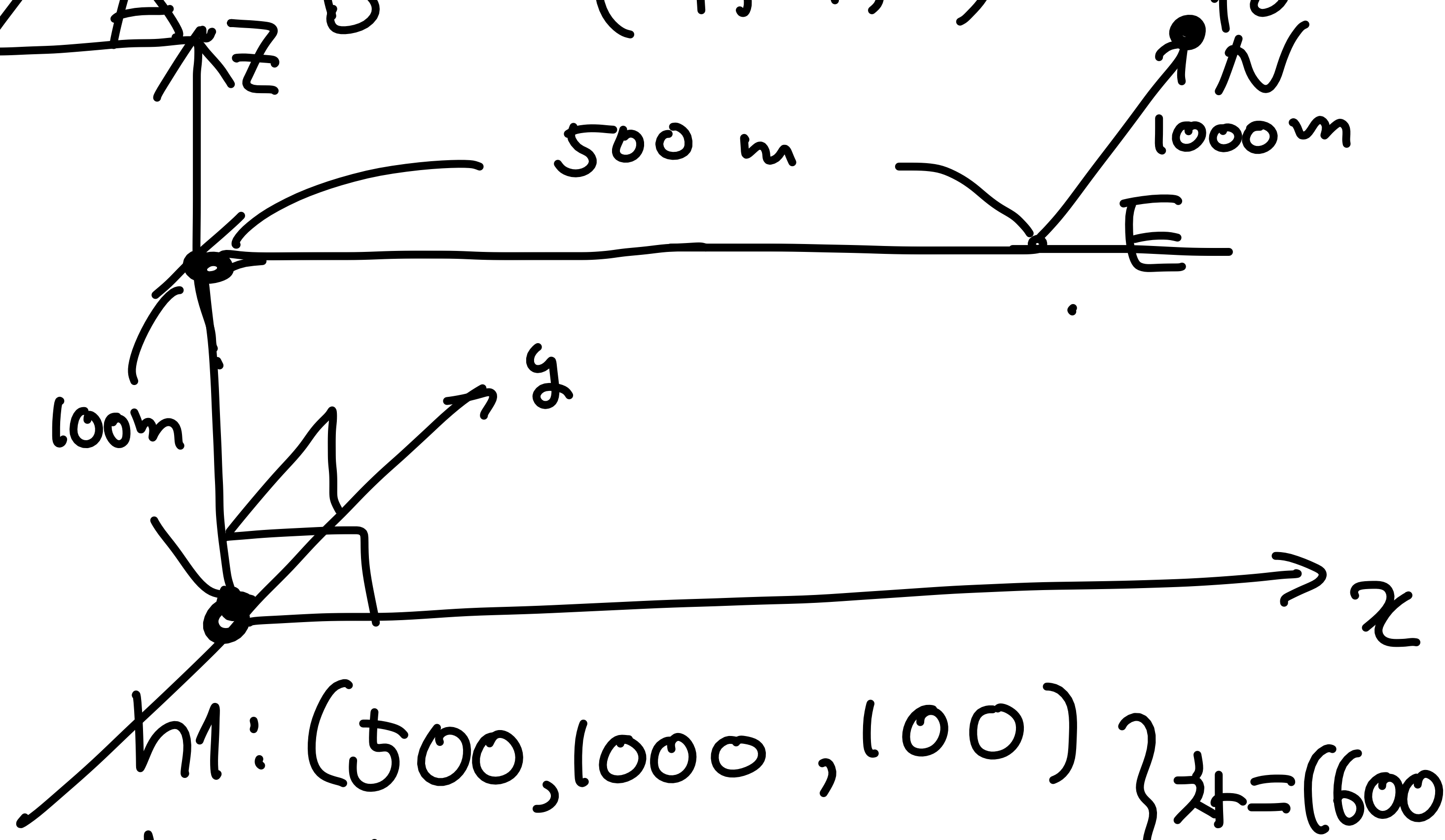
(Ex) 1.3.1.

$\vec{A}=(1,0,2)$, $\vec{B}=(0,1,1) \rightarrow \vec{A}+\vec{B}=(1,1,3)$

$|\vec{A}+\vec{B}|=\sqrt{1^2+1^2+3^2}=\sqrt{11}$

(Ex 1.3.2) $\vec{A}-\vec{B}=(1,-1,1)=\vec{i}-\vec{j}+\vec{k}$

(Ex 1.3.3)



h1: (500, 1000 , 100)

h2: (-100, 500 , 200)

차=(600, 500, -100)

거리= |(↙)| $=100\sqrt{6^2+5^2+1}=100\sqrt{62}$

1.4. scalar 곱

벡타•벡타 = 스칼라

"내적"(inner product)

$\cdot$ ×

$$\vec{A}\cdot\vec{B} = \underbrace{A_xB_x + A_yB_y + A_zB_z}_{스칼라}$$

임의의 차원

$$= \vec{B}\cdot\vec{A} \text{ (교환)}$$

분배 법칙

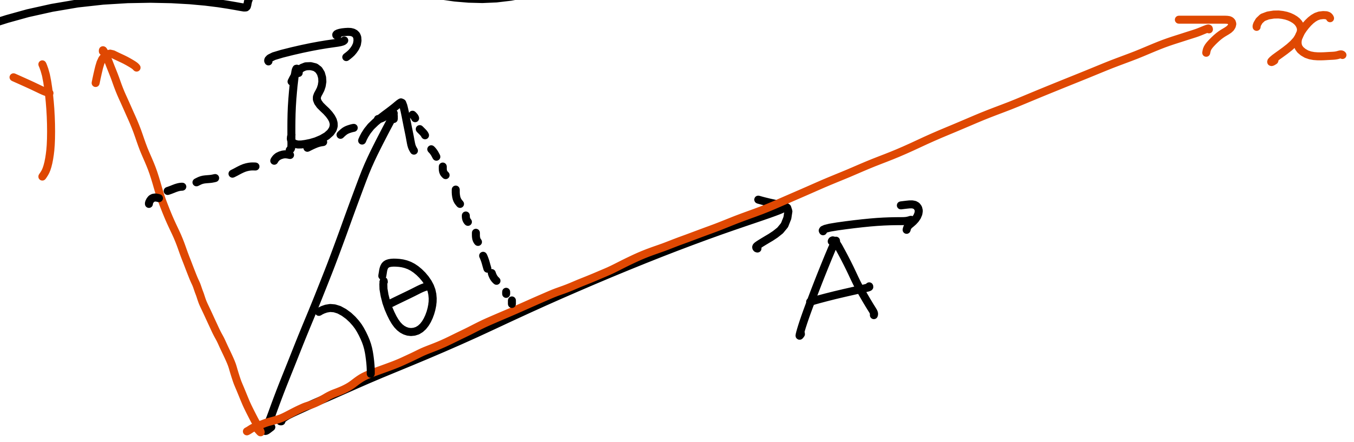
$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$$

$$(A_x,A_y,A_z)\cdot(B_x+C_x, B_y+C_y, B_z+C_z)$$

$$\|$$

$$A_x(B_x+C_x) + \cdots$$

$$A_xB_x+A_yB_y+A_zB_z + A_xC_x+A_yC_y+A_zC_z$$



$$\vec{A} = (A,0,0) \qquad A=|\vec{A}|$$

$$\vec{B} = (B\cos\theta, B\sin\theta, 0) \qquad B=|\vec{B}|$$

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

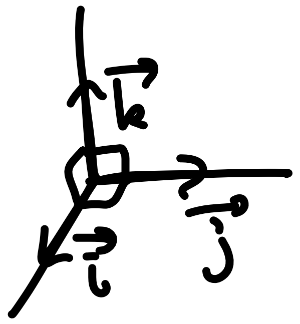
$$\therefore \boxed{\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}}$$

$$\vec{A}\cdot\vec{A} = A_x^2+A_y^2+A_z^2 = |\vec{A}|^2$$

$$|\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}}$$

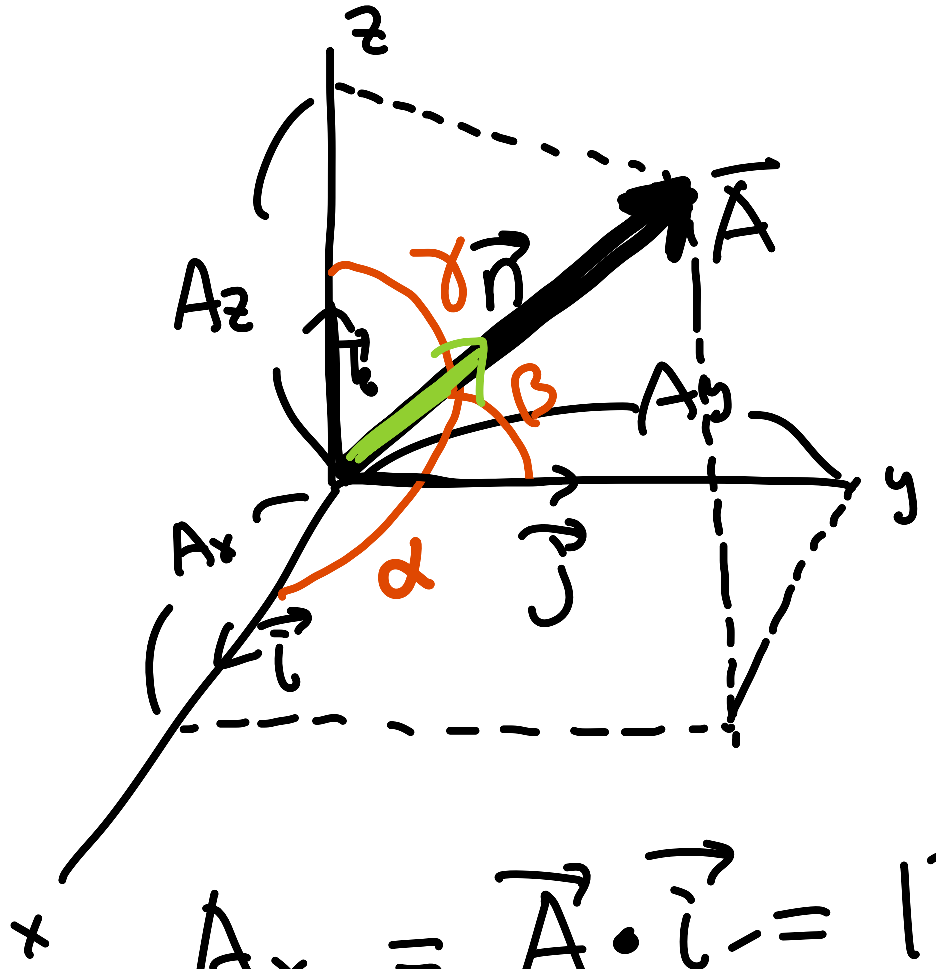
특히, $\theta = \frac{\pi}{2} = 90^\circ \longrightarrow \cos\theta = 0$

$$\therefore \vec{A}\cdot\vec{B} = 0$$



$\vec{i}\cdot\vec{i}=1$ $\quad\vec{i}\cdot\vec{j}=0$ $\quad\vec{i}\cdot\vec{k}=0$

$\vec{j}\cdot\vec{k}=0$



$$A_x = \vec{A}\cdot\vec{i} = |\vec{A}|\cos\alpha$$

$$\therefore \cos\alpha = \frac{A_x}{|\vec{A}|}$$

$$A_y = \vec{A}\cdot\vec{j} = |\vec{A}|\cos\beta$$

$$\therefore \cos\beta = \frac{A_y}{|\vec{A}|}$$

$$A_z = \vec{A}\cdot\vec{k} = |\vec{A}|\cos\gamma$$

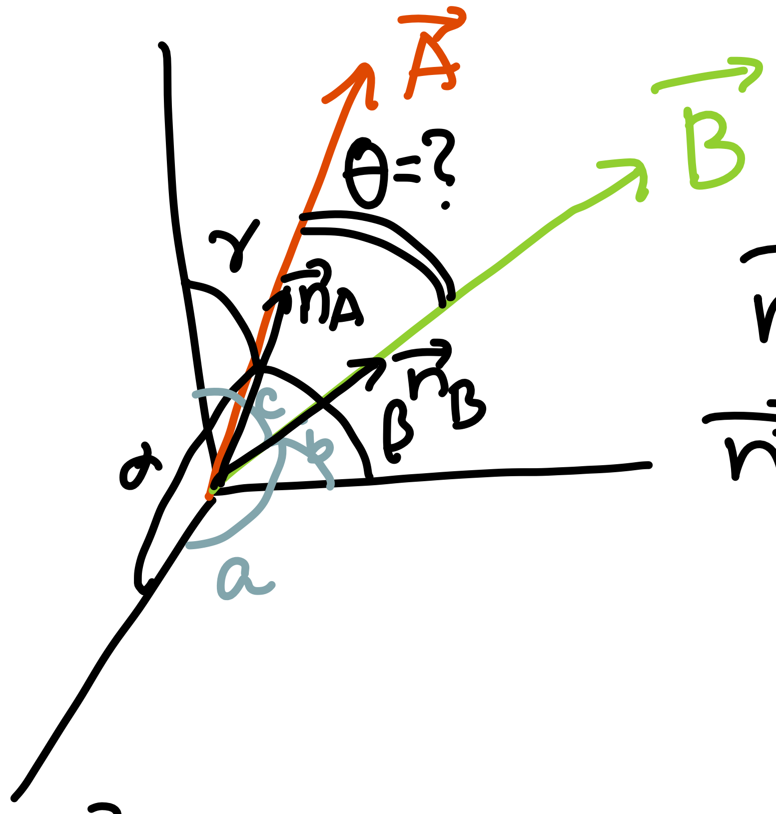
$$\therefore \cos\gamma = \frac{A_z}{|\vec{A}|}$$

$$(\cos\alpha, \cos\beta, \cos\gamma) = \vec{n}$$

$$= \left(\frac{A_x}{|\vec{A}|}, \frac{A_y}{|\vec{A}|}, \frac{A_z}{|\vec{A}|}\right)$$

$$= \frac{1}{|\vec{A}|}\underbrace{(A_x, A_y, A_z)}_{\vec{A}} = \underbrace{\frac{\vec{A}}{|\vec{A}|}}_{\substack{\text{길이} = 1 \\ \text{방향} \propto \vec{A}}}$$

$$\vec{A} = \underbrace{|\vec{A}|}_{}\underbrace{\vec{n}}_{} = A\vec{n}$$

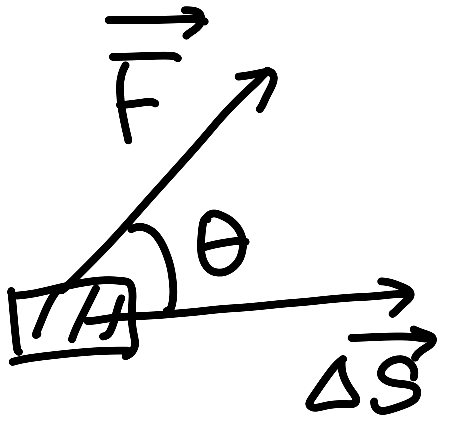


$$\vec{n}_A = (\cos\alpha, \cos\beta, \cos\gamma)$$

$$\vec{n}_B = (\cos a, \cos b, \cos c)$$

$$\vec{n}_A\cdot\vec{n}_B = \cos\theta = \cos\alpha\cos a + \cos\beta\cos b + \cos\gamma\cos c$$

[Ex 1.4.1] Work (일)

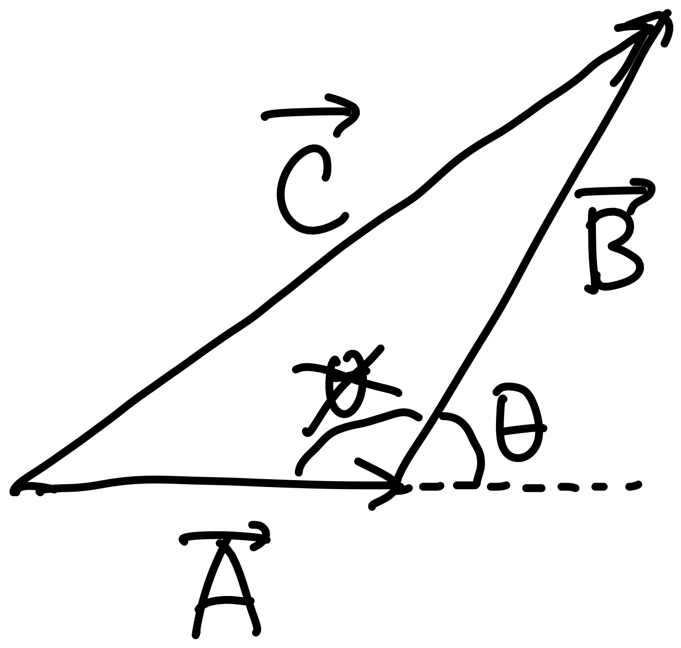


$$\Delta W = \vec{F}\cdot\Delta\vec{s}$$

↑ scalar

$$= F\,\Delta s\cos\theta$$

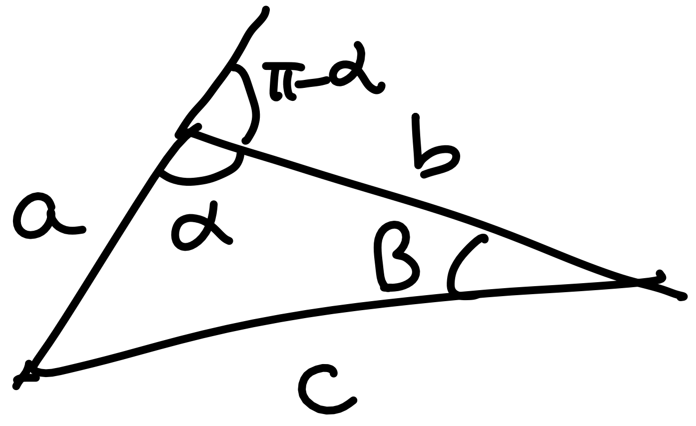
[Ex 1.4.2] Cosine Law



$$\vec{C} = \vec{A}+\vec{B} \qquad |\vec{C}|^2 = \vec{C}\cdot\vec{C} = (\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B})$$

$$|\vec{C}|^2 = \underset{\overset{\|}{\vec{A}\cdot\vec{A}}}{|\vec{A}|^2} + |\vec{B}|^2 + 2\vec{A}\cdot\vec{B} = |\vec{A}|^2 + |\vec{B}|^2 + 2|\vec{A}||\vec{B}|\cos\theta$$

$$\Rightarrow \cos\theta = \frac{C^2 - A^2 - B^2}{2AB}$$



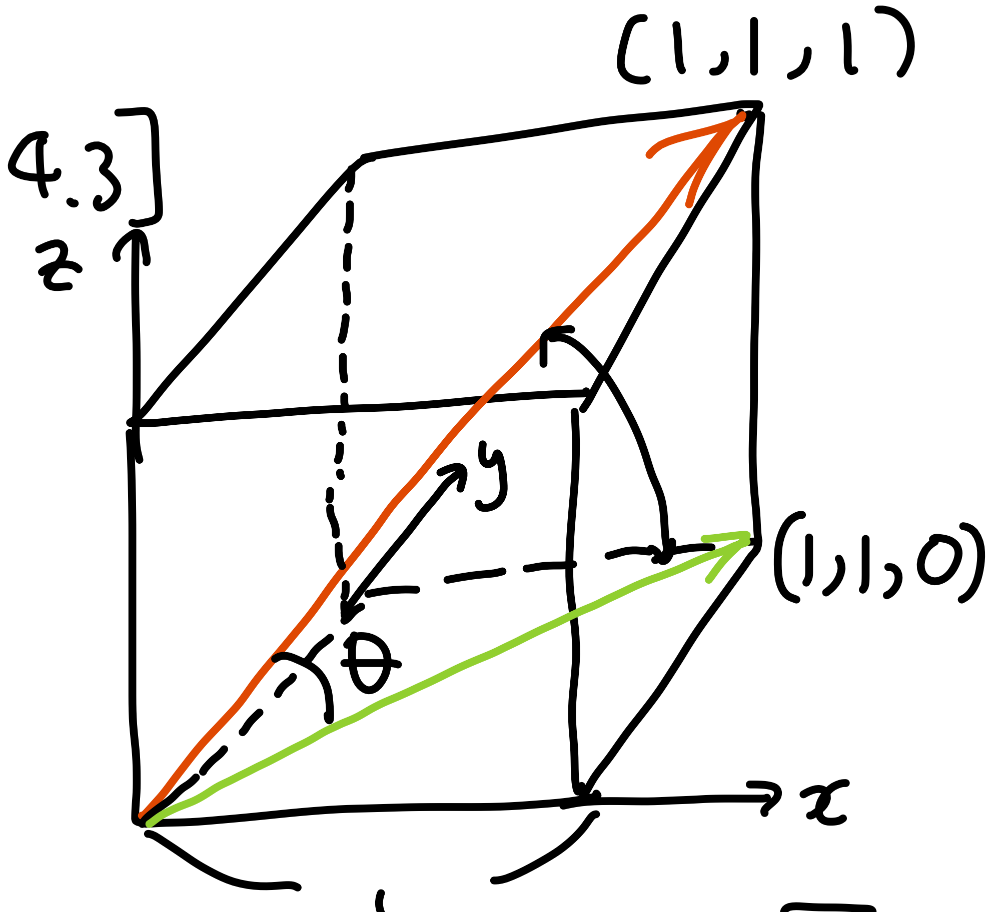
$$\theta \rightarrow \pi-\alpha$$

$$\rightarrow \cos(\pi-\alpha) = -\cos\alpha = \frac{c^2-a^2-b^2}{2ab}$$

$$\cos\alpha = \frac{a^2+b^2-c^2}{2ab}$$

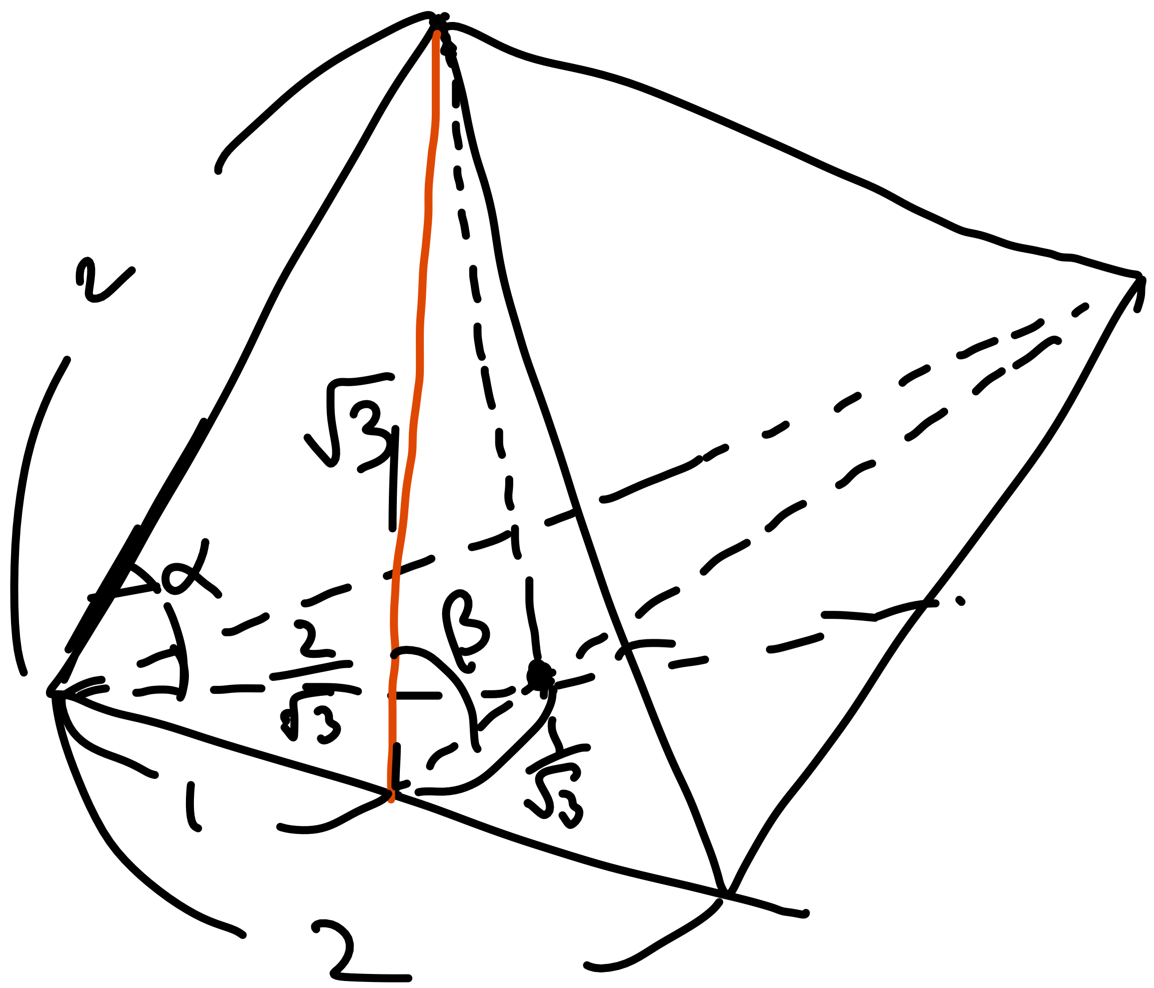
$$\cos\beta = \frac{b^2+c^2-a^2}{2bc}$$

[Ex 1.4.3]



$$\cos\theta = \frac{2}{\sqrt{3}\,\sqrt{2}} = \sqrt{\frac{2}{3}}$$

정 4면체



$$\cos\alpha = \frac{1}{\sqrt{3}} > \cos\beta = \frac{1}{3}$$

$$\alpha < \beta$$

[Ex 1.4.4]

$$\left.\begin{array}{l} a\vec{i}+\vec{j}-\vec{k} = (a,1,-1) \\ \vec{i}+2\vec{j}-3\vec{k} = (1,2,-3) \end{array}\right\} \text{수직}$$

$$a+2+3=0 \rightarrow a=-5$$

** 벡터의 외적 (outer product)
[3차원 벡터 only] vector "

$$\vec{A}\times\vec{B} = \text{벡터} \quad (\vec{A}\cdot\vec{B} = \text{스칼라})$$

$$= (A_yB_z - A_zB_y,\ A_zB_x - A_xB_z,\ A_xB_y - A_yB_x)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

A↔B "−"

$$= \vec{i}(A_yB_z - A_zB_y) + \vec{j}(A_zB_x - A_xB_z) + \vec{k}(A_xB_y - A_yB_x)$$

성질 ~~교환법칙~~ $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

$\neq \vec{B}\times\vec{A}$

결합 : $\cancel{\vec{A}\cdot\vec{B}\cdot\vec{C}}$

$$\boxed{(\vec{A}\times\vec{B})\times\vec{C} = \vec{A}\times(\vec{B}\times\vec{C})}$$

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$$

$$n(\vec{A}\times\vec{B}) = (n\vec{A})\times\vec{B} = \vec{A}\times(n\vec{B})$$

$$\vec{A}\times\vec{A} = -\vec{A}\times\vec{A} \rightarrow \vec{A}\times\vec{A} = \vec{0}$$

$$|\vec{A}\times\vec{B}|^2 = (\underbrace{\vec{A}\times\vec{B}})\cdot(\vec{A}\times\vec{B})$$

$$(|\vec{A}|^2 = \vec{A}\cdot\vec{A}) = (A_yB_z - A_zB_y)^2 + (A_zB_x - A_xB_z)^2 + (A_xB_y - A_yB_x)^2$$

$$\boxed{\therefore |\vec{A}\times\vec{B}|^2 = |\vec{A}|^2|\vec{B}|^2 - (\vec{A}\cdot\vec{B})^2}$$

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

$$= |\vec{A}|^2|\vec{B}|^2(\underbrace{1-\cos^2\theta}_{\sin^2\theta})$$

$$\therefore \boxed{|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\sin\theta}$$

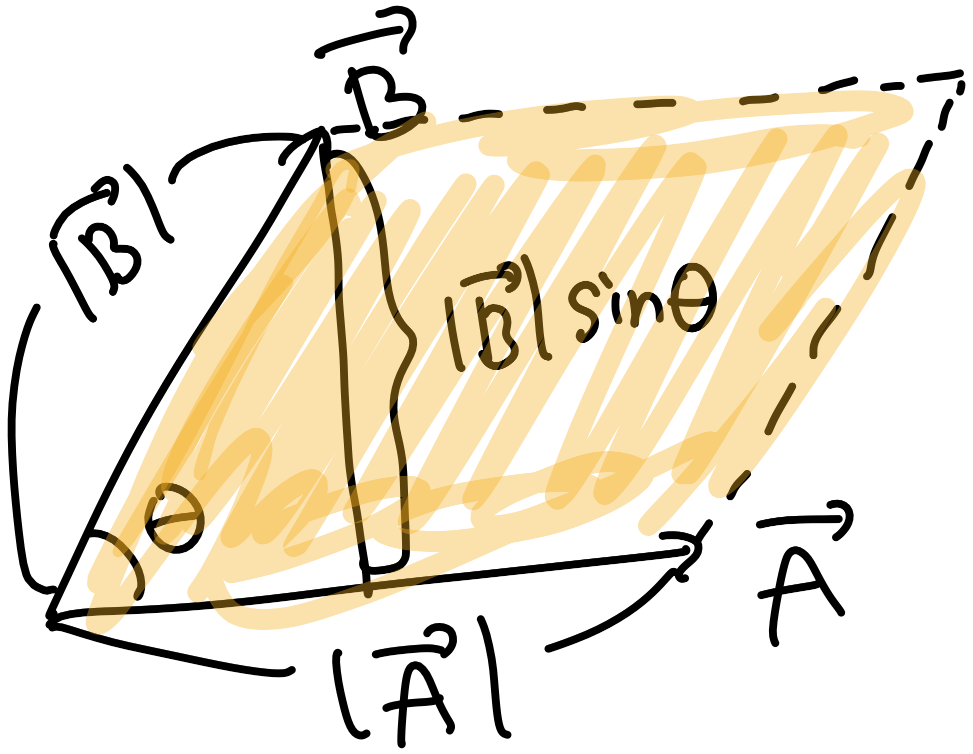
$$\begin{aligned}
= \; & A_y^2B_z^2 + A_z^2B_y^2 + A_x^2B_z^2 - 2A_yB_yA_zB_z \\
& + A_y^2B_x^2 + A_z^2B_x^2 + A_x^2B_y^2 - 2A_xB_xA_zB_z \\
& - 2A_xB_xA_yB_y
\end{aligned}$$

$$A_y^2|\vec{B}|^2 + A_z^2|\vec{B}|^2 + \overbrace{A_x^2|\vec{B}|^2}$$

$$\begin{aligned}
= \; & \boxed{A_y^2B_z^2 + A_y^2B_x^2 + A_y^2B_y^2} + A_z^2B_y^2 + A_z^2B_x^2 + A_z^2B_z^2 + A_x^2B_z^2 + A_x^2B_y^2 + A_x^2B_x^2 \\
& - 2A_yB_yA_zB_z - 2A_xB_xA_zB_z - 2A_xB_xA_yB_y - A_y^2B_y^2 - A_z^2B_z^2 - A_x^2B_x^2
\end{aligned}$$

$$= |\vec{A}|^2|\vec{B}|^2 - \underbrace{(A_xB_x + A_yB_y + A_zB_z)^2}_{\overset{\parallel}{(\vec{A}\cdot\vec{B})^2}}$$

$$|\vec{A}||\vec{B}|\sin\theta = |\vec{A}\times\vec{B}|$$



$$\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \quad \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}.$$

$$\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}.$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = (A_x, A_y, A_z)\cdot\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}.$$

Det : <u>순차적인 열, 행의 변화에 동일</u>

$$\det\begin{pmatrix} \vec{A} \\ \vec{B} \\ \vec{C} \end{pmatrix} = -\det\begin{pmatrix} \vec{B} \\ \vec{A} \\ \vec{C} \end{pmatrix}.$$

$\shortparallel$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \det\begin{pmatrix} \vec{B} \\ \vec{C} \\ \vec{A} \end{pmatrix}$$

If

$\underline{\vec{B} = \vec{A}}$

$$\vec{A}\cdot(\underset{\uparrow\ \textcircled{\vec{B}}}{\vec{A}}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A})$$

$$= \vec{C}\cdot\underbrace{(\vec{A}\times\underset{\underset{\vec{A}}{\shortparallel}}{\vec{B}})}_{\vec{0}} = 0.$$

행렬 (Matrix)

2차원 배열 : 사각형

$$\begin{pmatrix} A_1 & \cdots & A_n \\ B_1 & \cdots & B_n \\ C_1 & \cdots & C_n \end{pmatrix}$$

행 = 3

열 = n

$3 \times n$

$$\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ A_{21} & \cdots & A_{2n} \\ \vdots & & \\ A_{m1} & \cdots & A_{mn} \end{pmatrix} \equiv A \qquad m \times n$$

$$\boxed{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}} \quad 2\times 2$$

$$\det(A) = A_{11}A_{22} - A_{12}A_{21}$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Determinant

$n \times n$

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & & \cdots & A_{2n} \\ \vdots & & & \\ A_{n1} & & \cdots & A_{nn} \end{pmatrix} \equiv A \qquad n \times n$$

$\longmapsto$ 숫자

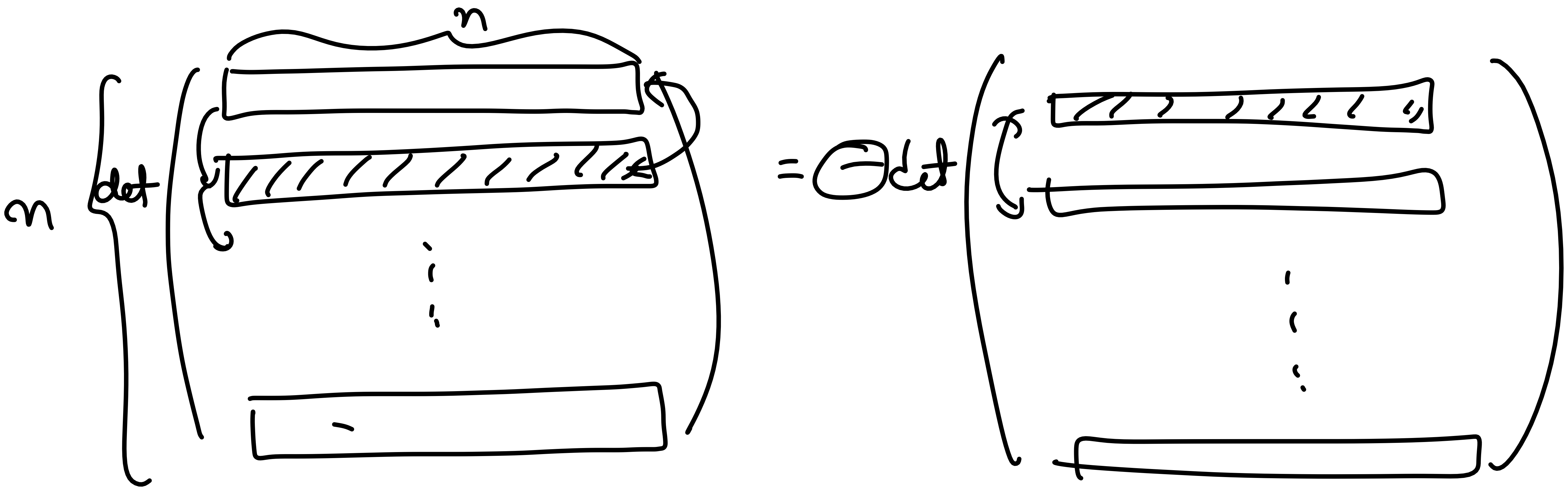
$(n-1) \times (n-1) \rightarrow A^{(11)}$

$A^{(12)}$

$$\boxed{\begin{aligned} \mathrm{Det}(A) &= 숫자 \\ &= A_{11}\,\mathrm{Det}(A^{(11)}) - A_{12}\,\mathrm{Det}(A^{(12)}) \\ &\quad + \cdots + (-1)^{n-1} A_{1n}\,\mathrm{Det}(A^{(1n)}) \end{aligned}}$$

(ex) $M = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} \rightarrow \det M = ?$

$$\det M = A_1 \underbrace{(B_2C_3 - B_3C_2)}_{(\vec{B}\times\vec{C})_1} - A_2 \overbrace{(B_1C_3 - B_3C_1)}^{-(\vec{B}\times\vec{C})_2} + A_3 \underbrace{(B_1C_2 - B_2C_1)}_{(\vec{B}\times\vec{C})_3} = \vec{A}\cdot(\vec{B}\times\vec{C})$$



$$\det\begin{pmatrix} \boxed{a \quad b} \\ \boxed{c \quad d} \end{pmatrix} = ad - bc = -\det\begin{pmatrix} c, & d \\ a & b \end{pmatrix} = bc - ad$$

$$\det\begin{pmatrix} \boxed{A_1 \quad A_2 \quad A_3} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} = \det\begin{pmatrix} \vec{A} \\ \vec{B} \\ \vec{C} \end{pmatrix} = -\det\begin{pmatrix} \vec{B} \\ \vec{A} \\ \vec{C} \end{pmatrix} = \det\begin{pmatrix} \vec{B} \\ \vec{C} \\ \vec{A} \end{pmatrix}$$

$$\det\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} = \vec{A}\cdot(\vec{B}\times\vec{C})$$

$$\det\begin{pmatrix} \vec{B} \\ \vec{C} \\ \vec{A} \end{pmatrix} = \vec{B}\cdot(\vec{C}\times\vec{A})$$

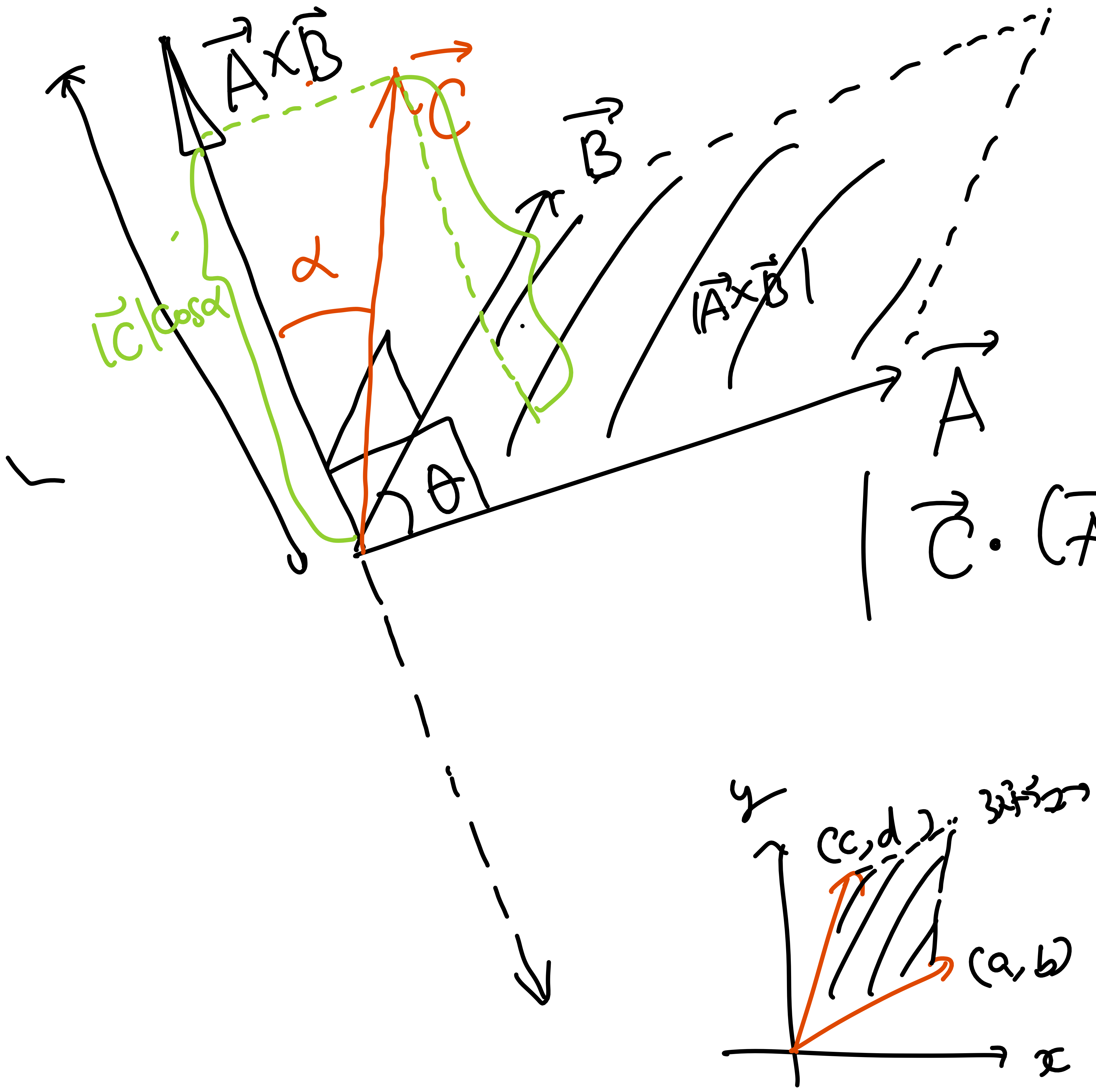
$$\Rightarrow \quad \begin{aligned} &\vec{A}\cdot(\vec{B}\times\vec{C}) \\ &= \vec{B}\cdot(\vec{C}\times\vec{A}) \\ &= \vec{C}\cdot(\vec{A}\times\vec{B}) \end{aligned} \qquad \begin{aligned} &= -\vec{A}\cdot(\vec{C}\times\vec{B}) \\ &= -\vec{C}\cdot(\vec{B}\times\vec{A}) \\ &= -\vec{B}\cdot(\vec{A}\times\vec{C}) \end{aligned}$$

$\vec{A} \perp (\vec{A}\times\vec{B})$

$\vec{B} \perp (\vec{A}\times\vec{B})$

$|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}| \sin\theta$

오른손 법칙

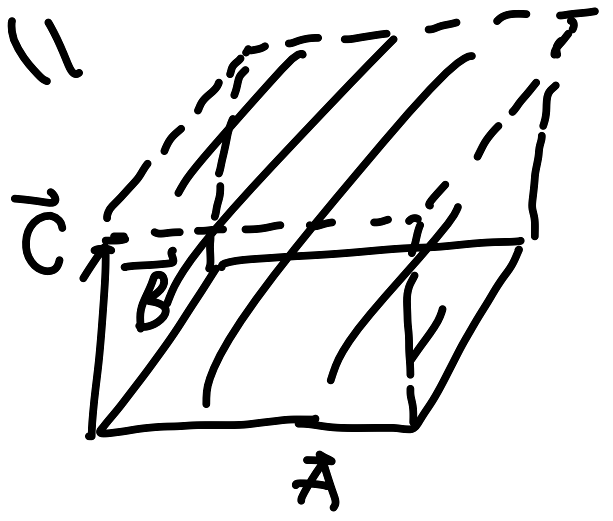


$\vec{A}\times\vec{B} \to$ $\vec{A}$에서 $\vec{B}$쪽으로 오른손 네손가락

$$|\vec{C}\cdot(\vec{A}\times\vec{B})| \to \left|\det\begin{pmatrix}\vec{C}\\ \vec{A}\\ \vec{B}\end{pmatrix}\right| = \left|\det\begin{pmatrix}\vec{A}\\ \vec{B}\\ \vec{C}\end{pmatrix}\right|$$

$\parallel$ 입방체의 부피

3차원 →



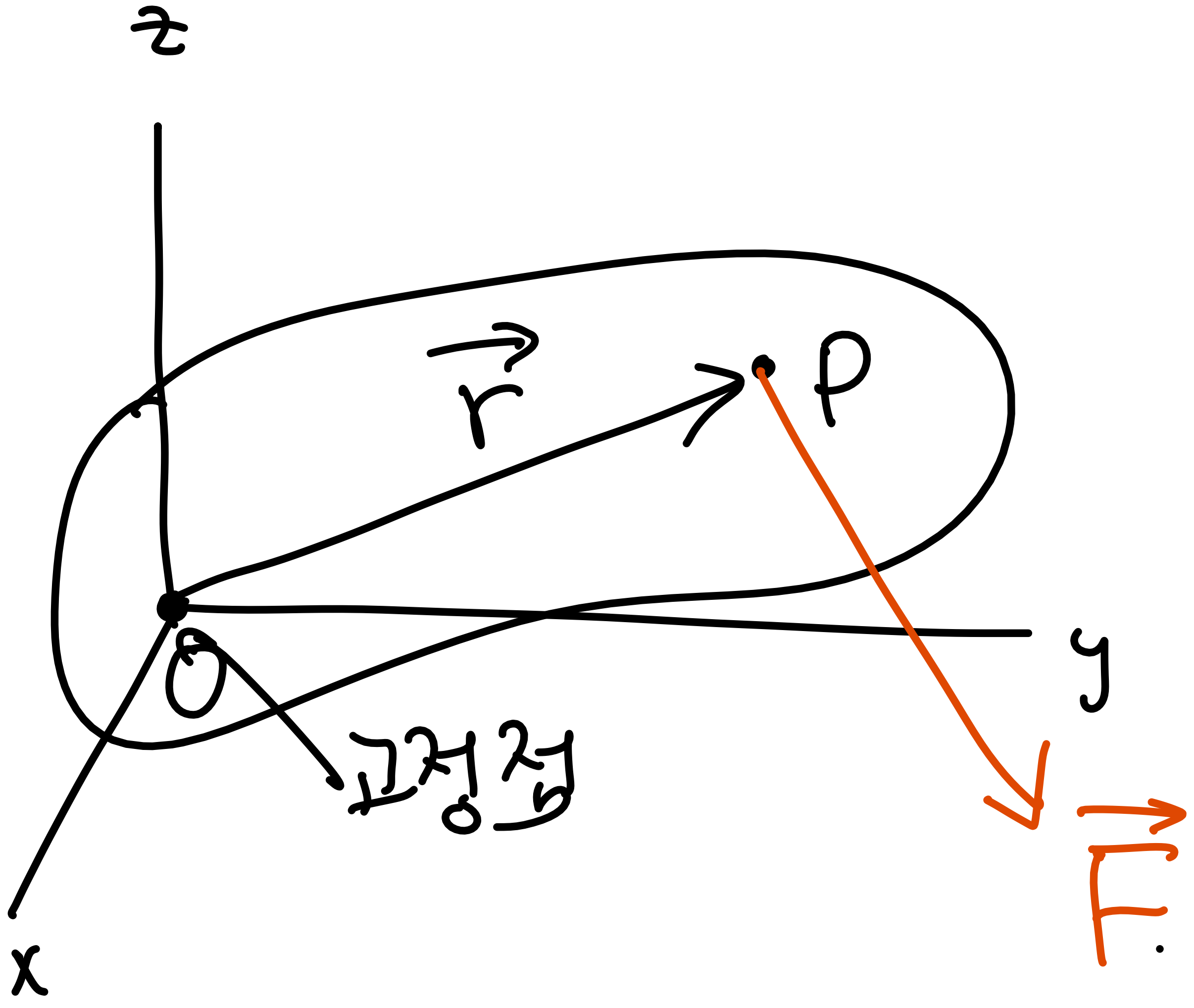
[ex 1.5.1] $\vec{A}\times\vec{B} = (1, -5, -3)$

[ex.1.5.2] $\vec{n} = \dfrac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|} = \dfrac{(1,-5,-3)}{\sqrt{35}}$

1.6. 외적 in physics

moment of a force : Torque $\underline{\vec{N}} = \underline{\vec{r}\times\vec{F}}$

if $\vec{F} \propto \vec{r} \rightarrow \vec{N} = 0$



$$|\vec{N}| = \underline{|\vec{r}|}\,\underline{|\vec{F}|}\,\underline{\underline{\sin\theta}}$$

Ex. 1.7.1 $\vec{A} = \vec{i}$, $\vec{B} = \vec{i}-\vec{j}$, $\vec{C} = \vec{k}$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} \cancel{1} & \cancel{0} & \cancel{0} \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1 \;//$$

Det 요령; 0이 많은 행이나 열은잡아
아까 공식적용

Ex. 1.7.2.

$\underline{\vec{A}}\times\underline{(\vec{B}\times\vec{C})} \rightarrow$ 3차원

$\vec{A}\times(\vec{A}\times\vec{C})$

$$\underbrace{\vec{A}}_{(A_1, A_2, A_3)} \times \underbrace{(\vec{B} \times \vec{C})}_{(B_2C_3 - B_3C_2,\ B_3C_1 - B_1C_3,\ B_1C_2 - B_2C_1)} =$$

$$= \big(A_2(B_1C_2 - B_2C_1) - A_3(B_3C_1 - B_1C_3),$$
$$A_3(B_2C_3 - B_3C_2) - A_1(B_1C_2 - B_2C_1),$$
$$A_1(B_3C_1 - B_1C_3) - A_2(B_2C_3 - B_3C_2) \big)$$

$$= \Big(B_1(\overset{+A_1C_1 - \cancel{A_1C_1}}{A_2C_2 + A_3C_3}) - C_1(\overset{A_1B_1 - \cancel{A_1B_1}}{A_2B_2 + A_3B_3}),$$
$$B_2(\underset{A_2C_2 - \cancel{A_2C_2}}{A_1C_1 + A_3C_3}) - C_2(\underset{A_2B_2 - \cancel{A_2B_2}}{A_1B_1 + A_3B_3}),$$
$$B_3(\underset{+A_3C_3 - \cancel{A_3C_3}}{A_1C_1 + A_2C_2}) - C_3(\underset{+A_3B_3 - \cancel{A_3B_3}}{A_1B_1 + A_2B_2}) \Big)$$

$\vec{A}\cdot\vec{C}$ $\vec{A}\cdot\vec{C}$ $\vec{A}\cdot\vec{C}$ $\vec{A}\cdot\vec{C}$

$\vec{A}\cdot\vec{B}$ $\vec{A}\cdot\vec{B}$ $\vec{A}\cdot\vec{B}$

$$\vec{A} \times (B \times \vec{C})$$

$$= \big(B_1 \underbrace{(\vec{A}\cdot\vec{C})},\ B_2 \underbrace{(\vec{A}\cdot\vec{C})},\ B_3 \underbrace{(\vec{A}\cdot\vec{C})} \big) - \big(C_1(\vec{A}\cdot\vec{B}),\ C_2(\vec{A}\cdot\vec{B}),\ C_3(\vec{A}\cdot\vec{B}) \big)$$

$$= (\vec{A}\cdot\vec{C}) \underbrace{(B_1, B_2, B_3)}_{\vec{B}} - (\vec{A}\cdot\vec{B}) \underbrace{(C_1, C_2, C_3)}_{\vec{C}}$$

$$\vec{B} \times (\vec{C} \times \vec{A}) \neq [illegible]$$

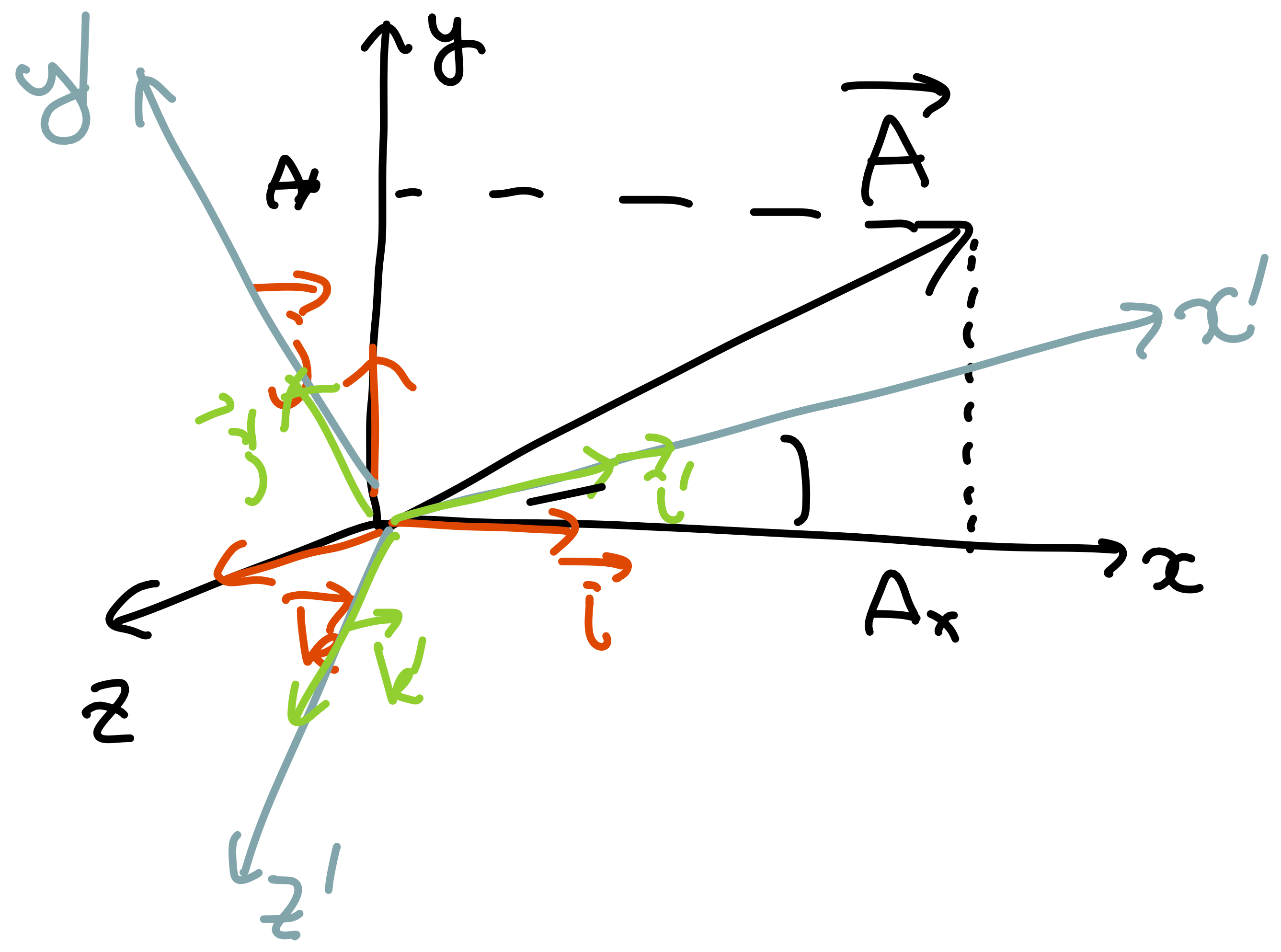
$$\therefore \boxed{\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\,(\vec{A}\cdot\vec{C}) - \vec{C}\,(\vec{A}\cdot\vec{B})}$$

$$(\vec{A} \times \vec{B})_i = \underbrace{\epsilon_{ijk}} A_j B_k$$

$$\epsilon_{ijk} = \begin{cases} \text{하나라도 같으면 } 0 \\ \epsilon_{123} = 1 \end{cases}$$
$$\epsilon_{213} = -1$$
$$\epsilon_{231} = 1$$

$$(\vec{A} \times \vec{B})_1 = A_2B_3 - A_3B_2$$
$$(\vec{A} \times \vec{B})_2 = A_3B_1 - A_1B_3$$
$$(\vec{A} \times \vec{B})_3 = A_1B_2 - A_2B_1$$

1.8. 좌표 변환



$$\vec{A}\cdot\vec{i}' = A_{x'} = A_{x'}(\vec{i}\cdot\vec{i}') + A_{y'}(\vec{j}\cdot\vec{i}') + A_{z'}(\vec{k}\cdot\vec{i}')$$

$$\vec{A}\cdot\vec{j}' = A_{y'} = A_{x'}(\vec{i}\cdot\vec{j}') + A_{y'}(\vec{j}\cdot\vec{j}') + A_{z'}(\vec{k}\cdot\vec{j}')$$

$$\vec{A}\cdot\vec{k}' = A_{z'} = A_{x'}(\vec{i}\cdot\vec{k}') + A_{y'}(\vec{j}\cdot\vec{k}') + A_{z'}(\vec{k}\cdot\vec{k}')$$

$$\boxed{\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} \vec{i}\cdot\vec{i}' & \vec{j}\cdot\vec{i}' & \vec{k}\cdot\vec{i}' \\ \vec{i}\cdot\vec{j}' & \vec{j}\cdot\vec{j}' & \vec{k}\cdot\vec{j}' \\ \vec{i}\cdot\vec{k}' & \vec{j}\cdot\vec{k}' & \vec{k}\cdot\vec{k}' \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}}$$

$\vec{A} \rightarrow$ 축 $\rightarrow$ 좌표

$$\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k} = (A_x, A_y, A_z)$$

$\updownarrow$ $(\vec{i}\ \vec{j}\ \vec{k}) \nearrow$

$$= A_{x'}\vec{i}' + A_{y'}\vec{j}' + A_{z'}\vec{k}' = (A_{x'}, A_{y'}, A_{z'})$$

$\vec{i}'\ \vec{j}'\ \vec{k}' \nearrow$

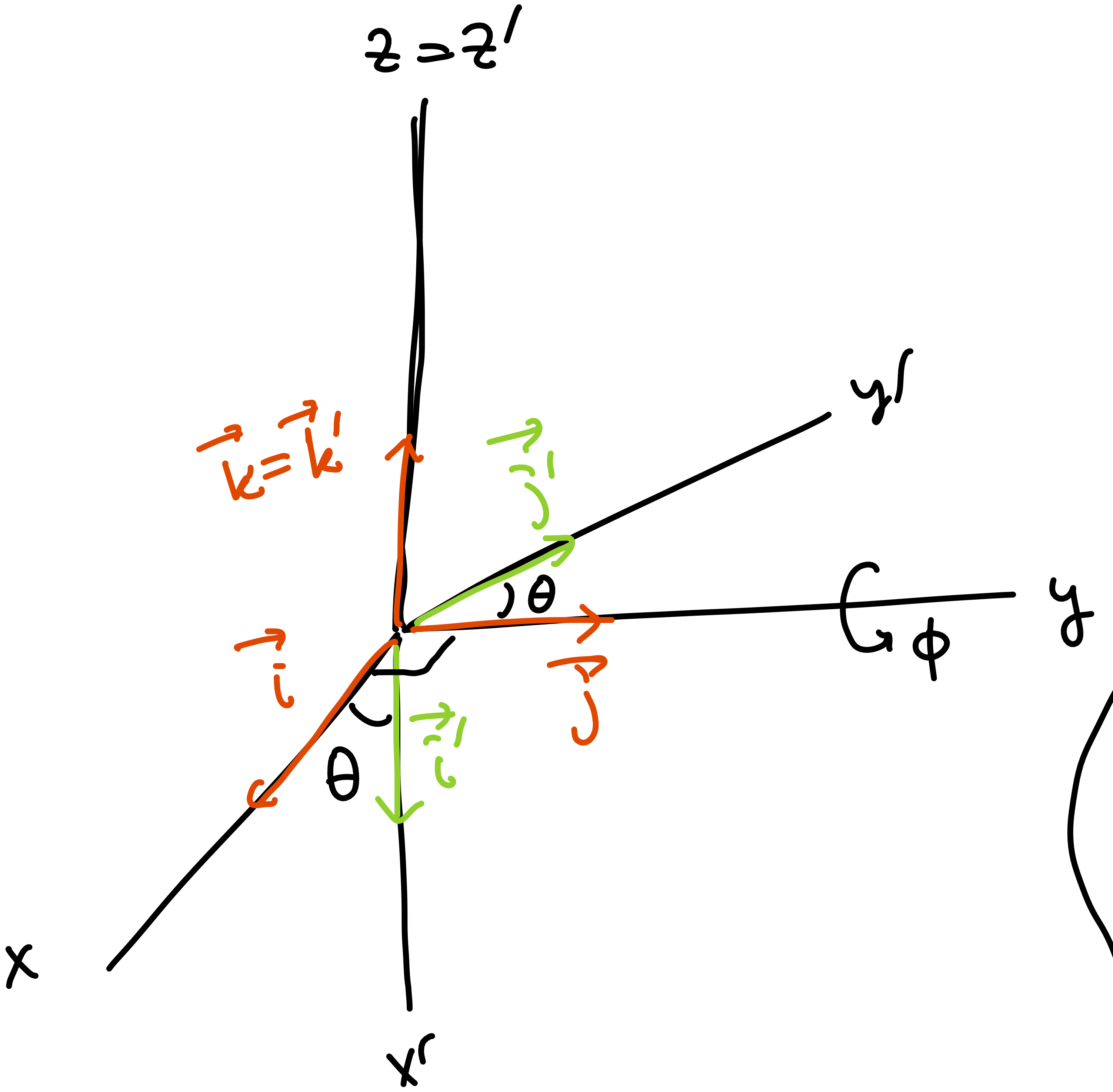
$$\vec{A}\cdot\vec{i} = A_x = A_{x'}(\vec{i}'\cdot\vec{i}) + A_{y'}(\vec{j}'\cdot\vec{i}) + A_{z'}(\vec{k}'\cdot\vec{i})$$

$$\vec{A}\cdot\vec{j} = A_y = A_{x'}(\vec{i}'\cdot\vec{j}) + A_{y'}(\vec{j}'\cdot\vec{j}) + A_{z'}(\vec{k}'\cdot\vec{j})$$

$$\vec{A}\cdot\vec{k} = A_z = A_{x'}(\vec{i}'\cdot\vec{k}) + A_{y'}(\vec{j}'\cdot\vec{k}) + A_{z'}(\vec{k}'\cdot\vec{k})$$

$$\boxed{\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \vec{i}'\cdot\vec{i} & \vec{j}'\cdot\vec{i} & \vec{k}'\cdot\vec{i} \\ \vec{i}'\cdot\vec{j} & \vec{j}'\cdot\vec{j} & \vec{k}'\cdot\vec{j} \\ \vec{i}'\cdot\vec{k} & \vec{j}'\cdot\vec{k} & \vec{k}'\cdot\vec{k} \end{pmatrix} \begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix}}$$

(ex)



$$\underbrace{\begin{pmatrix}A_x\\A_y\\A_z\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}A_{x'}\\A_{y'}\\A_{z'}\end{pmatrix}}$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\Rightarrow \boxed{\begin{aligned}A_x &= \cos\theta\, A_{x'} - \sin\theta\, A_{y'}\\ A_y &= \sin\theta\, A_{x'} + \cos\theta\, A_{y'}\\ A_{z'} &= A_z\end{aligned}}$$

$$\underbrace{\begin{pmatrix}A_{x'}\\A_{y'}\\A_{z'}\end{pmatrix}} = \begin{pmatrix}\vec{i}\cdot\vec{i}' & \vec{j}\cdot\vec{i}' & \vec{k}\cdot\vec{i}'\\ \vec{i}\cdot\vec{j}' & \vec{j}\cdot\vec{j}' & \vec{k}\cdot\vec{j}'\\ \vec{i}\cdot\vec{k}' & \vec{j}\cdot\vec{k}' & \vec{k}\cdot\vec{k}'\end{pmatrix}\begin{pmatrix}A_x\\A_y\\A_z\end{pmatrix} = \underbrace{\begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}}\underbrace{\begin{pmatrix}A_x\\A_y\\A_z\end{pmatrix}}$$

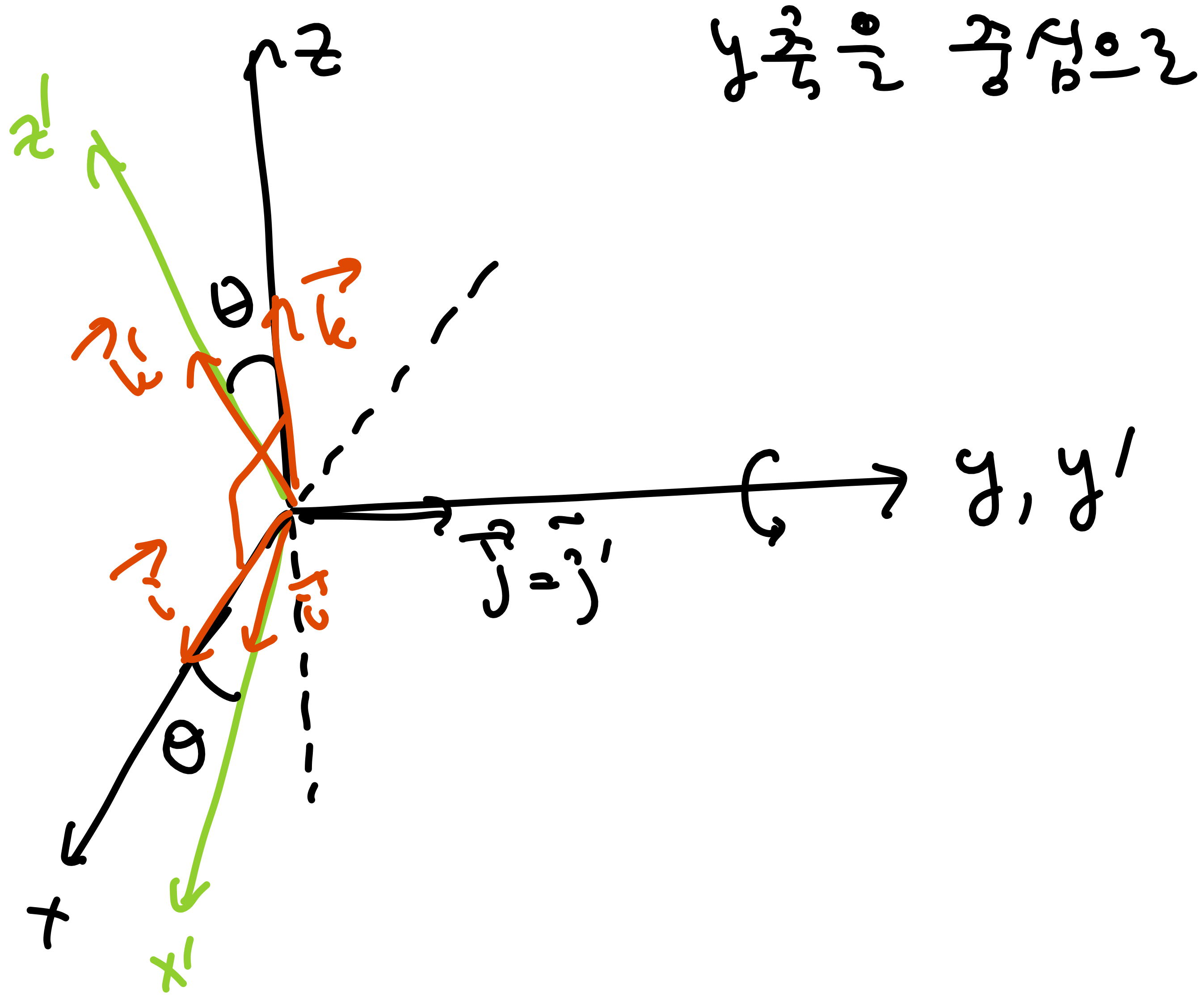
$$A_{x'} = \cos\theta\, A_x + \sin\theta\, A_y$$

$$A_{y'} = -\sin\theta\, A_x + \cos\theta\, A_y$$

$$A_{z'} = A_z$$

$$\begin{pmatrix}A_x\\A_y\\A_z\end{pmatrix} = \begin{pmatrix}\vec{i}'\cdot\vec{i} & \vec{j}'\cdot\vec{i} & \vec{k}'\cdot\vec{i}\\ \vec{i}'\cdot\vec{j} & \vec{j}'\cdot\vec{j} & \vec{k}'\cdot\vec{j}\\ \vec{i}'\cdot\vec{k} & \vec{j}'\cdot\vec{k} & \vec{k}'\cdot\vec{k}\end{pmatrix}\begin{pmatrix}A_{x'}\\A_{y'}\\A_{z'}\end{pmatrix}$$

$$= \underbrace{\begin{pmatrix}C & S & 0\\ -S & C & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}C & -S & 0\\ S & C & 0\\ 0 & 0 & 1\end{pmatrix}}\underbrace{\begin{pmatrix}A_{x'}\\A_{y'}\\A_{z'}\end{pmatrix}} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}A_{x'}\\A_{y'}\\A_{z'}\end{pmatrix}$$



y축을 중심으로 θ 회전.

$\vec{i}\cdot\vec{i}' = \cos\theta$

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} \vec{i}\cdot\vec{i}' & \vec{j}\cdot\vec{i}' & \vec{k}\cdot\vec{i}' \\ \vec{i}\cdot\vec{j}' & \vec{j}\cdot\vec{j}' & \vec{k}\cdot\vec{j}' \\ \vec{i}\cdot\vec{k}' & \vec{j}\cdot\vec{k}' & \vec{k}\cdot\vec{k}' \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

---

y축 θ → z축 φ 이면?

$$\begin{pmatrix} A_{x''} \\ A_{y''} \\ A_{z''} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi\cos\theta & \sin\phi & -\cos\phi\sin\theta \\ -\sin\phi\cos\theta & \cos\phi & \sin\phi\sin\theta \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

z축 φ → y축 θ

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

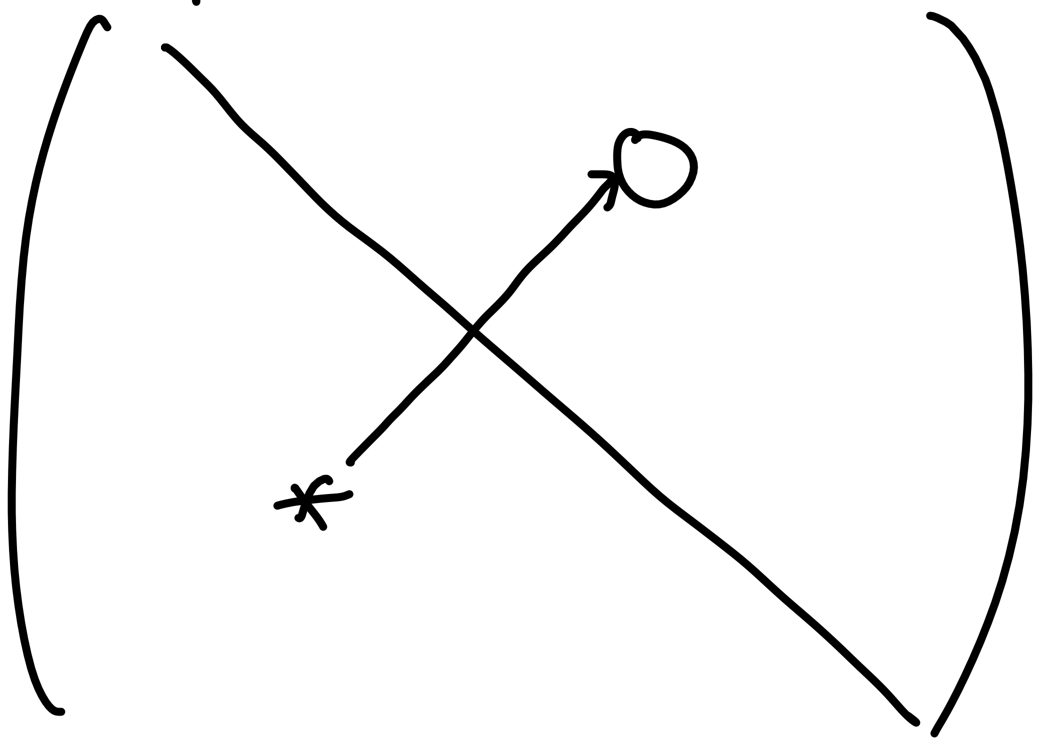
$\neq$

$$\begin{pmatrix}A_{x'}\\A_{y'}\\A_{z'}\end{pmatrix} = \underbrace{\begin{pmatrix} \vec{i}\cdot\vec{i}' & \vec{j}\cdot\vec{i}' & \vec{k}\cdot\vec{i}' \\ \vec{i}\cdot\vec{j}' & \vec{j}\cdot\vec{j}' & \vec{k}\cdot\vec{j}' \\ \vec{i}\cdot\vec{k}' & \vec{j}\cdot\vec{k}' & \vec{k}\cdot\vec{k}' \end{pmatrix}}_{\substack{\text{orthogonal matrix } M \\ (\text{직교, 수직})}} \cdot \underbrace{\begin{pmatrix} \vec{i}'\cdot\vec{i} & \vec{j}'\cdot\vec{i} & \vec{k}'\cdot\vec{i} \\ \vec{i}'\cdot\vec{j} & \vec{j}'\cdot\vec{j} & \vec{k}'\cdot\vec{j} \\ \vec{i}'\cdot\vec{k} & \vec{j}'\cdot\vec{k} & \vec{k}'\cdot\vec{k} \end{pmatrix}\begin{pmatrix}A_{x'}\\A_{y'}\\A_{z'}\end{pmatrix}}_{\begin{pmatrix}A_x\\A_y\\A_z\end{pmatrix}}$$

$$\Rightarrow \overset{\substack{\text{orthogonal matrix} \\ \uparrow}}{\boxed{MM^T = \mathbb{1}}}$$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

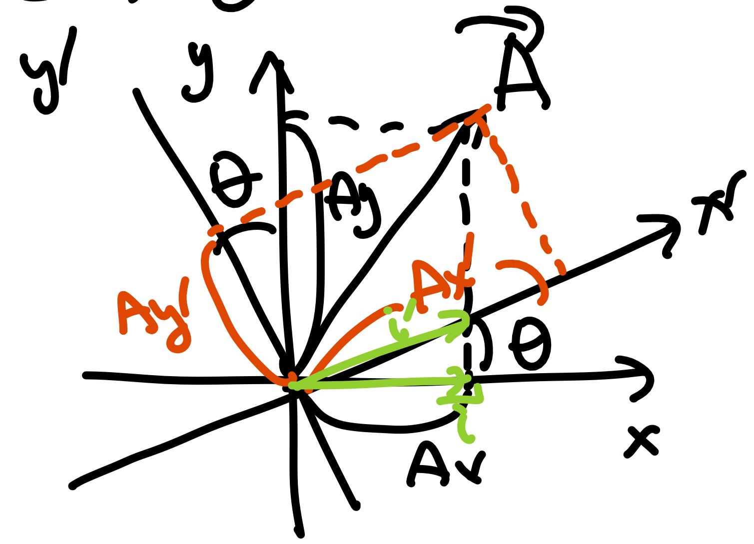
Transpose



$$\Rightarrow M^T$$

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}^T = \begin{pmatrix}a&c\\b&d\end{pmatrix}$$

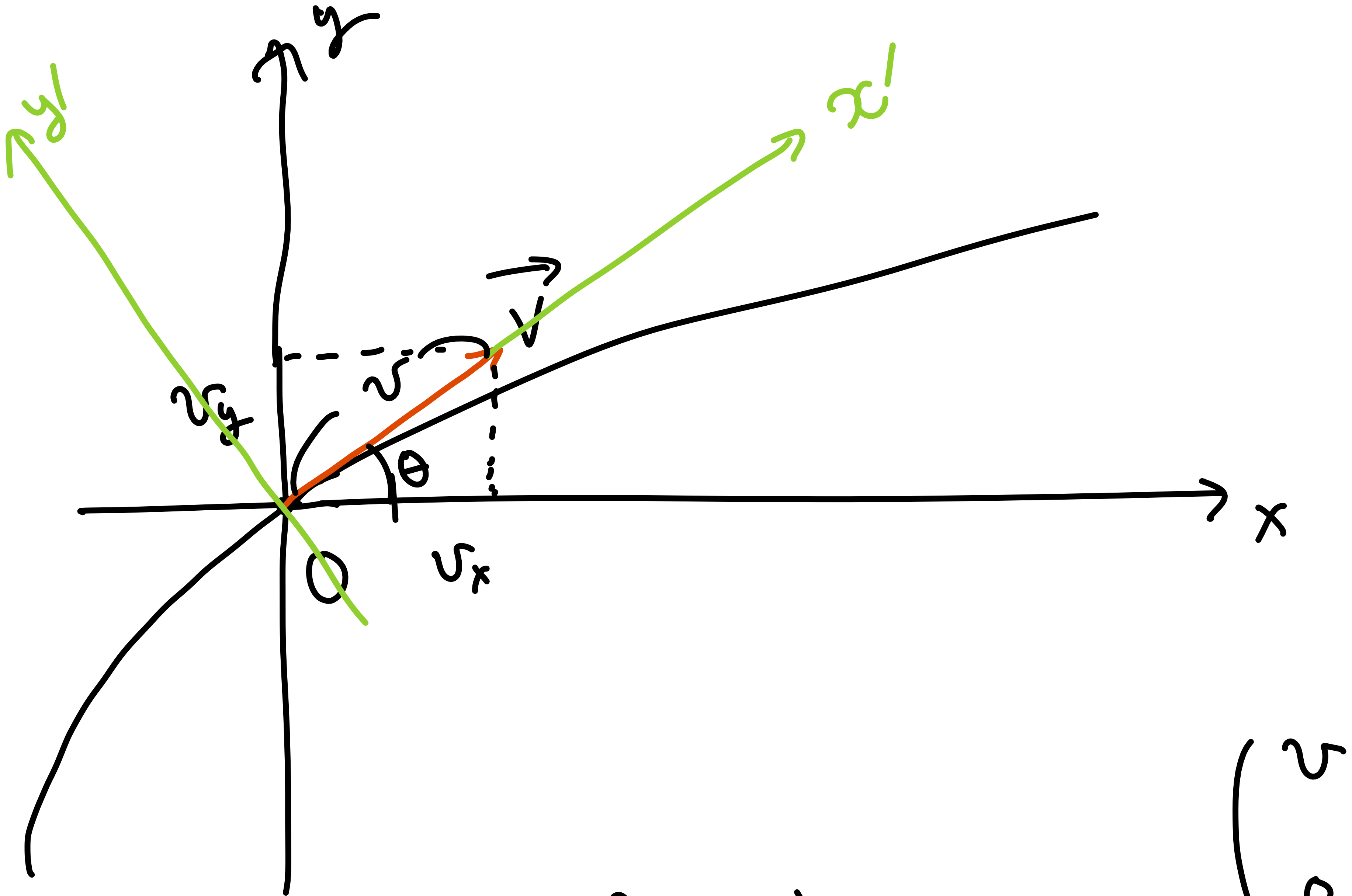
2차원 회전



$$\begin{pmatrix}A_x\\A_y\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}A_{x'}\\A_{y'}\end{pmatrix}$$

$$\begin{pmatrix}A_{x'}\\A_{y'}\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}A_x\\A_y\end{pmatrix}$$

[Ex 1.8.3]



$$\vec{V} = \begin{pmatrix} V_x \\ v_y \end{pmatrix}_{x-y}$$

$$\vec{v} = \begin{pmatrix} v \\ 0 \end{pmatrix}_{x'-y'}$$

$$\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A_{x'} \\ A_{y'} \end{pmatrix}$$

$$\begin{pmatrix} A_{x'} \\ A_{y'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

$$\begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

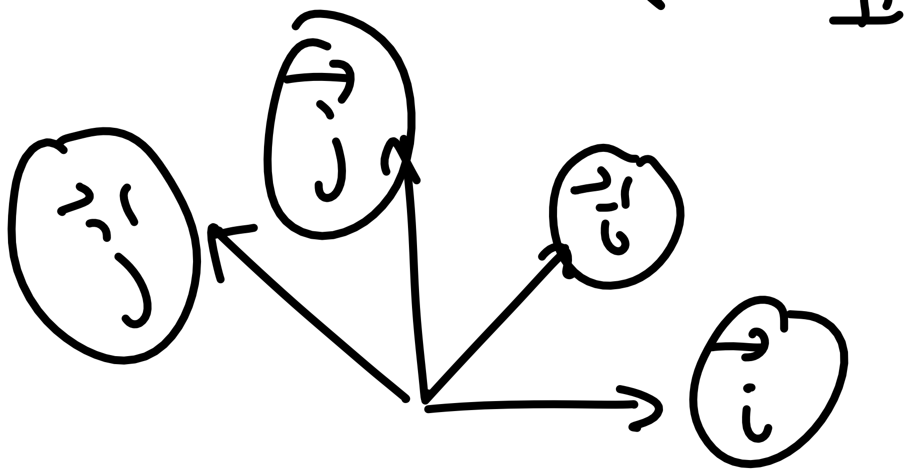
$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v \\ 0 \end{pmatrix}$$

$$v_x = v\cos\theta \ , \quad v_y = v\sin\theta$$

1.9. Vector 의 미분

$$\vec{A}(u) = \underline{\underline{\vec{i}}}\, A_x(u) + \underline{\underline{\vec{j}}}\, A_y(u) + \underline{\underline{\vec{k}}}\, A_z(u) = (A_x(u), A_y(u), A_z(u))$$

고정, 불변



$$\frac{d\vec{A}}{du} = \vec{i}\, A_x' + \vec{j}\, A_y' + \vec{k}\, A_z'$$

$$= (A_x', A_y', A_z')$$

$$\curvearrowleft \frac{dA_z}{du}$$

$$\lim_{\varepsilon \to 0} \frac{A_z(u+\varepsilon) - A_z(u)}{\varepsilon} \equiv A_z'(u)$$

$$\uparrow$$

$$\lim_{\varepsilon \to 0} \frac{\vec{A}(u+\varepsilon) - \vec{A}(u)}{\varepsilon} \equiv \frac{d\vec{A}}{du}(u)$$

$$\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{df}{dx} + \frac{dg}{dx}$$

$$\frac{d}{dx}\left(f(x)\, g(x)\right) = f'g + fg'$$

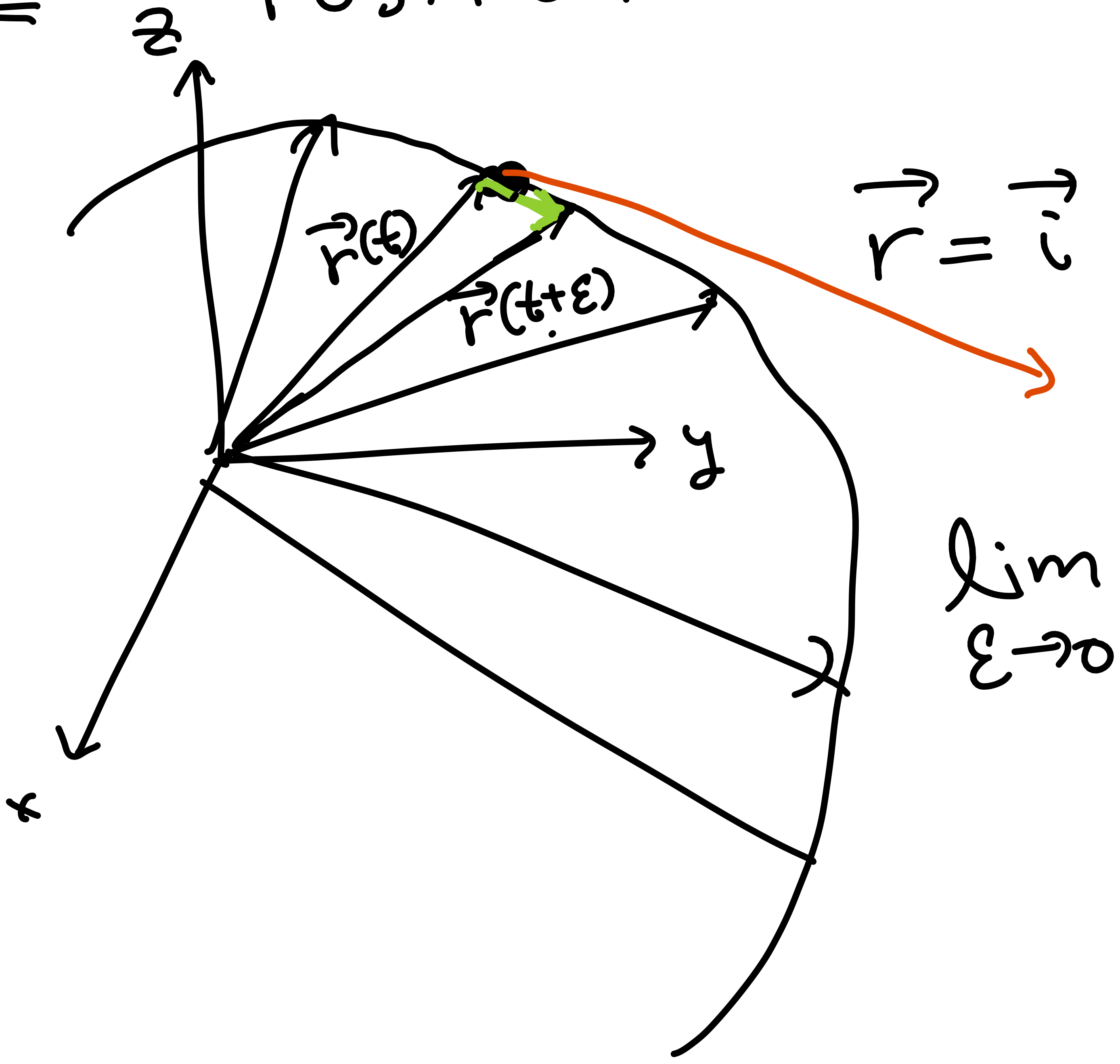
$$\frac{d}{du}\left(\vec{A}(u) + \vec{B}(u)\right) = \frac{d\vec{A}}{du} + \frac{d\vec{B}}{du}$$

$$\frac{d}{du}\left(n(u)\, \vec{A}(u)\right) = \frac{dn}{du}\vec{A} + n\frac{d\vec{A}}{du}$$

$$\frac{d}{du}(\vec{A} \cdot \vec{B}) = \frac{d\vec{A}}{du} \cdot \vec{B} + \vec{A} \cdot \frac{d\vec{B}}{du}$$

$$\frac{d}{du}(\vec{A} \times \vec{B}) = \frac{d\vec{A}}{du} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{du}$$

1.10. Position (위치) 벡터



$$\vec{r} = \vec{i}\,x + \vec{j}\,y + \vec{k}\,z$$

$$\lim_{\varepsilon \to 0} \frac{\vec{r}(t+\varepsilon) - \vec{r}(t)}{\varepsilon} = \frac{d\vec{r}}{dt} = \vec{v}$$

velocity vector

$$= \vec{i}\,\dot{x} + \vec{j}\,\dot{y} + \vec{k}\,\dot{z}$$

$$\dot{x} \equiv \frac{dx}{dt}$$

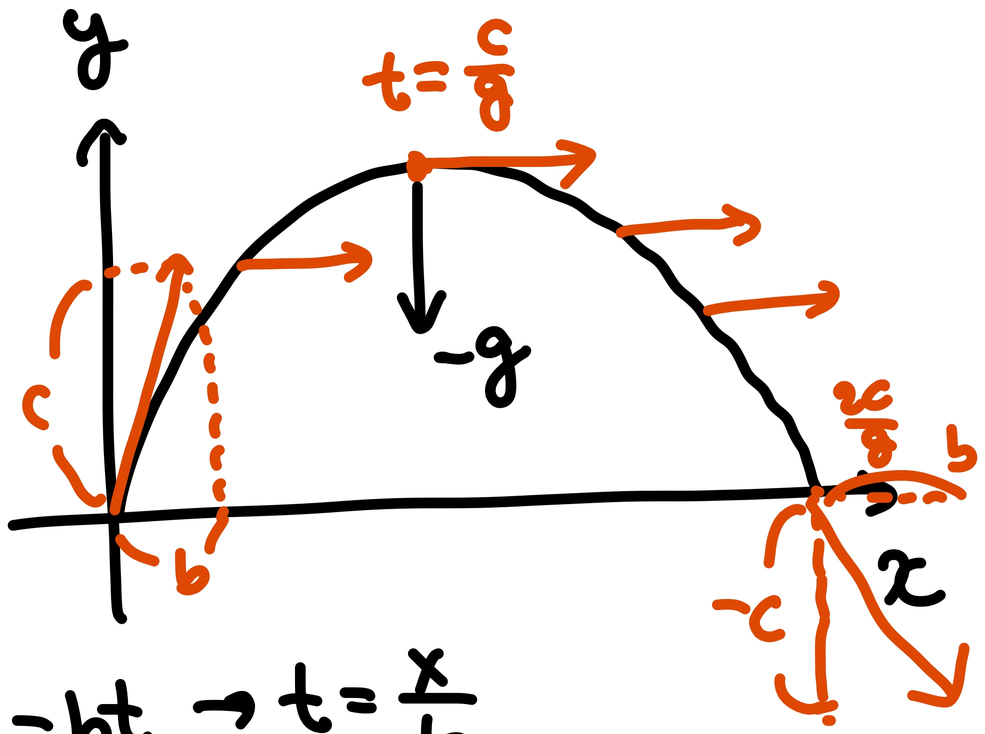
$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d^2\vec{r}}{dt^2} = \vec{i}\,\ddot{x} + \vec{j}\,\ddot{y} + \vec{k}\,\ddot{z}$$

[ex 1.10.1]

$$\vec{r} = \vec{i}\,(\overset{x}{bt}) + \vec{j}\left(\overbrace{ct - \frac{gt^2}{2}}^{y}\right) + \vec{k}\cdot 0$$

$$\vec{V} = \frac{d\vec{r}}{dt} = \vec{i}\,\underset{\overset{\|}{v_x}}{b} + \vec{j}\,(\underbrace{c - gt}_{v_y})$$

$$\vec{a} = -g\,\vec{j}$$



$$x = bt \rightarrow t = \frac{x}{b}$$

$$y = c\,\frac{x}{b} - \frac{g}{2}\left(\frac{x}{b}\right)^2$$

$$|\vec{v}| = v = \sqrt{v_x^2 + v_y^2} = \sqrt{b^2 + (c-gt)^2}$$

[Ex 1.10.2] 원운동 (평면 x-y)

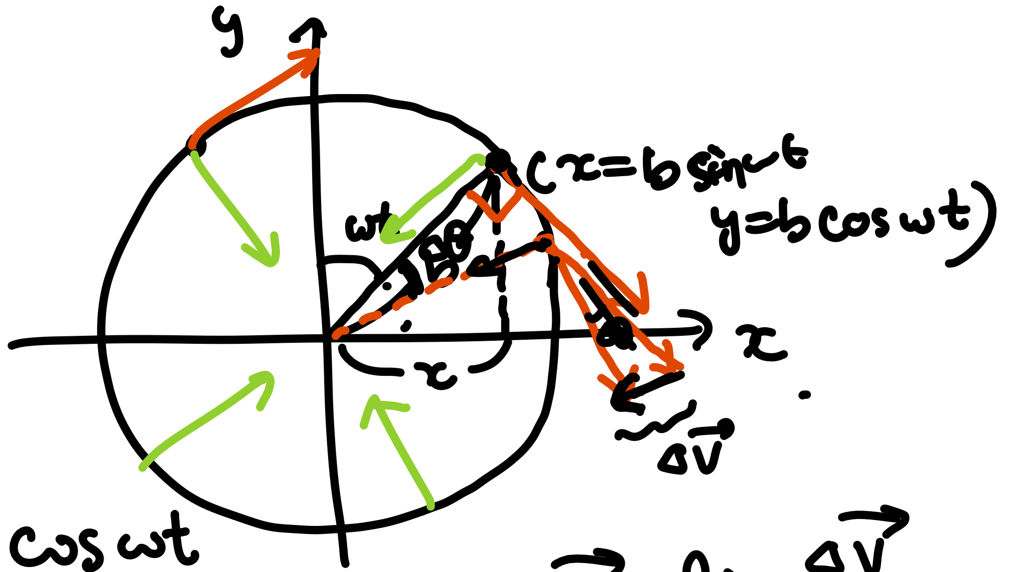
$$\vec{r} = \vec{i}\, b\sin\omega t + \vec{j}\, b\cos\omega t$$

$$|\vec{r}| = b, \quad \vec{v} = \vec{i}\,\omega b\cos\omega t - \vec{j}\,\omega b\sin\omega t$$

$$\vec{r}\cdot\vec{v} = 0 \longrightarrow \vec{r} \perp \vec{v}$$

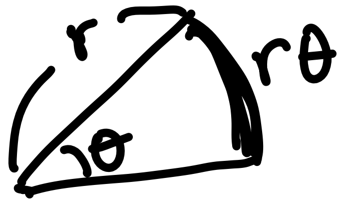
$$|\vec{v}| = \omega b, \quad \vec{a} = -\vec{i}\,\omega^2 b\sin\omega t - \vec{j}\,\omega^2 b\cos\omega t$$

$$= -\omega^2 \vec{r}$$



구심가속도

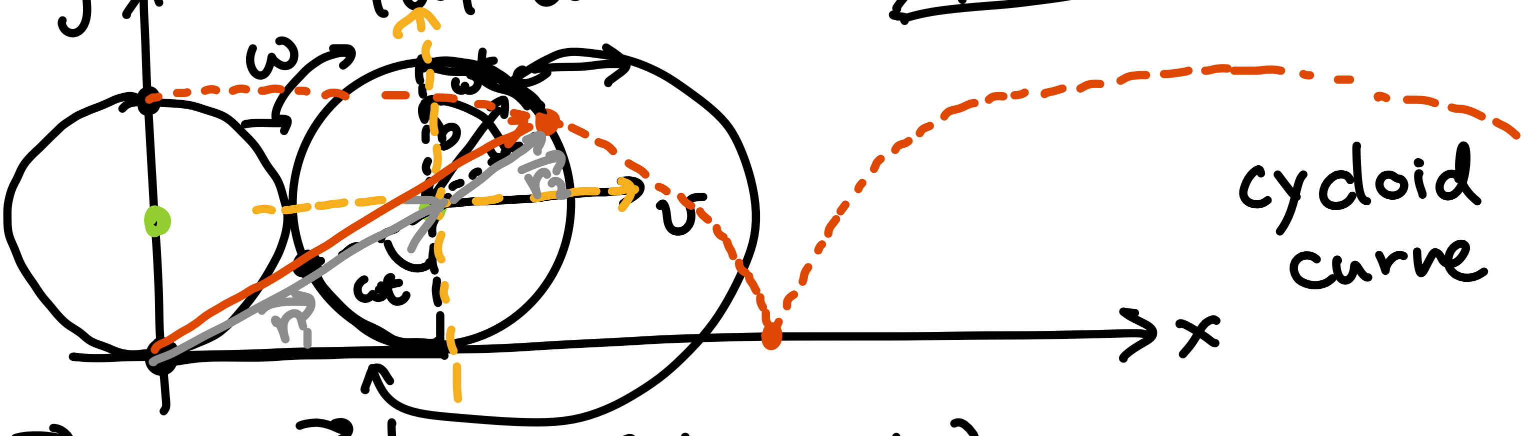
$$\vec{a} = \lim_{\Delta t\to 0}\frac{\Delta\vec{v}}{\Delta t}$$



$$\omega\,\Delta t = \Delta\theta$$

$$\rightarrow |\vec{a}| = a = \omega^2 b$$

[Ex. 1.10.3]



$$a = \frac{\Delta v}{\Delta t} = \frac{\Delta\theta\cdot\omega b}{\Delta t}$$

$$= \omega b\frac{\Delta\theta}{\Delta t} = \omega^2 b \,//$$

$$\vec{r}_1 = \vec{i}\, b\omega t + \vec{j}\, b = (b\omega t, b)$$

↑ 바퀴중심

$$\vec{v}_1 = \vec{i}(b\omega)$$

$$\vec{r} = \vec{r}_1 + \vec{r}_2$$

$$= (b(\omega t + \sin\omega t), b(1+\cos\omega t))$$

$$\vec{r}_2 = (b\sin\omega t, b\cos\omega t)$$

접점이 바퀴중심

$$\vec{v}_2 = (b\omega\cos\omega t, -b\omega\sin\omega t)$$

$$\omega t = 2\pi \overset{\swarrow \text{정수}}{n} + \pi$$

$$\vec{v} = 0$$

$$\vec{v} = (b\omega(1+\cos\omega t), -b\omega\sin\omega t)$$

1.11. Polar coordinate (극좌표)

3차원 x-y-z 좌표계

구면 spherical 좌표계

원통 Cylindrical 〃

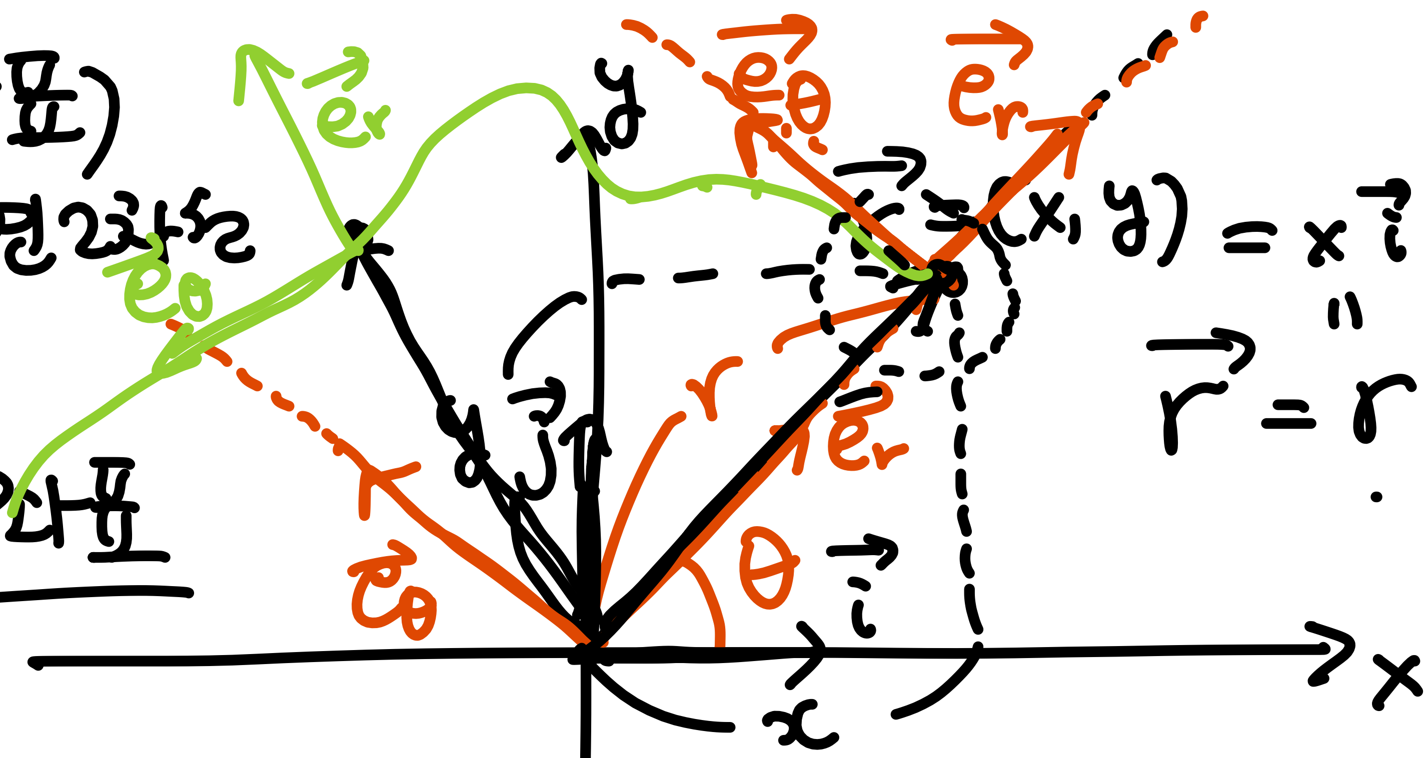
(x, y) 평면 2차원

↓

(r, θ) 극좌표

장점. 간단. (r 만)

단점. 축의 방향이 위치마다 변함

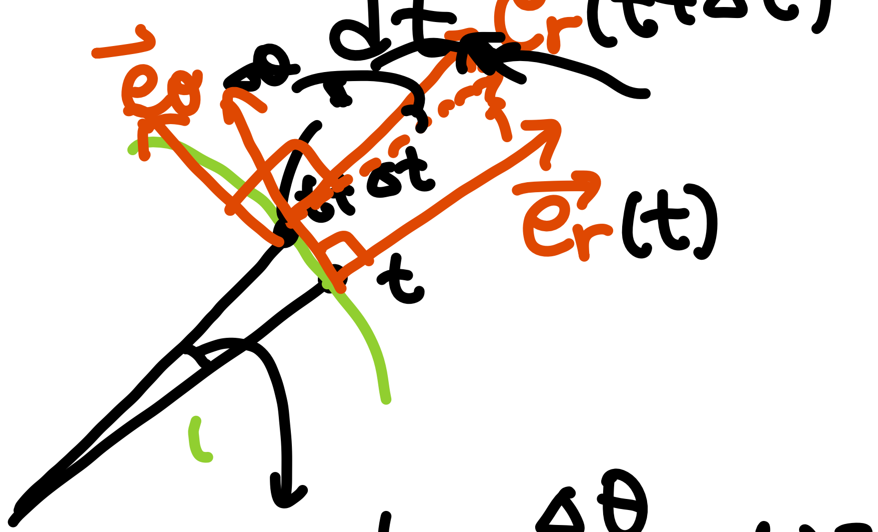


$\vec r = (x, y) = x\vec i + y\vec j$

$$\vec r \overset{\shortparallel}{=} r\vec e_r$$

$$\vec v = \frac{d\vec r}{dt} = \dot x \underline{\vec i} + \dot y \underline{\vec j} = \frac{d}{dt}(\underset{\uparrow\, t}{r}\,\vec e_r) = \dot r \vec e_r + r\frac{d\vec e_r}{dt}$$

$$\lim_{\Delta t \to 0}\frac{\vec e_r(t+\Delta t) - \vec e_r(t)}{\Delta t} = \vec e_\theta \frac{\Delta\theta}{\Delta t} = \dot\theta$$



$\omega\Delta t = \Delta\theta$ $\omega = \frac{\Delta\theta}{\Delta t} = \dot\theta$

$$\therefore \boxed{\frac{d\vec e_r}{dt} = \dot\theta \vec e_\theta, \quad \frac{d\vec e_\theta}{dt} = -\dot\theta \vec e_r \qquad \vec v = \underset{\sim\sim\sim}{\dot r \vec e_r} + r\dot\theta \vec e_\theta}$$

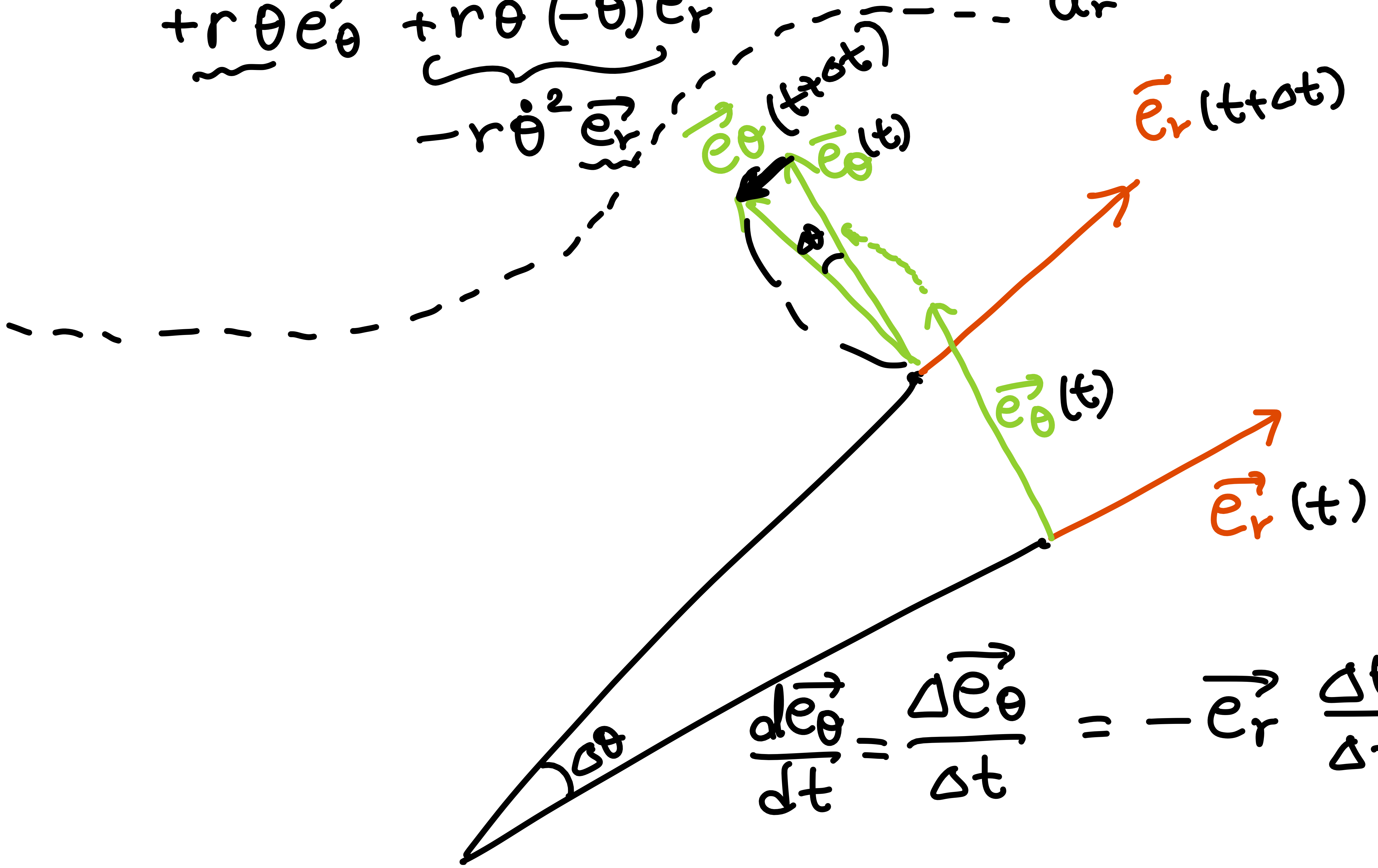
$$\left(\Rightarrow \text{if } \dot r = 0^{\,r=b} \quad \underline{\vec v = b\omega \vec e_\theta}\right);$$

$$\vec a = \frac{d\vec v}{dt} = \ddot r \vec e_r + 2\dot r\dot\theta \vec e_\theta + r\ddot\theta \vec e_\theta + r\dot\theta\frac{d\vec e_\theta}{dt}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{r}\vec{e}_r + 2\dot{r}\dot{\theta}\vec{e}_\theta + r\ddot{\theta}\vec{e}_\theta + \underbrace{r\dot{\theta}(-\dot{\theta})\vec{e}_r}_{}$$

$$-r\dot{\theta}^2\vec{e}_r$$

$$= \vec{e}_r(\underbrace{\ddot{r} - r\dot{\theta}^2}_{a_r}) + \vec{e}_\theta(\underbrace{r\ddot{\theta} + 2\dot{r}\dot{\theta}}_{a_\theta})$$



$$\frac{d\vec{e}_\theta}{dt} = \frac{\Delta\vec{e}_\theta}{\Delta t} = -\vec{e}_r\frac{\Delta\theta}{\Delta t} = -\vec{e}_r\dot{\theta}$$

$$a_r = \ddot{r} - r\dot{\theta}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

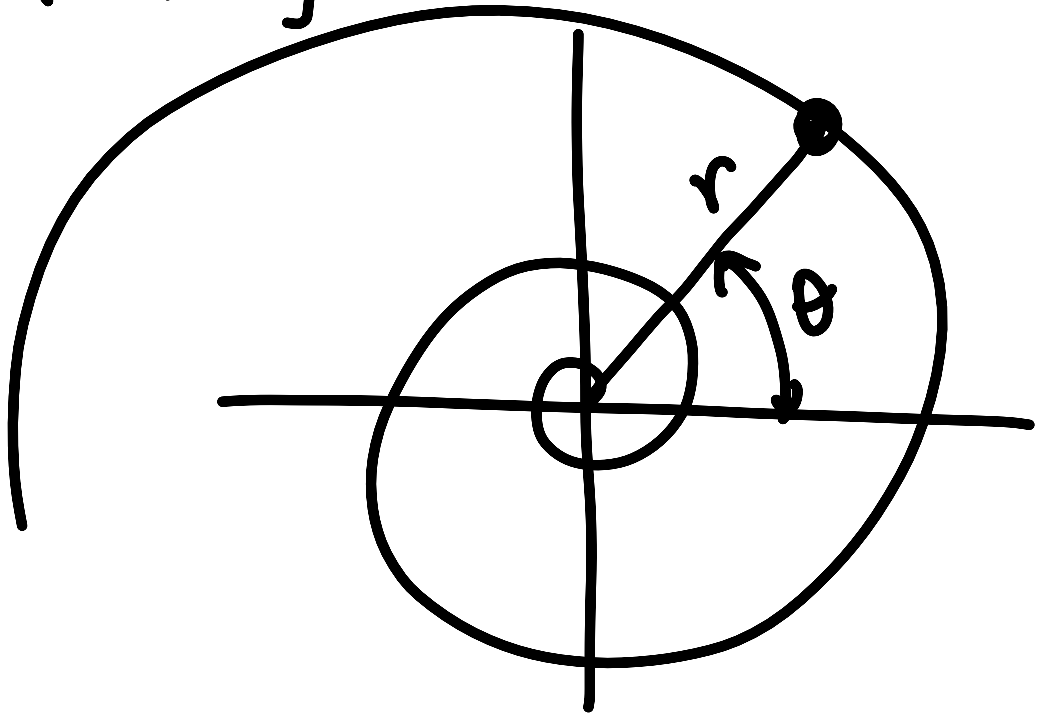
원운동. $\dot{r}=0$, $\ddot{r}=0$, $\dot{\theta}=\omega$

$r=b$

$a_r = -b\omega^2$, $a_\theta = 0$

$\vec{a} = -b\omega^2\vec{e}_r$

[Ex. 1.11.1]



$\dot{\theta} = kt$

$r = b - ct$

speed?

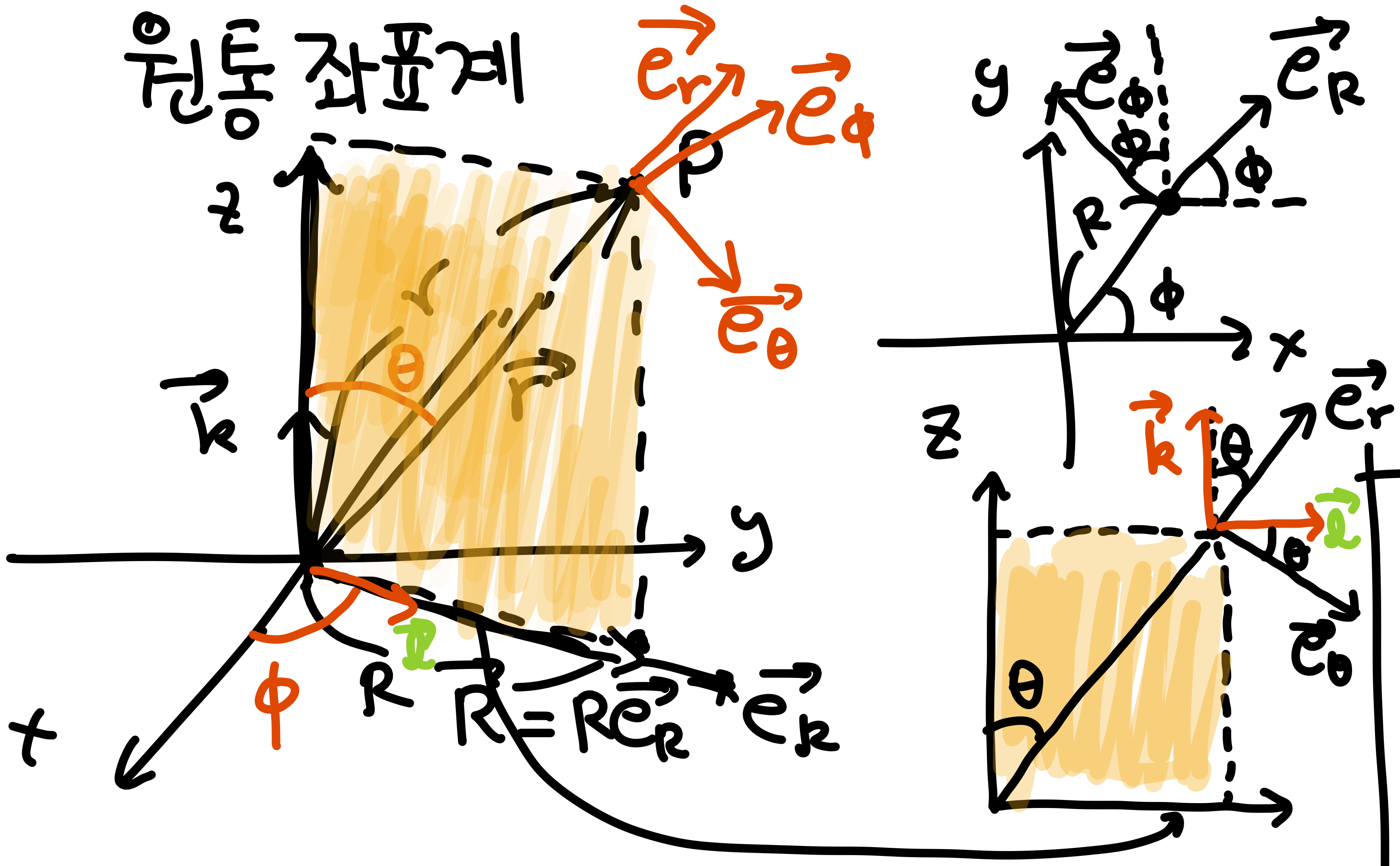
$$\vec{v} = \dot{r}\vec{e_r} + r\dot{\theta}\vec{e_\theta} = -c\vec{e_r} + \underbrace{\quad}_{(b-ct)} kt\,\vec{e_\theta}$$

$$v = \sqrt{c^2 + (b-ct)^2\,(kt)^2} \,,,$$

$$\vec{a} = \vec{e_r}\left(\underbrace{\ddot{r} - r\dot{\theta}^2}\right) + \vec{e_\theta}\left(\underbrace{r\ddot{\theta} + 2\dot{r}}\dot{\theta}\right)$$

$$= \vec{e_r}\left(-(b-ct)(kt)^2\right) + \vec{e_\theta}\left(k(b-ct) + (-2c)kt\right)$$

§ 1.12. 원통 좌표계



$(\underbrace{R,\phi}, z)$

$\vec e_z = \vec k$

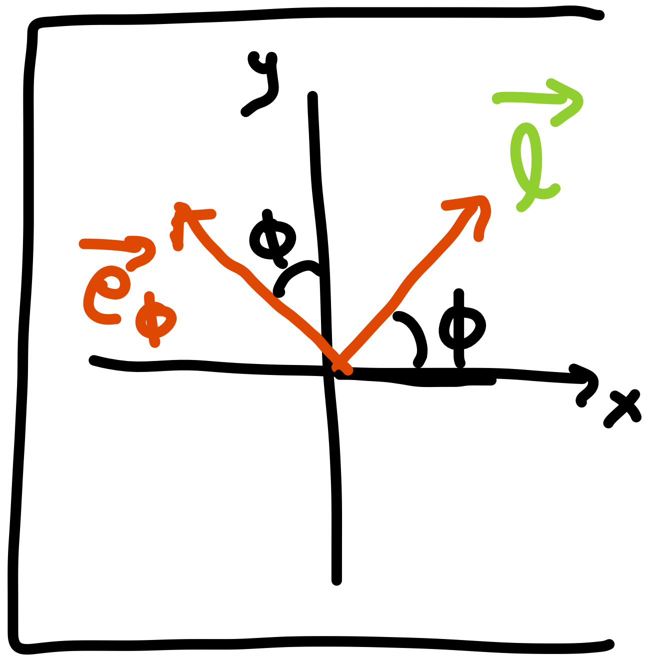
$$\boxed{\begin{aligned} \vec e_R &= \vec i\cos\phi + \vec j\sin\phi \\ \vec e_\phi &= -\vec i\sin\phi + \vec j\cos\phi \\ \vec e_z &= \vec k \end{aligned}}$$

$$\vec r = R\vec e_R + z\vec k$$

$$\vec v = \frac{d\vec r}{dt} = \dot R\vec e_R + R\dot\phi\vec e_\phi + \dot z\vec k$$

$$\vec a = \frac{d^2\vec r}{dt^2} = \frac{d\vec v}{dt} = (\ddot R\vec e_R + \dot R\dot\phi\vec e_\phi) + (\dot R\dot\phi\vec e_\phi + R\ddot\phi\vec e_\phi - R\dot\phi^2\vec e_R) + \ddot z\vec k$$

$$= (\ddot R - R\dot\phi^2)\vec e_R + (2\dot R\dot\phi + R\ddot\phi)\vec e_\phi + \ddot z\vec k$$



구면좌표계 (spherical coordinates) : (r, θ, ϕ)

$\vec r = r\vec e_r$ $\qquad \vec v = \dot r\vec e_r + r\dfrac{d\vec e_r}{dt}$

$\vec e_r = \sin\theta\,\vec l + \cos\theta\,\vec k$, $\quad \vec e_\theta = -\sin\theta\,\vec k + \cos\theta\,\vec l$

$\vec e_\phi = -\sin\phi\,\vec i + \cos\phi\,\vec j$ $\qquad \vec l = \cos\phi\,\vec i + \sin\phi\,\vec j$

$$\overrightarrow{e_\theta} = -\sin\theta\,\vec{k} + \cos\theta(\cos\phi\,\vec{i} + \sin\phi\,\vec{j})$$

$$\cos\theta\,\overrightarrow{e_\theta} + \sin\theta\,\overrightarrow{e_r} = \cos\phi\,\vec{i} + \sin\phi\,\vec{j}$$

$$\overrightarrow{e_\phi} = -\sin\phi\,\vec{i} + \cos\phi\,\vec{j} \leftarrow$$

$$\overrightarrow{e_r} = \sin\theta(\cos\phi\,\vec{i} + \sin\phi\,\vec{j}) + \cos\theta\,\vec{k}$$

$$\frac{d\overrightarrow{e_r}}{dt} = (\dot\theta\cos\theta\cos\phi - \dot\phi\sin\theta\sin\phi)\vec{i} + (\dot\theta\cos\theta\sin\phi + \dot\phi\sin\theta\cos\phi)\vec{j} - \dot\theta\sin\theta\,\vec{k}$$

$$= \dot\theta\,(\underbrace{\cos\theta\cos\phi\,\vec{i} + \cos\theta\sin\phi\,\vec{j} - \sin\theta\,\vec{k}}_{\overrightarrow{e_\theta}}) + \dot\phi\sin\theta(\underbrace{-\sin\phi\,\vec{i} + \cos\phi\,\vec{j}}_{\overrightarrow{e_\phi}})$$

$$\therefore \boxed{\frac{d\overrightarrow{e_r}}{dt} = \dot\theta\,\overrightarrow{e_\theta} + \dot\phi\sin\theta\,\overrightarrow{e_\phi}}$$

$$\frac{d\overrightarrow{e_\theta}}{dt} = (-\dot\theta\sin\theta\cos\phi - \dot\phi\cos\theta\sin\phi)\vec{i} + (-\dot\theta\sin\theta\sin\phi + \dot\phi\cos\theta\cos\phi)\vec{j} - \dot\theta\cos\theta\,\vec{k}$$

$$= -\dot\theta\,(\underbrace{\sin\theta\cos\phi\,\vec{i} + \sin\theta\sin\phi\,\vec{j} + \cos\theta\,\vec{k}}_{\overrightarrow{e_r}}) + \dot\phi\cos\theta(\underbrace{-\sin\phi\,\vec{i} + \cos\phi\,\vec{j}}_{\overrightarrow{e_\phi}})$$

$$\therefore \boxed{\frac{d\overrightarrow{e_\theta}}{dt} = -\dot\theta\,\overrightarrow{e_r} + \dot\phi\cos\theta\,\overrightarrow{e_\phi}}$$

$$\frac{d\overrightarrow{e_\phi}}{dt} = -\dot\phi\cos\phi\,\vec{i} - \dot\phi\sin\phi\,\vec{j}$$

$$= -\dot\phi\,(\underbrace{\cos\phi\,\vec{i} + \sin\phi\,\vec{j}}_{\cos\theta\overrightarrow{e_\theta} + \sin\theta\overrightarrow{e_r}}) = -\dot\phi(\cos\theta\,\overrightarrow{e_\theta} + \sin\theta\,\overrightarrow{e_r}) \;//$$

$$\vec{v} = \dot{r}\vec{e}_r + r\left(\dot{\theta}\vec{e}_\theta + \dot{\phi}\sin\theta\,\vec{e}_\phi\right) = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + r\dot{\phi}\sin\theta\,\vec{e}_\phi$$

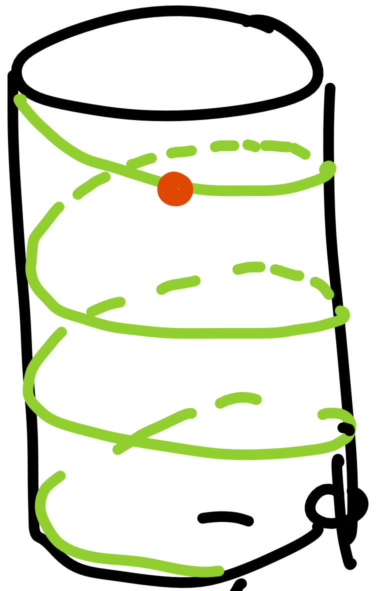
$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{r}\vec{e}_r + \dot{r}\underbrace{\dot{\vec{e}}_r}_{\dot{\theta}\vec{e}_\theta + \dot{\phi}\sin\theta\,\vec{e}_\phi} + \dot{r}\dot{\theta}\vec{e}_\theta + r\ddot{\theta}\vec{e}_\theta + r\dot{\theta}\underbrace{\dot{\vec{e}}_\theta}_{-\dot{\theta}\vec{e}_r + \dot{\phi}\cos\theta\,\vec{e}_\phi} +$$

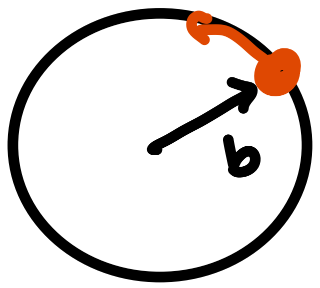
$$+\dot{r}\dot{\phi}\sin\theta\,\vec{e}_\phi + r\ddot{\phi}\sin\theta\,\vec{e}_\phi + r\dot{\phi}\dot{\theta}\cos\theta\,\vec{e}_\phi + r\dot{\phi}\sin\theta\underbrace{\dot{\vec{e}}_\phi}_{-\dot{\phi}\left(\cos\theta\,\vec{e}_\theta + \sin\theta\,\vec{e}_r\right)}$$

$$\frac{d}{dx}f(g(x)) = f'(g)\,g'(x)$$ "chain-rule"

$$\boxed{\begin{aligned}\vec{a} &= \vec{e}_r\left(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta\right) + \vec{e}_\theta\left(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta\right)\\ &\quad + \vec{e}_\phi\left(2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\ddot{\phi}\sin\theta\right)\end{aligned}}$$

[ex 1.12.1] 원통





$R = b$

$\phi = \omega t$

$z = ct$

$\dot{R} = 0 \quad \ddot{R} = 0$

$$\vec{v} = \dot{R}\vec{e}_R + R\dot{\phi}\vec{e}_\phi + \dot{z}\vec{k}$$

$$\vec{v} = b\omega\vec{e}_\phi + c\vec{k}$$

$$v = \sqrt{c^2 + b^2\omega^2}$$

$$\vec{a} = \left(\cancel{\ddot{R}} - R\dot{\phi}^2\right)\vec{e}_R + \left(2\cancel{\dot{R}}\dot{\phi} + R\cancel{\ddot{\phi}}\right)\vec{e}_\phi + \cancel{\ddot{z}}\vec{k}$$

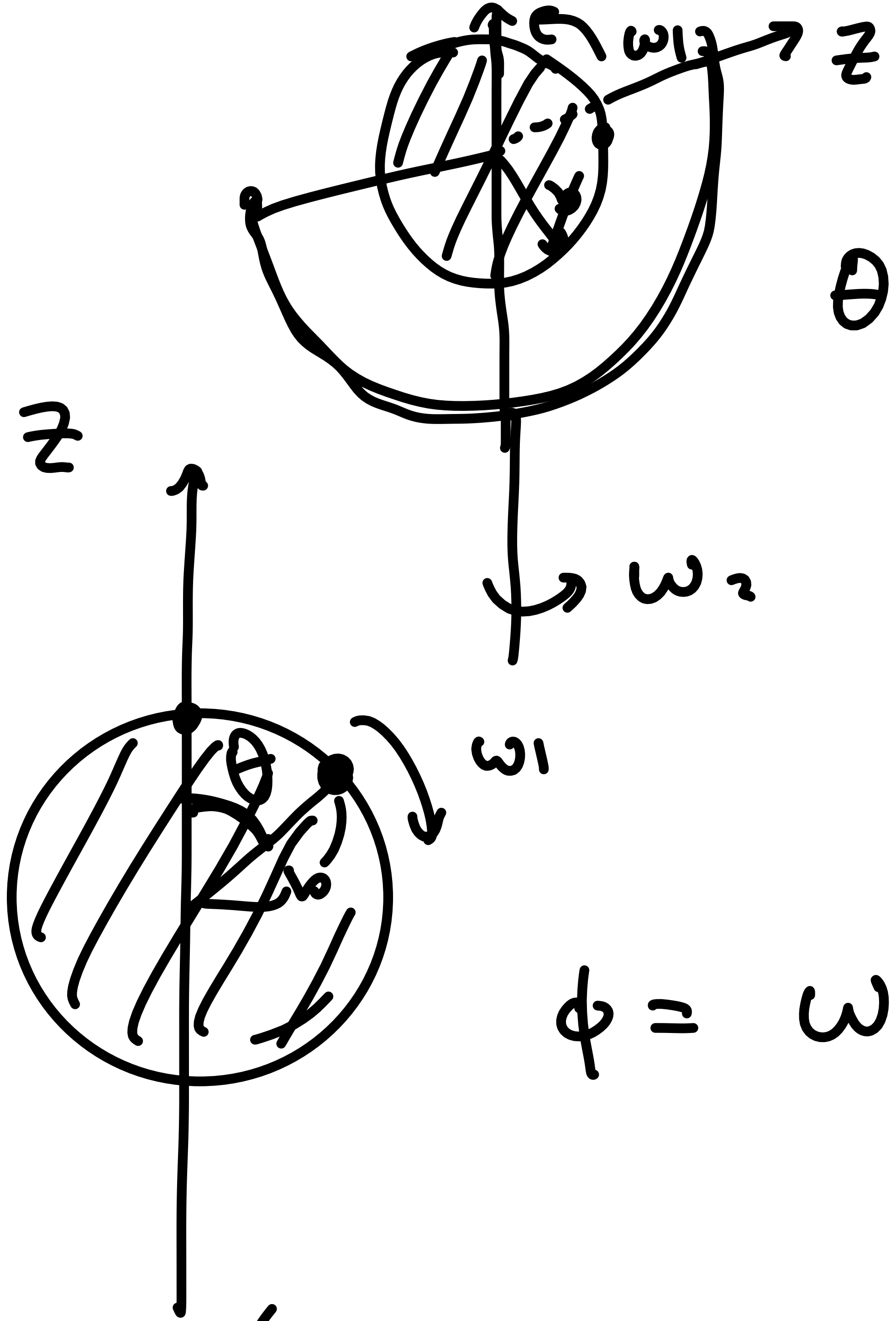
$$= -b\omega^2\vec{e}_R$$

$$\frac{d\vec{v}}{dt} = b\omega\dot{\vec{e}}_\phi$$

$$'' \quad -\dot{\phi}\vec{e}_R$$

''

[Ex. 1.12.2]



$\theta = \omega_1 t \longrightarrow \dot\theta = \omega_1 \quad \ddot\theta = 0$

$\dot\phi = \omega_2 \quad \ddot\phi = 0$

$\phi = \omega_2 t$

$$\vec{V} = \cancel{\dot r}\,\vec{e_r} + r\dot\theta\,\vec{e_\theta} + r\dot\phi\sin\theta\,\vec{e_\phi}$$

$$\vec{v} = b\omega_1\vec{e_\theta} + b\omega_2\sin(\omega_1 t)\vec{e_\phi}$$

$$\vec{a} = \vec{e_r}\left(\cancel{\ddot r} - r\dot\theta^2 - r\dot\phi^2\sin^2\theta\right) + \vec{e_\theta}\left(\cancel{2\dot r\dot\theta} + \cancel{r\ddot\theta} - r\dot\phi^2\sin\theta\cos\theta\right)$$

$$+\vec{e_\phi}\left(\cancel{2\dot r\dot\phi}\sin\theta + 2r\dot\theta\dot\phi\cos\theta + \cancel{r\ddot\phi}\sin\theta\right)$$

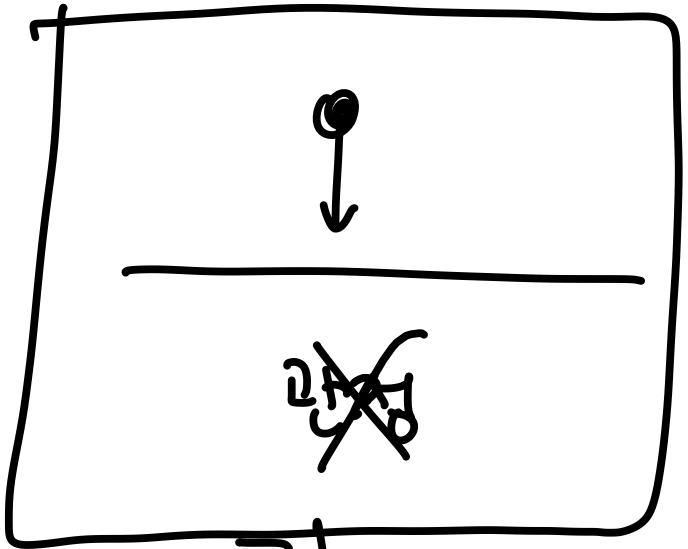
$\theta = 0$

$$= -b\omega_1^2\vec{e_r} + 2b\omega_1\omega_2\vec{e_\phi}$$

다음 주 화요일 (4/5) 휴강.
〃 목 〃 (4/7) 6:30~7:45 보강

2장. Newtonian Mechanics

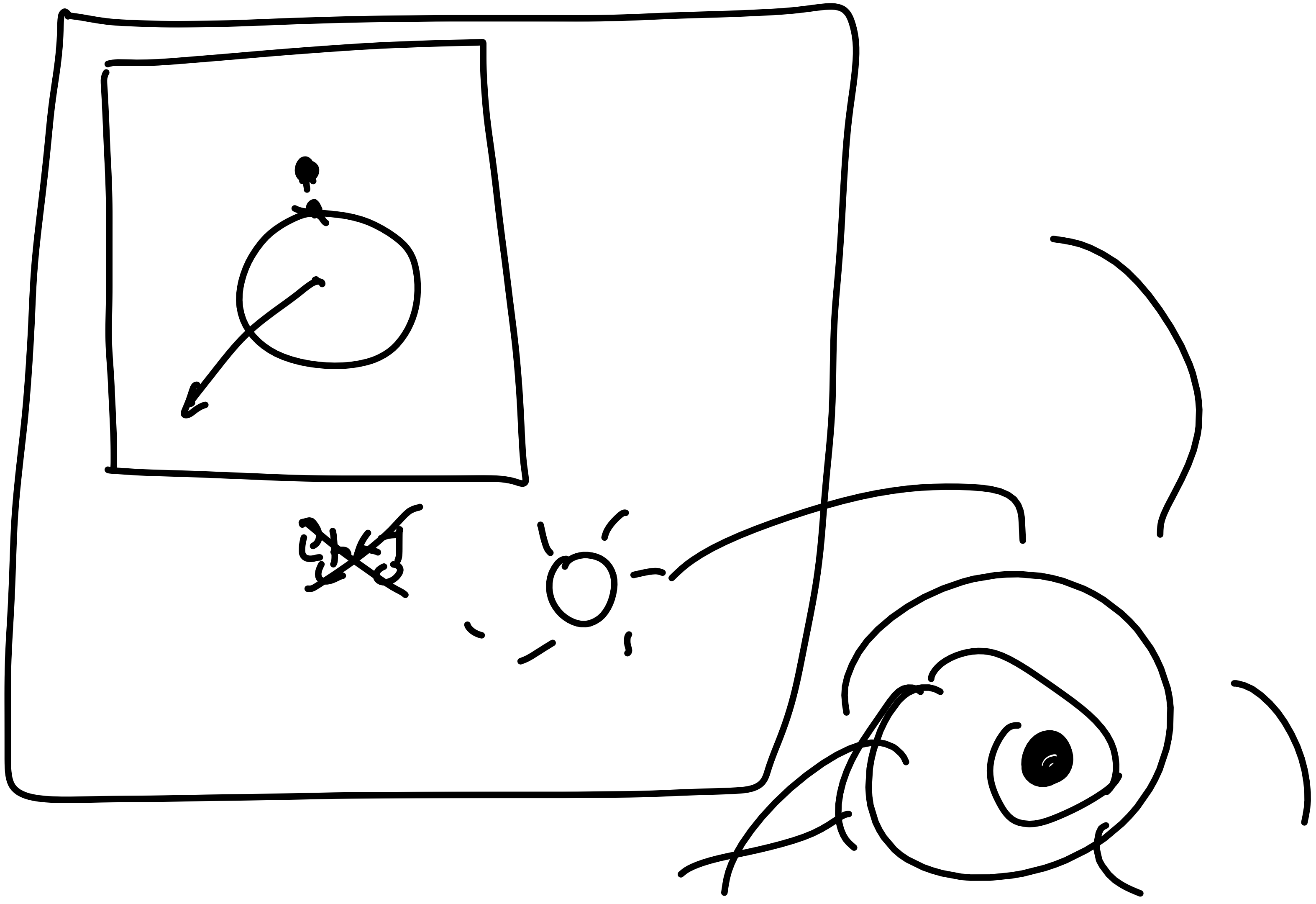
Inertial reference frame (관성좌표계)



$F \neq mg$

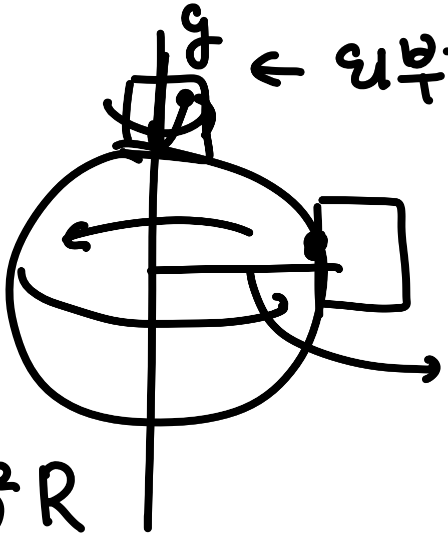
mg

10^{-100}



[Ex. 2.1.1]

(a)



지구자전 : $a = \omega^2 R$

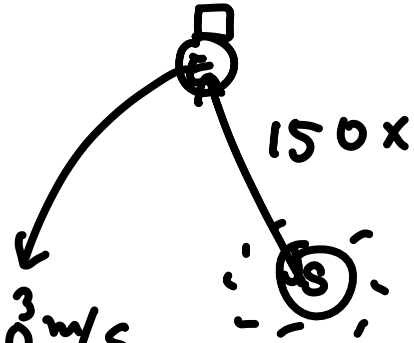
$= \left(\frac{2\pi}{T}\right)^2 6.4 \times 10^6$

$T = 1\text{day} = 24 \times 3600$

$\longrightarrow \frac{a}{\boxed{g}} = 3.4 \times 10^{-3} \ll 1$

g: $9.8 m/s^2$

(b) 지구공전

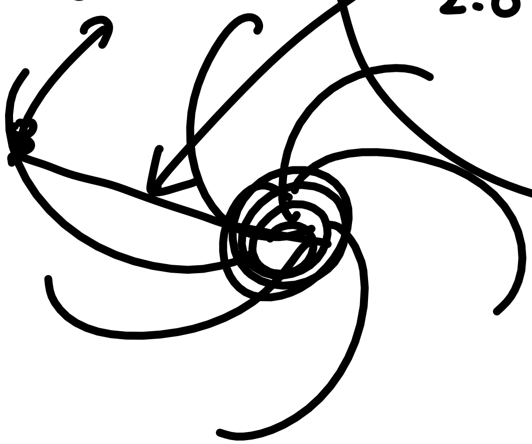


150×10^6 km

$a = \left(\frac{2\pi}{T}\right)^2 R \longrightarrow \frac{a}{g} = 6 \times 10^{-4} \ll 1$

T: $365 \times 24 \times 3600$

(c)



$v = 220 \times 10^3 m/s$

2.8×10^4 LY

$365 \times 24 \times 3600 \times 3 \times 10^8$ m

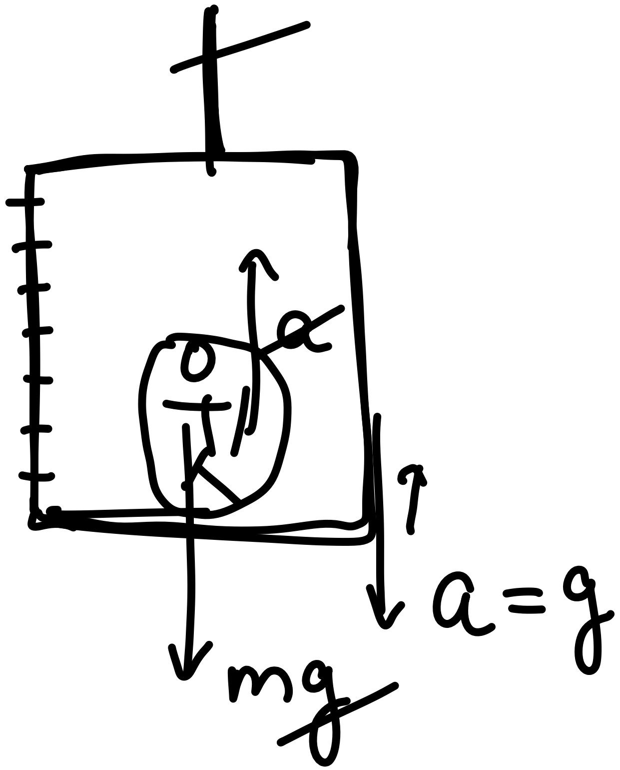
$a = \frac{v^2}{R} = 1.5 \times 10^{-12} \ll 1$

등가원리 (Equivalence principal)

→ 일반상대성 이론

→ 뉴턴

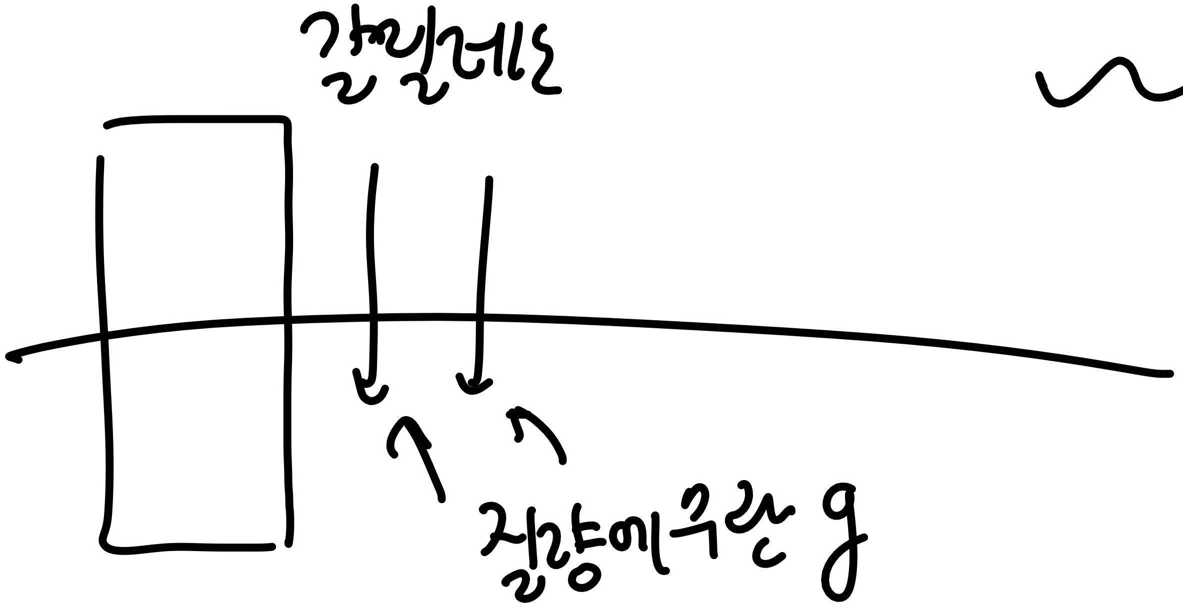
가속 = 중력



$$\vec{F} = m\vec{a}$$

= ↑ =

관성질량



$$\vec{F}_{\text{중력}} = G\frac{\boxed{M_1}M_2}{r^2}$$

중력질량.

운동량 $\vec{p} = \underline{\underline{m}}\,\vec{v}$

$\downarrow$ 상수(시간에 무관) $\to |\vec{v}| \ll c$

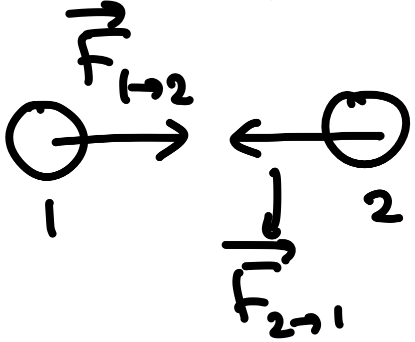
2법칙 : $\vec{F} = \frac{d\vec{p}}{dt} = \underline{m(\cancel{t})}\,\frac{d\vec{v}}{dt} = m\,\vec{a}$

$\uparrow$

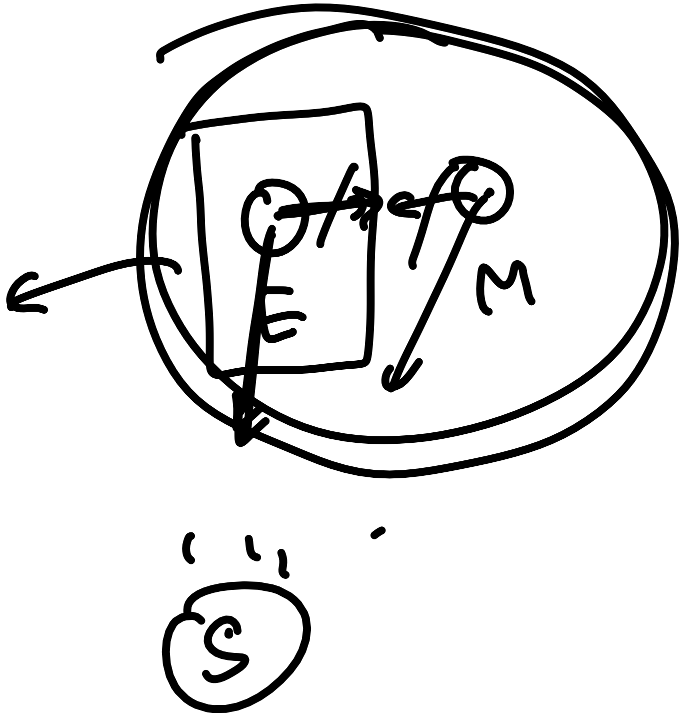
특수상대성이론

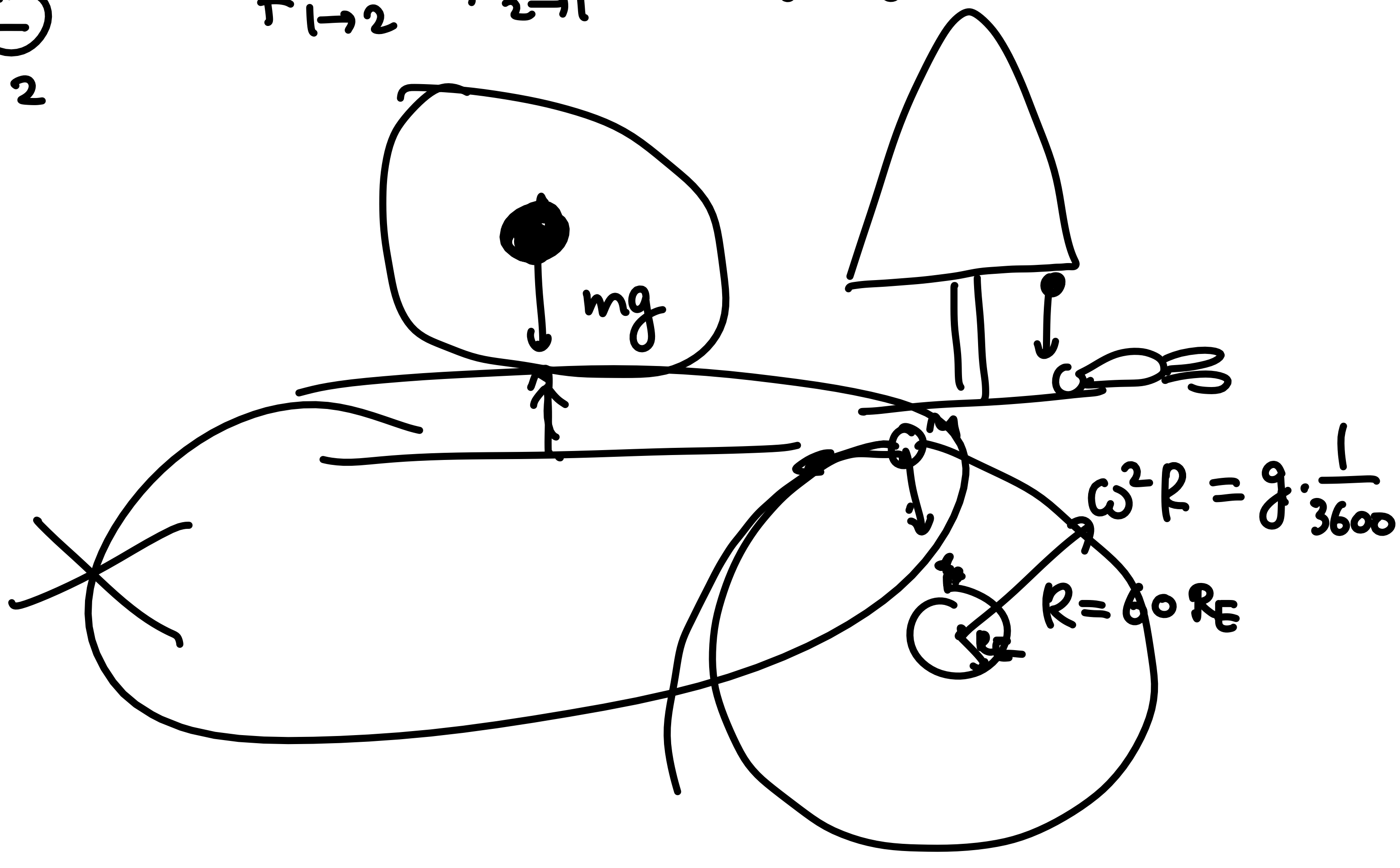
1법칙 : $\vec{F} = 0 \to \vec{p}$ = 보존. (관성의 법칙)

3법칙 :



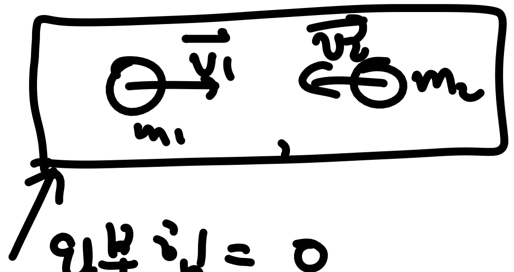
$\vec{F}_{1\to2} + \vec{F}_{2\to1} = 0$ (작용과 반작용)





$\omega^2 R = g \cdot \frac{1}{3600}$

$R = 60\,R_E$



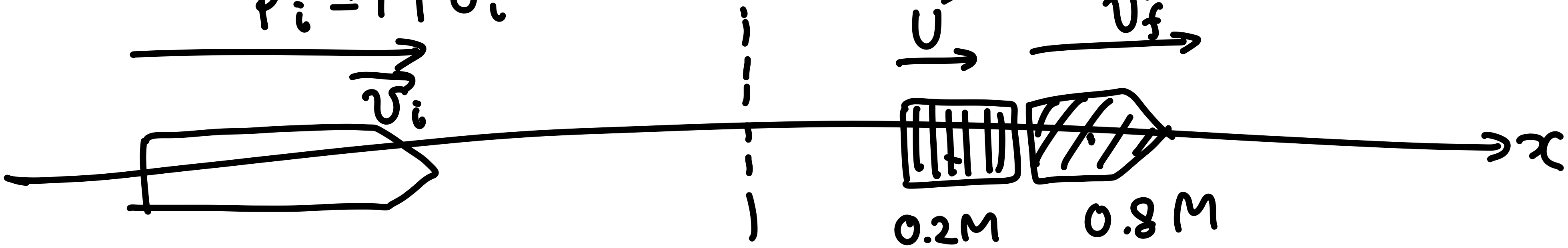
$$\frac{d}{dt}\left(\vec{p}_1+\vec{p}_2 = m_1\vec{v}_1+m_2\vec{v}_2\right) = \vec{F} = 0$$

$$\therefore \boxed{\vec{p}_1+\vec{p}_2 = \text{일정}} \longrightarrow \vec{p}_1, \vec{p}_2 \neq \text{일정}.$$

$$\frac{d}{dt}\vec{p}_1 = \vec{F}_{2\to1} \neq 0$$

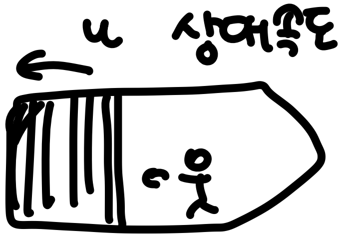
[Ex 2.1.2]

$$\vec{p}_i = M\vec{v}_i$$



외력 = 0

$$M v_i = 0.2 M U + 0.8 M v_f$$



$$u = v_f - U > 0$$
$$\downarrow$$
$$U = v_f - u$$

$$v_i = 0.2(v_f - u) + 0.8 v_f$$

v_i = 20km/s

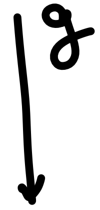
u = 5km/s

→ v_f = 21km/s

§2.2.

지표면

1차원운동 x



$$F(x, \dot{x}, t) = m\ddot{x} = F\,(\text{상수}) \longrightarrow \ddot{x} = \frac{dv}{dt} = \frac{F}{m} = a$$

$$\frac{dv}{dt} = a \longrightarrow v = at + v_0 = \frac{dx}{dt} \longrightarrow \boxed{x = \frac{1}{2}at^2 + v_0 t + x_0}$$

$v = \dot{x}$

$$\downarrow$$

$$t = \frac{1}{a}(v - v_0)$$

$v_0 = v(t=0) \longleftarrow$ 초기조건 $\longrightarrow x_0 = x(t=0)$

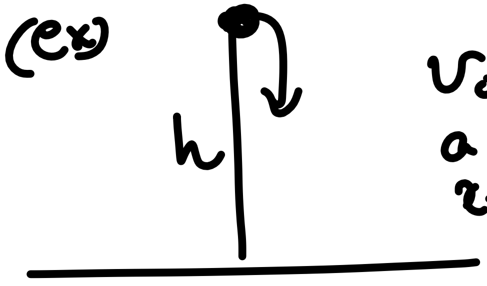
$$x - x_0 = \frac{1}{2}\cancel{a}\frac{1}{a^{\cancel{2}}}(v-v_0)^2 + \frac{v_0}{a}(v-v_0)$$

$$= \frac{1}{2a}(v-v_0)\underbrace{(v-v_0+2v_0)}_{v+v_0}$$

$$\boxed{v^2 - v_0^2 = 2a(x-x_0)}$$

$a=g$

(ex)



$v_0 = 0$

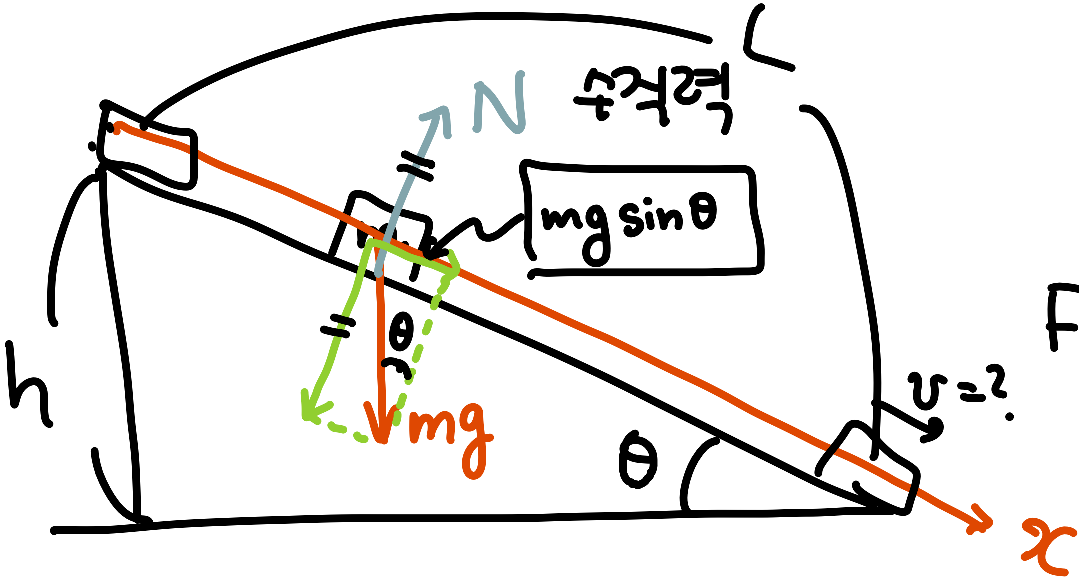
$a = g$

$x - x_0 = h$

$\rightarrow v = \sqrt{2gh}$

[Ex. 2.2.1]

마찰력 = 0



$$F = mg\sin\theta = ma$$

$$\therefore\ a = g\sin\theta$$

$$\frac{h}{L} = \sin\theta$$

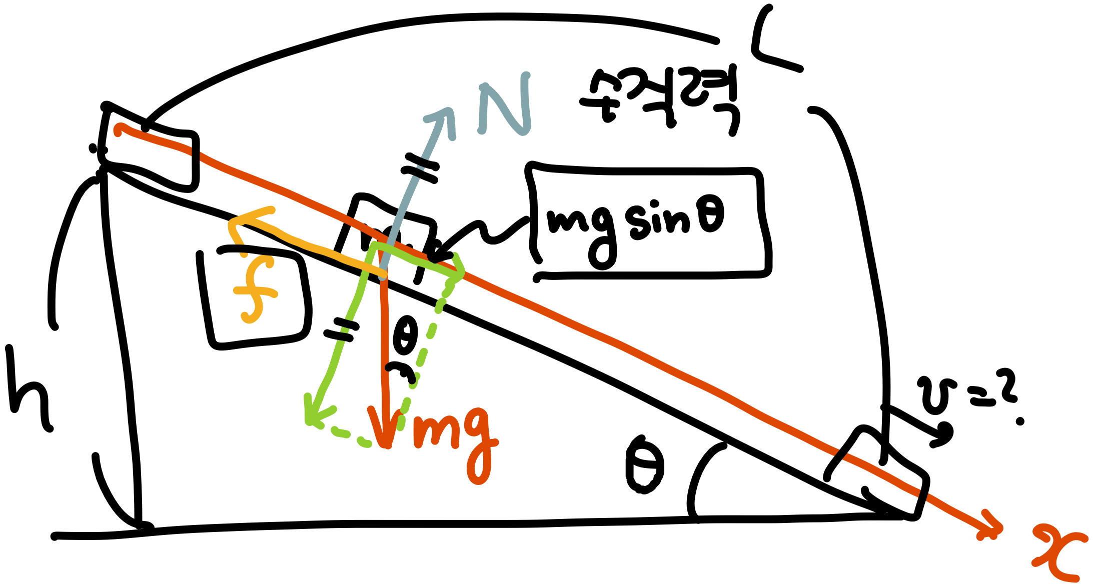
$$v^2 = 2g\cancel{\sin\theta}\cdot L = 2gh$$

$$L = \frac{h}{\cancel{\sin\theta}}$$

$$\boxed{v = \sqrt{2gh}}$$

[Ex. 2.2.1]

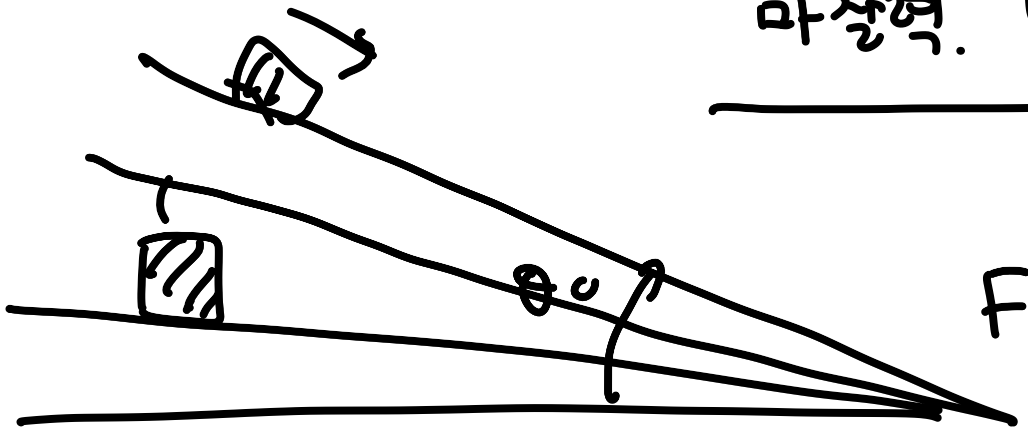
마찰력 $\neq 0$



$$f = \mu_k N$$

마찰력. ↑ 운동마찰계수 (μ_s 정지마찰계수)

$\mu_k < \mu_s$



$$F = mg\sin\theta - \mu_k \overset{\uparrow}{N} = ma$$

$$N = mg\cos\theta$$

$$\therefore\ a = g\left(\underbrace{\sin\theta - \mu_k\cos\theta}_{\theta > \theta_c}\right) > 0$$

$(\tan\theta_c = \mu_k)$

§2.3. Energy

$$\text{Energy}\begin{cases}\text{kinetic (운동E)}\\ \text{potential (위치E)}\end{cases}$$

$$\underset{\substack{\uparrow\\ (\dot{x},\,t\text{ 에 무관})}}{F = F(x)} = m\ddot{x} = m\frac{dv}{dt} \longrightarrow \underbrace{\overset{}{\underset{\frac{dx}{dt}}{\underset{\shortparallel}{v}}} F(x)}_{\displaystyle \downarrow} = m v \frac{dv}{dt} = \frac{1}{2} m \underbrace{\frac{d(v^2)}{dt}}_{2v\cdot\dot{v}}$$

$$\frac{dG(x)}{dt} = \dot{x}\,G'(x)$$

$$G' = F$$

$$-\frac{dV}{dt} = \frac{d}{dt}\left(\frac{m}{2}v^2\right)$$

$$\Downarrow$$

$$\frac{d}{dt}\Big(\underbrace{\frac{1}{2}mv^2}_{T\,=\,\text{운동에너지}} + V\Big) = 0$$

$$\underset{\substack{\uparrow\\ \text{위치에너지}}}{-V} = G = \int_{x_0}^{x} F(x)\,dx$$

$$E = T + V \longrightarrow \frac{dE}{dt} = 0 \rightarrow E = \text{일정} = \text{에너지 보존 법칙}$$

(역학적)

$\underset{\sim\sim}{t=t_0}$, $x(t_0)=x_0$, $\dot{x}(t_0)=v_0$

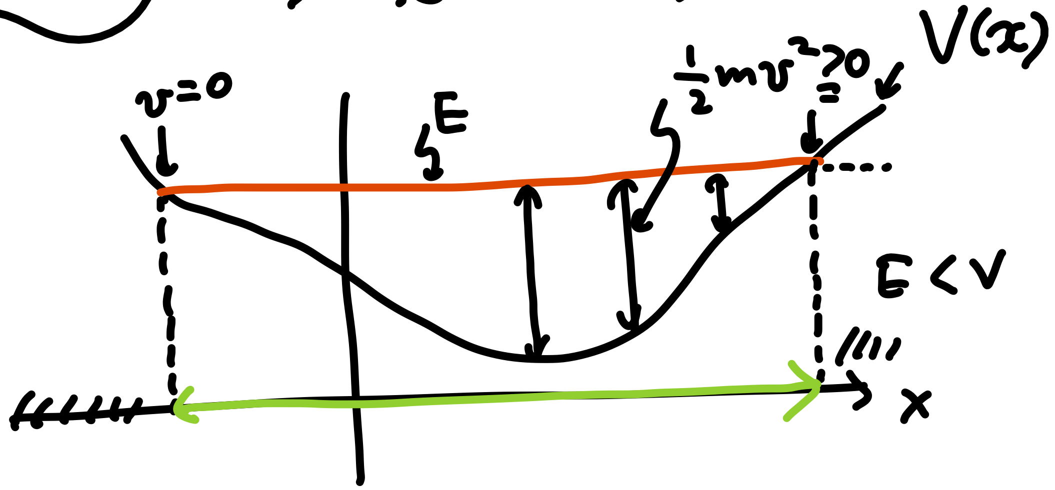
$$E = \frac{1}{2}mv_0^2 + V(x_0) = \frac{1}{2}mv^2 + V(x)$$

$$\frac{1}{2}mv^2 = \underset{\uparrow \text{상수}}{E} - V(x) \rightarrow v^2 = \frac{2}{m}(E-V(x))$$

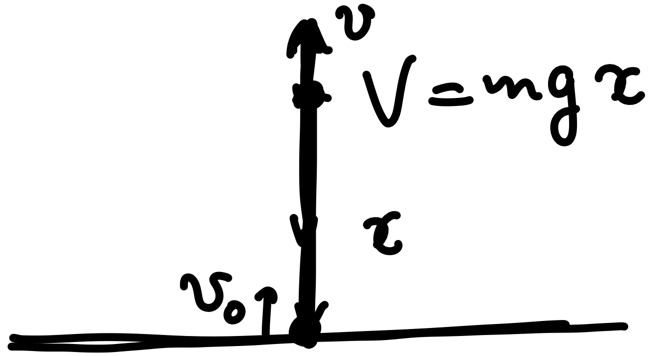
$$v = \pm \underbrace{\sqrt{\frac{2}{m}(E-V(x))}}_{x \text{ 만의 함수}} = \frac{dx}{dt}$$

$$\therefore \underbrace{\int_{t_0}^{t} dt}_{t-t_0} = \pm \underbrace{\underbrace{\int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2}{m}(E-V(x))}}}_{\text{적분 이가능}}}_{H(x)} = t-t_0$$

$$\rightarrow x = x(t)$$



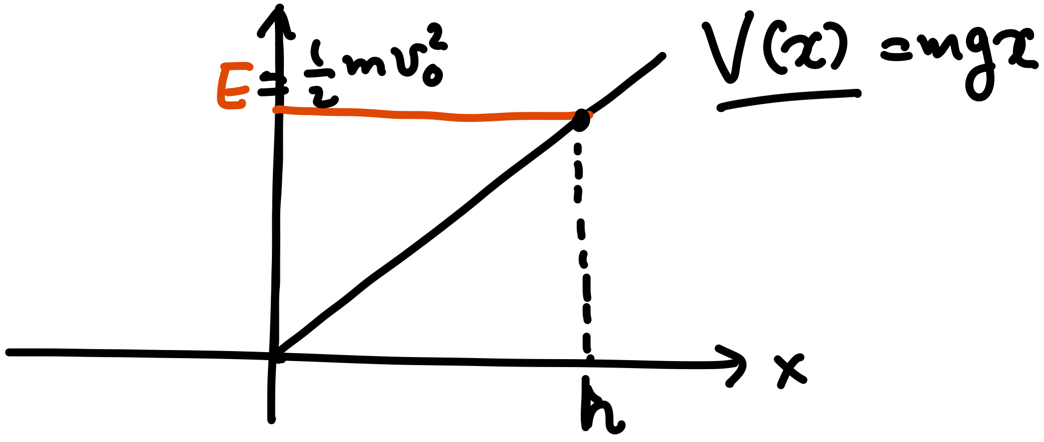
(ex)



$$E = \frac{1}{2}mv^2 + \underline{mgx}$$

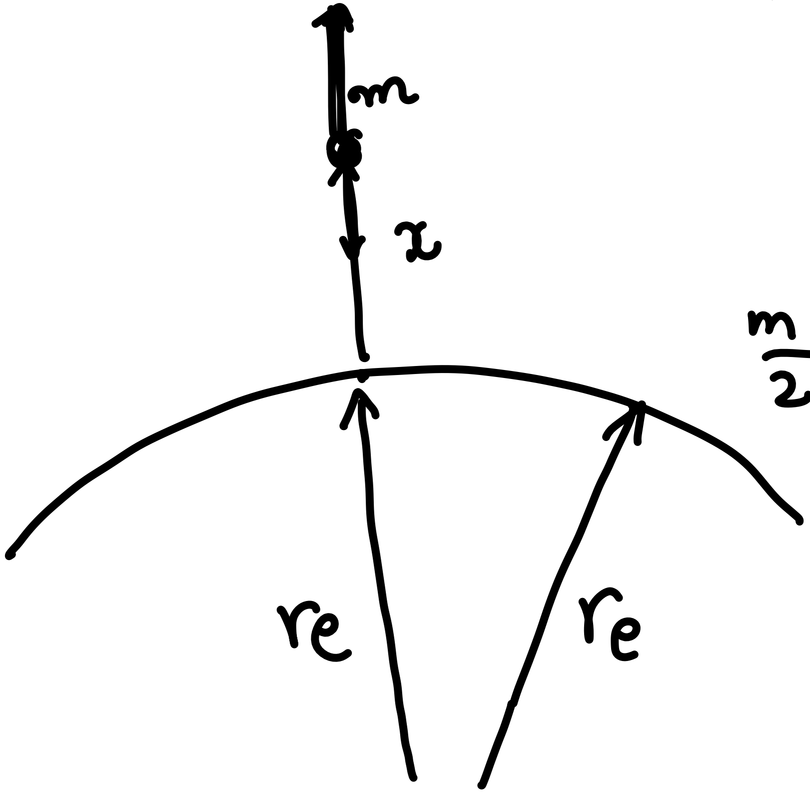
$$= \frac{1}{2}mv_0^2 + mg\cdot 0$$

$$\rightarrow \underline{v^2 - v_0^2 = -2gx}$$



$$mgh = \frac{1}{2}mv_0^2 \rightarrow h = \frac{v_0^2}{2g} = x_{max}$$

(Ex)



$$F_r = -\frac{GMm}{r^2} \qquad r = r_e + x$$

$$m\ddot{r} = F_r \rightarrow m\ddot{x} = -\frac{GMm}{(r_e+x)^2}$$

양쪽에 $\underset{\overset{\shortparallel}{v}}{\dot{x}}$ 을 곱하면

$$\frac{m}{2}(v^2 - v_0^2) = \int_{v_0}^{v} m\,v\,\frac{dv}{\cancel{dt}} = -\dot{x}\frac{GMm}{(r_e+x)^2}$$

그후 적분

$$\frac{d}{dx}\left(\frac{1}{r_e+x}\right) = -\frac{1}{(r_e+x)^2} \quad \Big| \quad \int \overbrace{\frac{d}{\cancel{dt}}\left(\frac{GMm}{r_e+x}\right)}$$

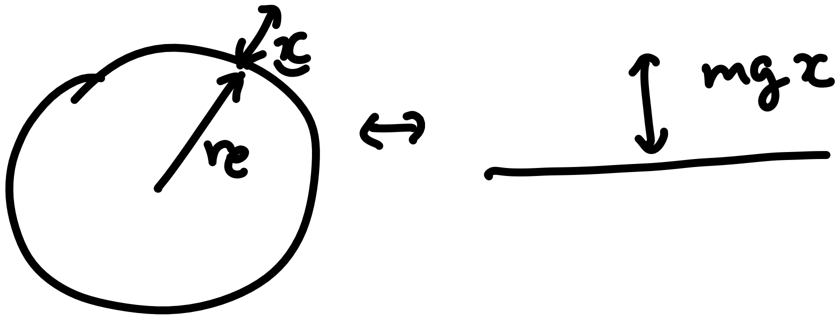
$$\frac{m}{2}(v^2-v_0^2) = \int d\left(\frac{GMm}{r_e+x}\right) = GMm\left(\frac{1}{r_e+x} - \frac{1}{r_e+x_0}\right)$$

$$\frac{m}{2}v^2 \underbrace{- \frac{GMm}{r_e+x}}_{\text{위치에너지} = V(x)} = \frac{m}{2}v_0^2 \underbrace{- \frac{GMm}{r_e+x_0}} = E$$

$$V(x) = -\frac{GMm}{r_e+x} \underset{r_e \gg x}{=} -\frac{GMm}{r_e}\cdot\frac{1}{1+\frac{x}{r_e}} \approx -\frac{GMm}{r_e}\left(1-\frac{x}{r_e}\right)$$

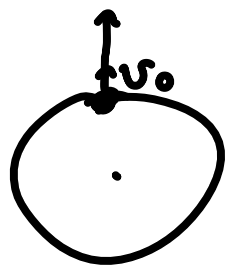
$$\left(\frac{x}{r_e} \ll 1 \quad \longrightarrow \quad \frac{1}{1+\epsilon} \approx 1-\epsilon \ \ \text{Taylor 전개}\right)$$

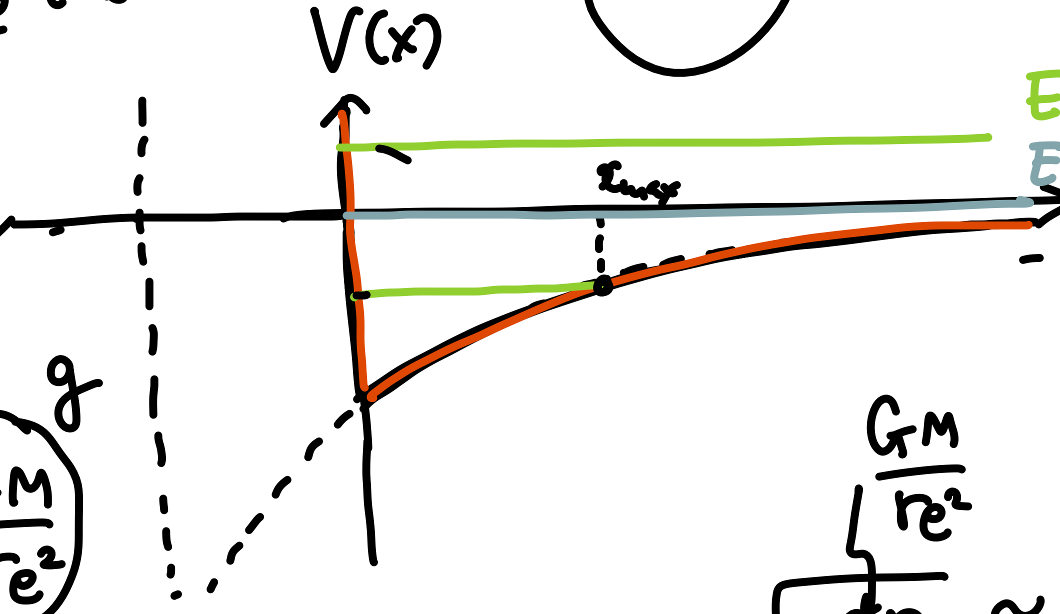
$$= \underbrace{-\frac{GMm}{r_e}}_{\text{상수}} + \underbrace{\frac{GMmx}{r_e^2}}_{mgx}$$



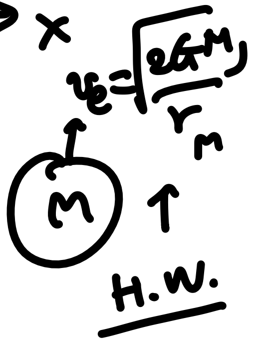
$$g \simeq 9.8\,m/s^2 = \frac{GM}{r_e^2}$$

$r_e \sim x$ $\qquad V(x) = -\frac{GMm}{r_e + x'}$





x

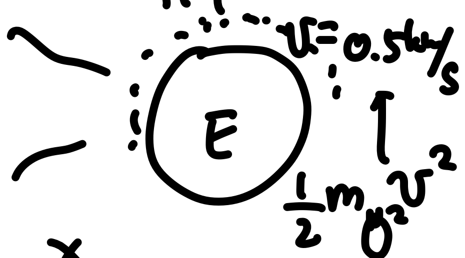


$$E = T + V$$

탈출조건

$$= \frac{1}{2} m v_0^2 - \frac{GMm}{r_e} \overset{\downarrow}{=} 0$$

$$v_0^2 = \frac{2GM}{\underset{\overset{\uparrow}{\text{earth}}}{r_e}} = 2 r_e \overset{g}{\boxed{\frac{GM}{r_e^2}}} = 2 g r_e \rightarrow \text{탈출속도 } \underset{\overset{\uparrow}{\text{escape}}}{v_e} = \sqrt{2 \overset{\frac{GM}{r_e^2}}{g} r_e} \simeq 11\,\text{km/s}$$



$$\underbrace{\frac{1}{2} m_H v^2 = \frac{3}{2} kT}$$

$$E<0 \quad E = \frac{1}{2} m v_0^2 - \frac{GMm}{r_e} = 0 - \frac{GMm}{r_e + x_{max}}$$

$$\therefore \frac{1}{2} v_0^2 = GM\left(\frac{1}{r_e} - \frac{1}{r_e + x_{max}}\right) = GM \frac{x_{max}}{r_e(r_e + x_{max})}$$

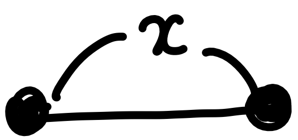
$$= \frac{GM}{r_e} \frac{1}{\left(\frac{r_e}{x_{max}} + 1\right)}$$

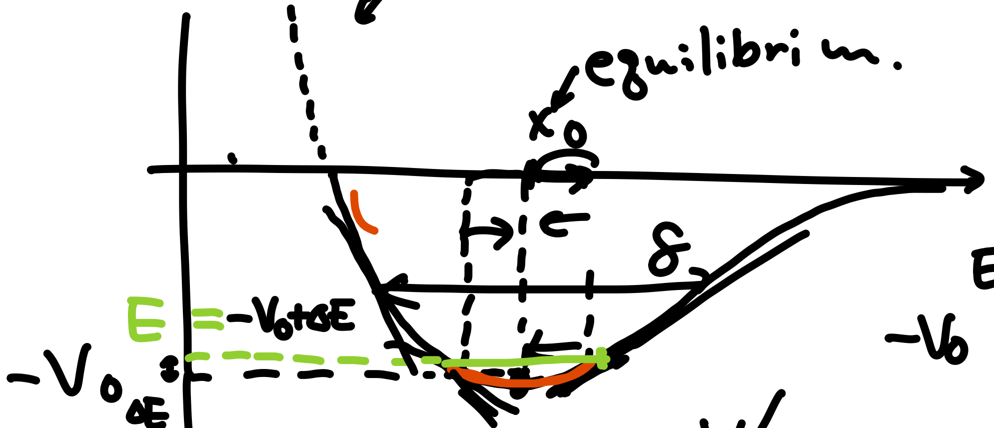
$$\frac{r_e}{x_{max}} + 1 = \frac{2GM}{r_e v_0^2} = \frac{2 g r_e}{v_0^2}$$

$$\frac{x_{max}}{r_e} = \left(\frac{r_e}{x_{max}}\right)^{-1} = \left(\frac{2 g r_e}{v_0^2} - 1\right)^{-1}$$

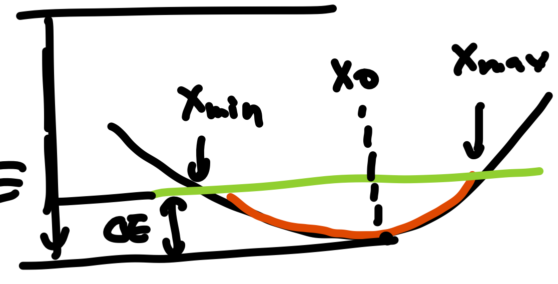
$$\therefore x_{max} = r_e \left(\frac{2 g r_e}{v_0^2} - 1\right)^{-1} \quad //$$

[Ex3] $V(x) = V_0\left[1 - e^{-(x-x_0)/\delta}\right]^2 - V_0$





$x = x_0$ $\quad V'(x) = 0 = -F$



[Ex4] $\quad x - x_0 \ll \delta$

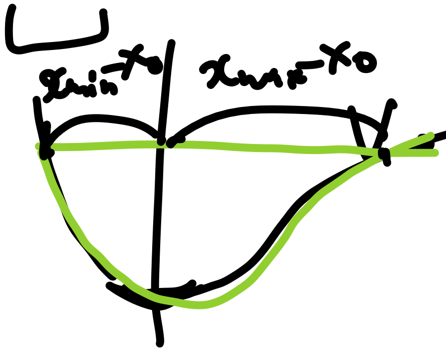
$$e^{-\frac{x-x_0}{\delta}} \quad \frac{x-x_0}{\delta} = \varepsilon \ll 1$$

$$e^{-\varepsilon} \approx 1 - \varepsilon$$

$$1 - e^{-\varepsilon} \approx 1 - (1-\varepsilon) = \varepsilon$$

$$V \approx V_0 \varepsilon^2 - V_0 = V_0\left(\frac{x-x_0}{\delta}\right)^2 - V_0$$

$$V(x_{max}) = E$$



$$\left(\frac{x_{max}-x_0}{\delta}\right)^2 - V_0 = -V_0 + \Delta E$$

$$\therefore \quad x_{max} - x_0 = \delta\sqrt{\frac{\Delta E}{V_0}}$$

$$x_{min} - x_0 = -\delta\sqrt{\frac{\Delta E}{V_0}}$$

$\Downarrow$

열팽창

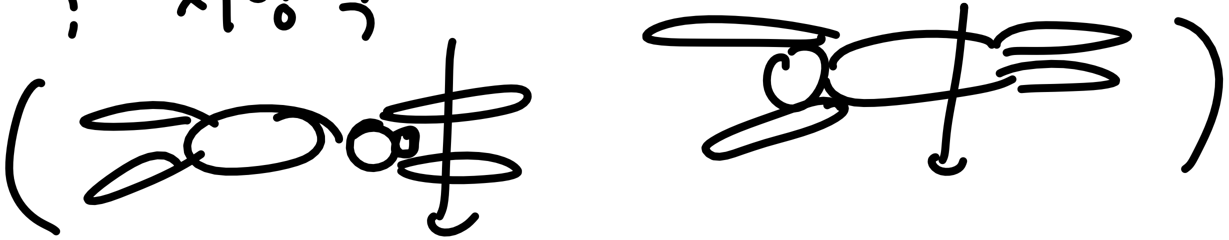
$$\underline{F_0 + F(v)} \leftarrow \left(F(x, v, t) \right)$$

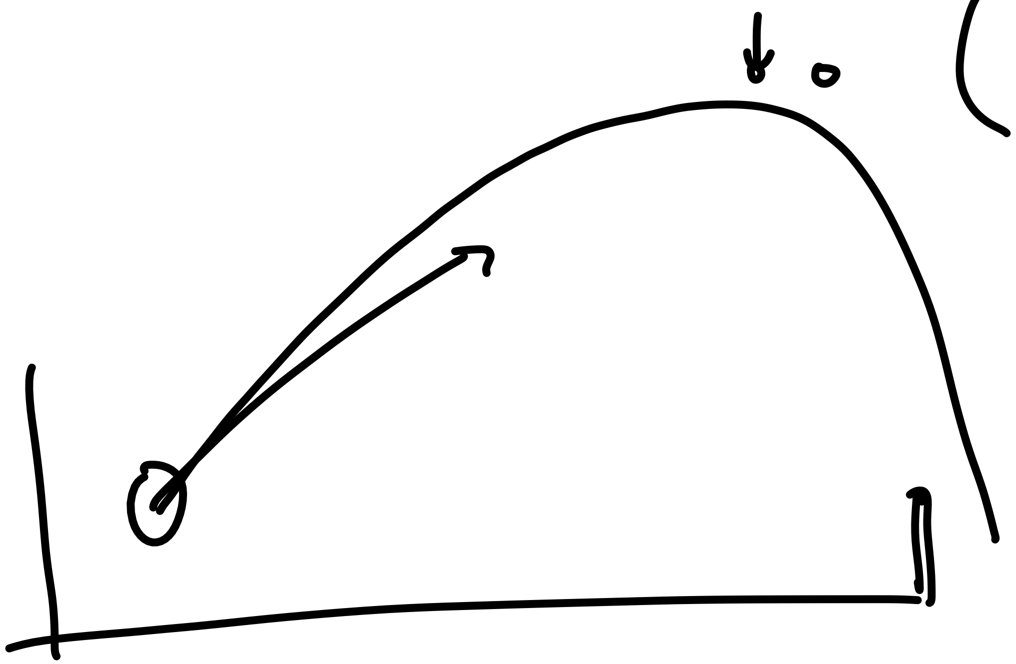
$$\overset{\shortparallel}{F}$$

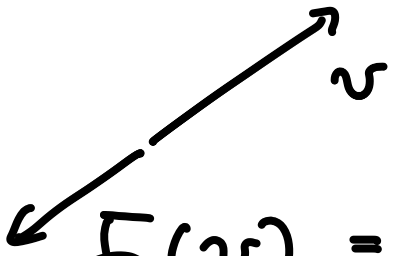
$$F = m\frac{dv}{dt} = F_0 + F(v)$$

$F_0 = 0$

유체 (Fluid) 역학 ; 저항력







$$\underline{F(v) = \overbrace{-c_1 v}^{F_1} \overbrace{-C_2 v|v|}^{F_2}}$$

$C_1 = 1.55 \times 10^{-4} D \leftarrow$ meter.

$C_2 = 0.22 \times D^2$

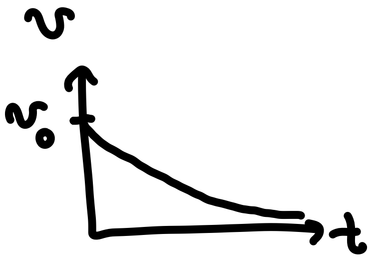
$$\frac{F_2}{F_1} = 1.4 \times 10^3 |v| \underset{\uparrow\, 7\times 10^{-2}}{D} \gg 1$$

$$\ll 1$$

$$F \approx \begin{cases} F_2 \leftarrow \text{사격, 야구공} \\ F_1 \leftarrow \text{빗방울} \end{cases}$$

linear

$$-c_1 v = F = m\frac{dv}{dt}$$

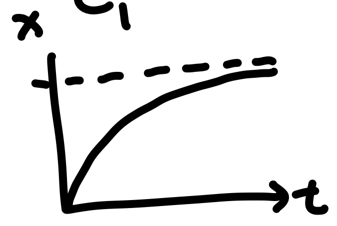


$$-\frac{c_1}{m}dt = \frac{1}{v}dv$$

$$\downarrow$$

$$-\frac{c_1}{m}t = -\frac{c_1}{m}\int_0^t dt = \int_{v_0}^{v}\frac{dv}{v} = \overset{\ln}{\log}\frac{v}{v_0} \rightarrow v = \underline{v_0 e^{-\frac{c_1}{m}t} = \frac{dx}{dt}}$$

$$x = x_0 - \frac{mv_0}{c_1}e^{-\frac{c_1}{m}t} \qquad x(t=0) = 0 = x_0 - \frac{mv_0}{c_1} \rightarrow x_0 = \frac{mv_0}{c_1}$$



$$\therefore x = \frac{mv_0}{c_1}\left(1 - e^{-\frac{c_1}{m}t}\right) \qquad t\rightarrow\infty \quad x_{max} = \frac{mv_0}{c_1}$$

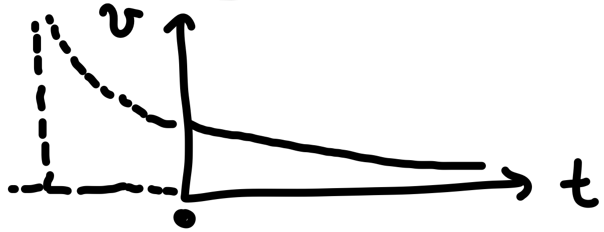
[Ex. 2.4.2] $F = -c_2 v^2 = m\frac{dv}{dt} \longrightarrow -\int_0^t \frac{c_2}{m}dt = \int_{v_0}^{v}\frac{dv}{v^2}$

$$\qquad\qquad -\frac{c_2}{m}t \qquad \left[-\frac{1}{v}\right]_{v_0}^{v} = \frac{1}{v_0} - \frac{1}{v}$$

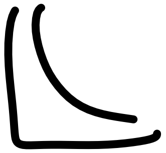
$$\therefore \frac{1}{v} = \frac{1}{v_0} + \frac{c_2}{m}t = \frac{1 + \overset{=k}{\boxed{\frac{c_2 v_0}{m}}}t}{v_0} = \frac{1+kt}{v_0}$$

$$\therefore \boxed{v(t) = \frac{v_0}{1+kt}} = \frac{dx}{dt}, \quad x - x_0 = \int_{x_0}^{x}dx = \int_0^t \frac{v_0}{1+kt}dt = \frac{v_0}{k}\log(1+kt)$$

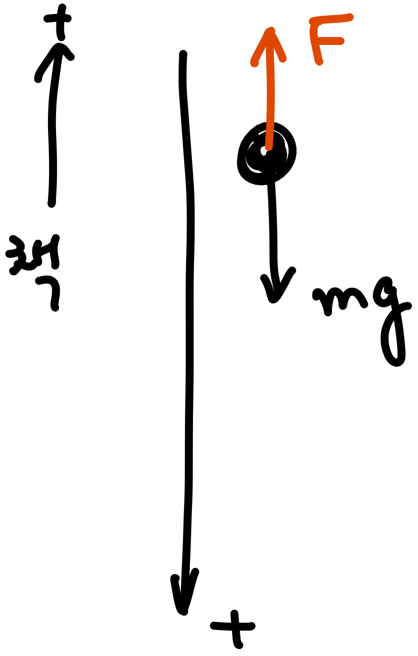
$$x_0 = x(0) = 0 \longrightarrow x = \frac{v_0}{k}\log(1+kt)$$



$$\frac{v_0}{k(t+\frac{1}{k})}$$



Vertical Fall : 종단 속도 (terminal velocity)



$$F = mg - c_1 v = m\frac{dv}{dt}$$

$$t = \int_0^t dt = \int_{v_0}^{v} \frac{m}{mg - c_1 v}\, dv = \frac{m}{c_1}\left[-\log\left(g - \frac{c_1}{m}v\right)\right]_{v_0}^{v}$$

$$t = -\frac{m}{c_1}\log\left(\frac{g - \frac{c_1}{m}v}{g - \frac{c_1}{m}v_0}\right)$$

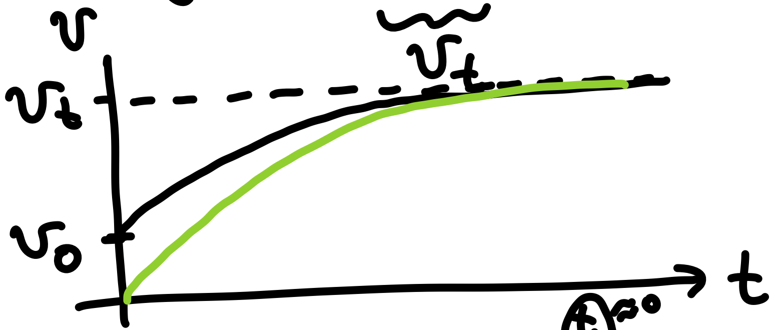
$$g - \frac{c_1}{m}v(t) = \left(g - \frac{c_1}{m}v_0\right)e^{-\frac{c_1}{m}t}$$

$$\to v(t) = \frac{mg}{c_1} - \underbrace{\frac{m}{c_1}\left(g - \frac{c_1}{m}v_0\right)}\, e^{-\frac{c_1}{m}t}$$

$$= \frac{mg}{c_1} + \left(v_0 - \underbrace{\frac{mg}{c_1}}_{v_t}\right)e^{-\frac{c_1}{m}t} \qquad \frac{m}{c_1} \equiv \tau$$

$$t \to \infty \qquad v \to \frac{mg}{c_1} = v_t$$

$$v < v_t$$



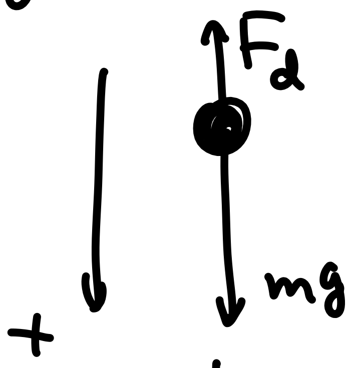
$$v(t) = v_t + (v_0 - v_t)e^{-\frac{t}{\tau}}$$

$$= v_0 e^{-\frac{t}{\tau}} + v_t\left(1 - e^{-\frac{t}{\tau}}\right)$$

If $v_0 = 0$, $\quad v(t) = v_t\left(1 - e^{-\frac{t}{\tau}}\right)$

$$\begin{cases} t \ll \tau & e^{-\boxed{\frac{t}{\tau}}^{\to 0}} \approx 1 \quad v \approx v_0 \\ t \gg \tau & e^{-\frac{t}{\tau}} \approx 0 \quad v \approx v_t \end{cases}$$

quadratic : $F = mg - C_2 v^2 = m\frac{dv}{dt}$



$t\to\infty \quad v \to v_t \Rightarrow \frac{dv}{dt}\to 0 \quad \therefore\ mg - C_2 v_t^2 = 0$

$$v_t = \sqrt{\frac{mg}{C_2}}$$

$$\frac{dv}{dt} = g - \frac{C_2}{m}v^2 = g\left(1 - \overset{\equiv \frac{1}{v_t^2}}{\boxed{\frac{C_2}{mg}}}\, v^2\right) = g\left(1 - \frac{v^2}{v_t^2}\right)$$

$$\int_{t_0}^{t} dt = \int_{v_0}^{v} \frac{dv}{g\left(1-\frac{v^2}{v_t^2}\right)}$$

$y = \tanh^{-1}\left(\frac{v}{v_t}\right)$

$\uparrow$

$\frac{v}{v_t} \equiv \tanh y$

$$= \int_{y_0}^{y} \frac{v_t\, d(\tanh y)}{g(\underbrace{1-\tanh^2 y}_{\cancel{\operatorname{sech}^2 y}})} = \frac{v_t}{g}(y - y_0)$$

$d(\tanh y) = \cancel{\operatorname{sech}^2 y}\, dy$

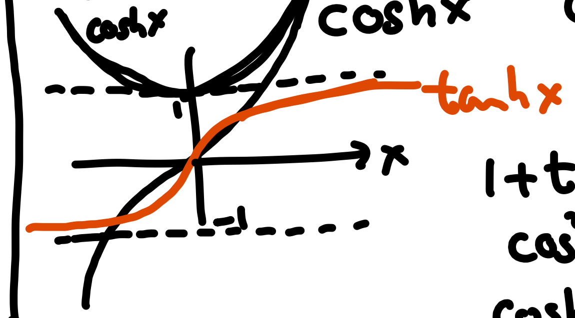
$d(\tan x) = \sec^2 x\, dx$

$$\therefore\ t - t_0 = \underset{\overset{\shortparallel}{\tau}}{\boxed{\frac{v_t}{g}}}\left(\tanh^{-1}\frac{v}{v_t} - \tanh^{-1}\frac{v_0}{v_t}\right)$$

$\cos x = \frac{e^{ix}+e^{-ix}}{2}, \quad \sin x = \frac{e^{ix}-e^{-ix}}{2i}$

$\cosh x = \frac{e^{x}+e^{-x}}{2}, \quad \sinh x = \frac{e^{x}-e^{-x}}{2}$

$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$



$1 + \tan^2 x = \sec^2 x$

$\cos^2 x + \sin^2 x = 1$

$\cosh^2 x - \sinh^2 x = 1$

$1 - \tanh^2 x = \operatorname{sech}^2 x$

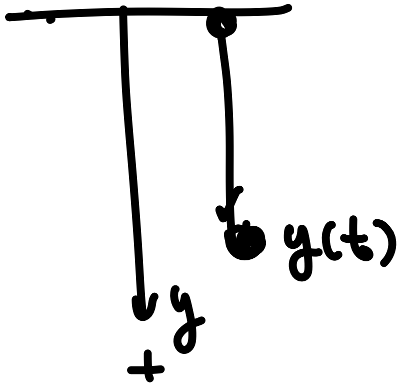
$$\tau \equiv \sqrt{\frac{m}{gC_2}}$$

$$\tanh^{-1}\frac{v}{v_t} = \frac{t-t_0}{\tau} + \tanh^{-1}\frac{v_0}{v_t} \Rightarrow \frac{\overset{\text{time}}{\overset{\downarrow}{v(t)}}}{\underset{\text{terminal}}{\underset{\uparrow}{v_t}}} = \tanh\left(\frac{t-t_0}{\tau} + \tanh^{-1}\frac{v_0}{v_t}\right)$$

$t = t_0 = 0 \rightarrow v_0 = 0 \quad \underline{v(t) = v_t \tanh\left(\frac{t}{\tau}\right)}$

$$\frac{dv}{dt} = \underline{g\left(1-\frac{v^2}{v_t^2}\right)} = \frac{dv}{dy}v = \underline{\frac{1}{2}\frac{d}{dy}v^2}$$

$$\frac{dv}{dt} = \frac{dv}{dy}\underbrace{\frac{dy}{dt}}_{v}$$



$$v = \frac{dy}{dt}$$

$$v(t) \rightarrow v(y)$$

$$v^2 \equiv v_2$$

$$\therefore \frac{dv_2}{dy} = 2g\underbrace{\left(1-\frac{v_2}{v_t^2}\right)}_{\equiv u}$$

$$\frac{dv_2}{dy} = -v_t^2\frac{du}{dy}$$

$$\frac{du}{dy} = -\frac{1}{v_t^2}\frac{dv_2}{dy}$$

$$\therefore \frac{du}{dy} = -\frac{2g}{v_t^2}u \Rightarrow u = \overset{\uparrow}{u_0} e^{-\frac{2g}{v_t^2}y}$$

$$u = 1-\frac{v^2}{v_t^2} \qquad u_0 = u(y=0)$$

$$u_0 = 1-\frac{v_0^2}{v_t^2}$$

$$\rightarrow v_t^2\left(1-u_0 e^{-\frac{2g}{v_t^2}y}\right) = v^2(y)$$

$$\underline{v(y) = v_t\sqrt{1-\left(1-\frac{v_0^2}{v_t^2}\right)e^{-\frac{2g}{v_t^2}y}}}$$

(Ex) (a) 빗방울 ; $0.1\text{mm} = 10^{-4}\text{m}$

$C_1 = 1.55 \times 10^{-4} D = 1.55 \times 10^{-8}$

$m = 0.52 \times 10^{-9}\text{kg}$

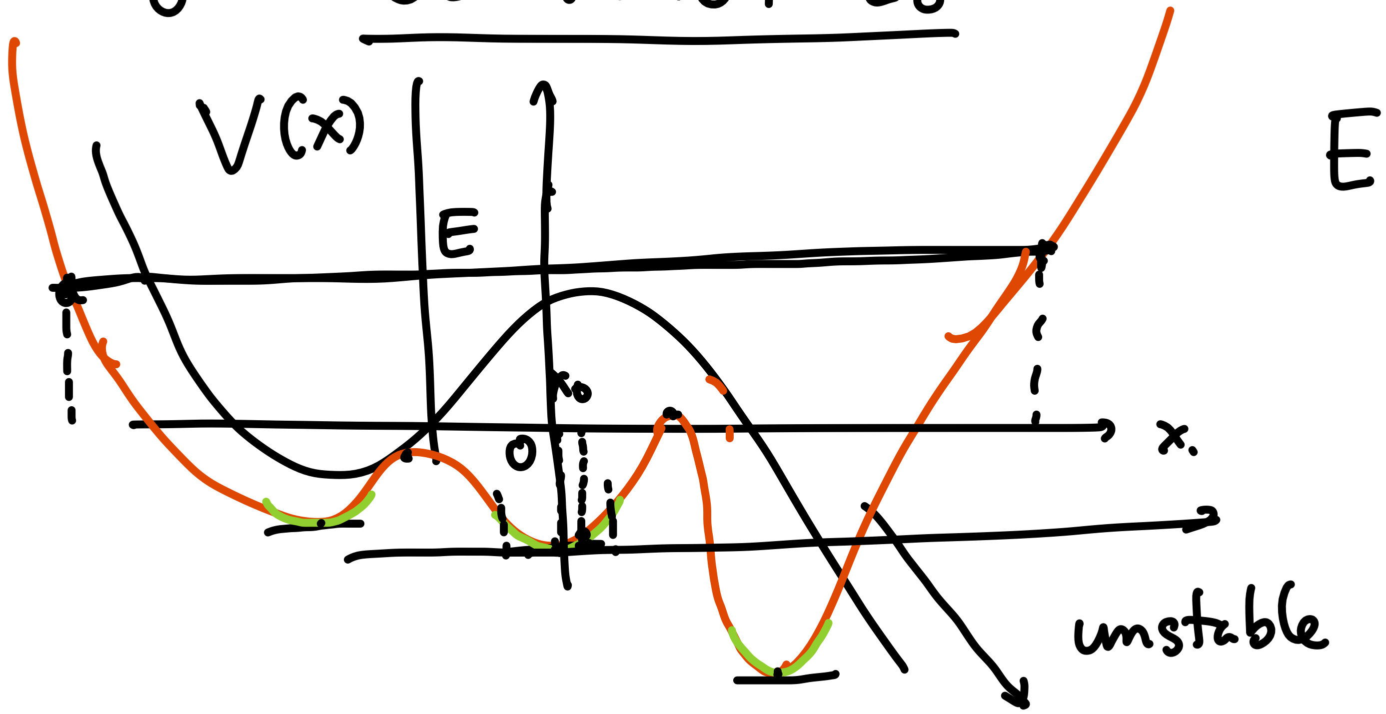
$\Big\}\ v_t = \frac{mg}{C_1} = 0.33\text{m/s}$, $\tau = \frac{v_t}{g} = 0.034\text{s}$

(b) $C_2 = 0.22 D^2 = 0.0138$

$m = 0.6\text{kg}$

$\underline{v_t = \sqrt{\frac{mg}{C_2}}} = 20.6\text{m/s}$, $\tau = \frac{v_t}{g} = 2.1\text{s}$

3장. Oscillation 진동



$$E = T + V(x)$$

$V(x)$ $\quad V'(x) = 0$

↓

$x = x_0$

Taylor 전개 $x = x_0$ 근처

$$V(x) = V(x_0) + \underbrace{V'(x_0)}_{\overset{\shortparallel}{0}}(x-x_0) + \frac{1}{2}\underbrace{V''(x_0)}_{\overset{\shortparallel}{k}}(x-x_0)^2 + \dots$$

$$V(x) \approx V(x_0) + \underline{\frac{1}{2}k(x-x_0)^2}$$

$$\rightarrow \boxed{V(x) = \frac{1}{2}kx^2} \Rightarrow F = -\frac{dV}{dx} = -kx = m\frac{d^2x}{dt^2}$$

$$\therefore \ddot{x} + \underbrace{\frac{k}{m}}_{\overset{\shortparallel}{\omega_0^2}} x = 0 \Longrightarrow \underbrace{\ddot{x}}_{} = -\omega_0^2 x$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

↑

t에대한 <u>2차 미방</u>

초기조건 : <u>두개</u>

$$\boxed{\begin{aligned} &\text{초기위치} \\ &x(0) = x_0 \\ &\dot{x}(0) = v_0 \ \text{초기속도} \end{aligned}}$$

$$\boxed{\begin{aligned} x &= \underline{A}\sin\omega_0 t \\ &+ \underline{B}\cos\omega_0 t \end{aligned}}$$

$$\left(+ \ e^{\omega_0 t} + e^{-\omega_0 t} \right) \times$$

$$x(t) = A\sin\omega_0 t + B\cos\omega_0 t$$

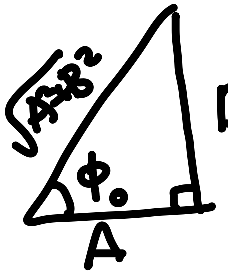
$$x(0) = x_0 = B$$

$$\dot{x}(t) = A\omega_0\cos\omega_0 t - B\omega_0\sin\omega_0 t$$

$$\dot{x}(0) = v_0 = A\omega_0 \rightarrow A = \frac{v_0}{\omega_0}$$

$$\therefore \boxed{x(t) = \frac{v_0}{\omega_0}\underset{\sim\sim\sim}{\sin\omega_0 t} + x_0\underset{\sim\sim\sim}{\cos\omega_0 t}}$$

$$x(t) = \sqrt{A^2+B^2}\left(\underbrace{\frac{A}{\sqrt{A^2+B^2}}}_{\cos\phi_0}\sin\omega_0 t + \underbrace{\frac{B}{\sqrt{A^2+B^2}}}_{\sin\phi_0}\cos\omega_0 t\right)$$

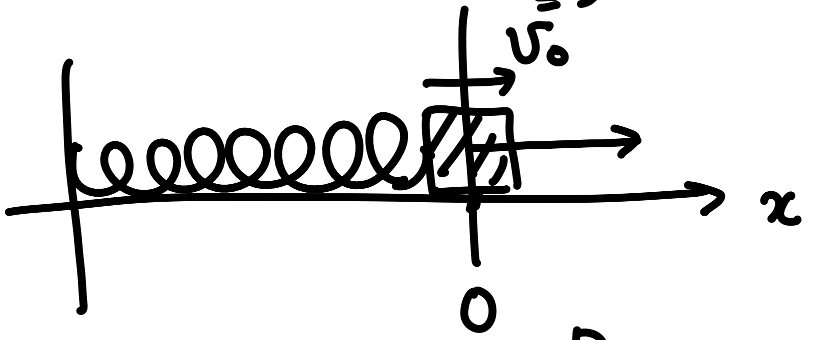


$$\rightarrow \tan\phi_0 = \frac{B}{A}$$

$$x(t) = \underset{=A}{\boxed{\sqrt{A^2+B^2}}}\sin(\omega_0 t + \phi_0) = \sqrt{\frac{v_0^2}{\omega_0^2} + x_0^2}\,\sin(\omega_0 t + \phi_0)$$

$$\phi_0 = \tan^{-1}\left(\frac{x_0\omega_0}{v_0}\right)$$

(ex)



$$x_0 = 0 \rightarrow x(t) = \frac{v_0}{\omega_0}\sin(\omega_0 t)$$

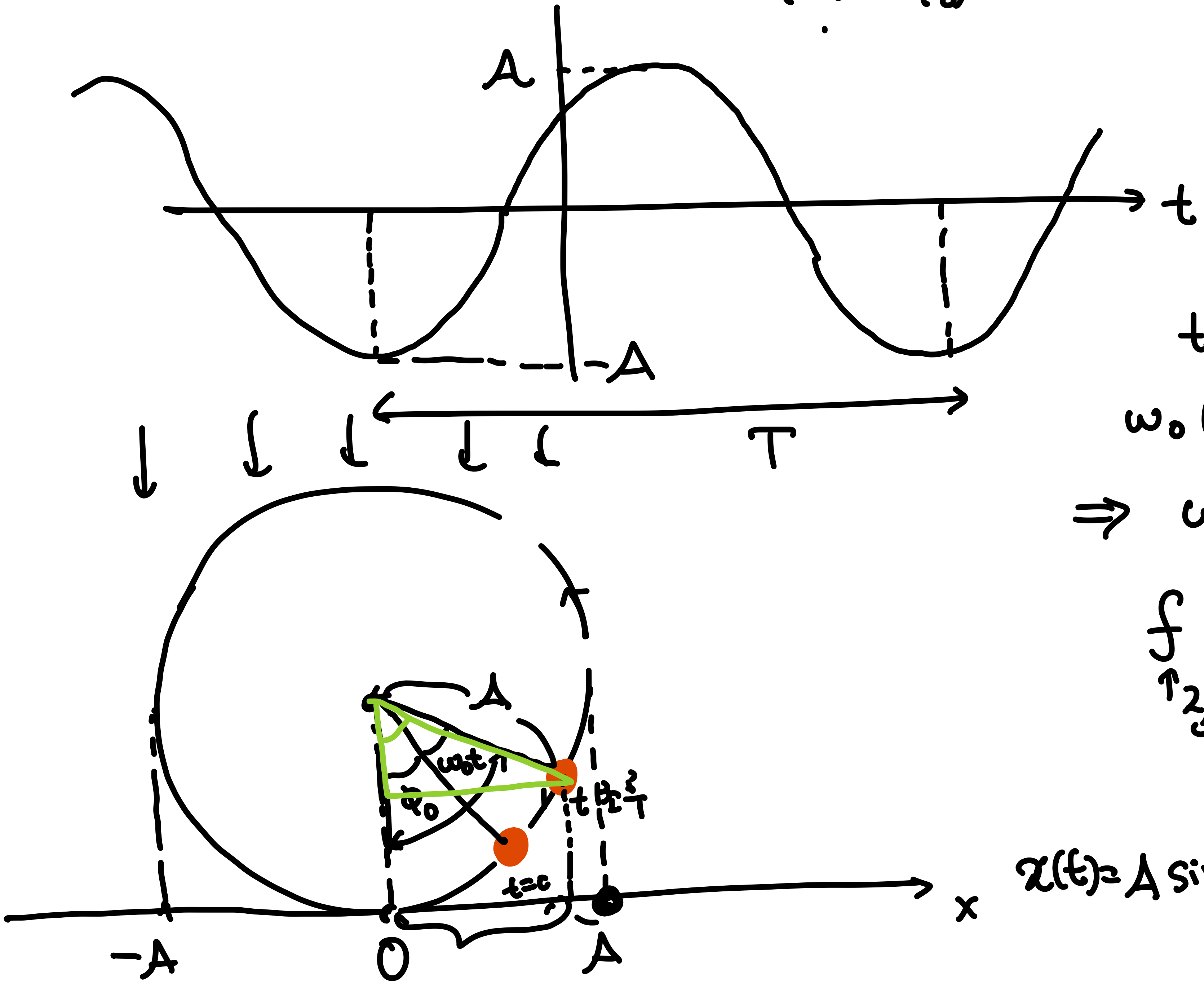
$x_0 = D$

가만히 놓은 경우

$$\phi_0 = \frac{\pi}{2}$$

$$x(t) = D\sin\left(\omega_0 t + \frac{\pi}{2}\right) = D\cos\omega_0 t$$

$$x = A\sin(\omega_0 t+\phi_0)$$



$$t \rightarrow t+T$$

$$\omega_0(\cancel{t}+T)+\cancel{\phi_0} - \cancel{\omega_0 t}+\cancel{\phi_0}$$

$$+2\pi$$

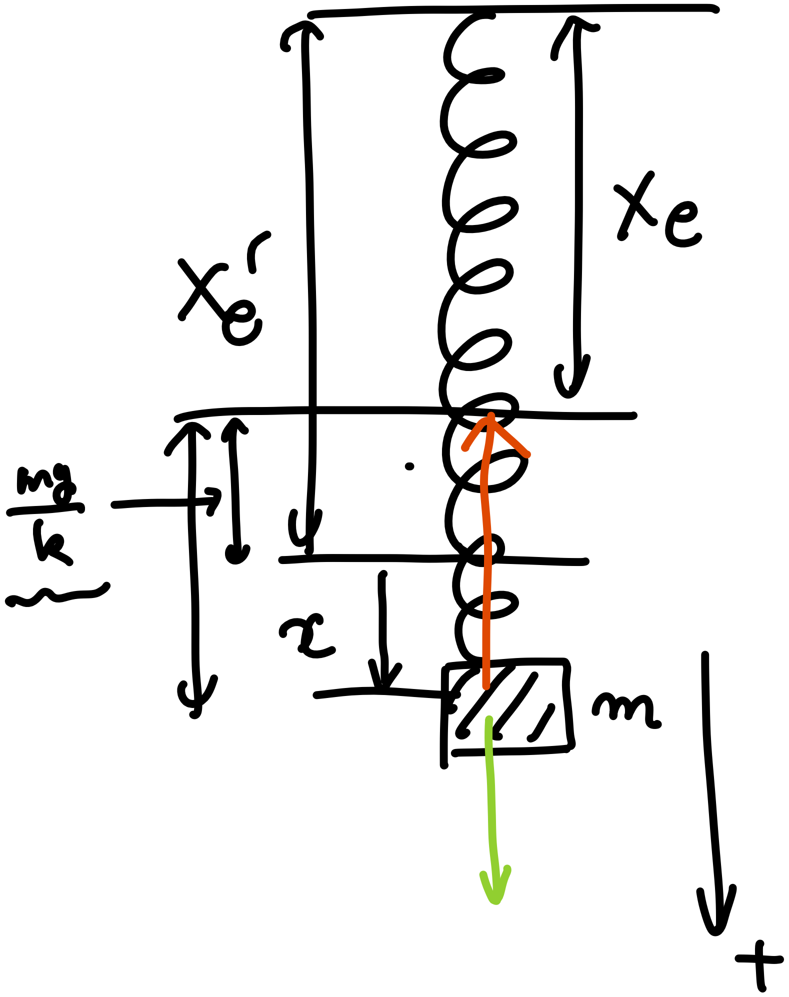
$$\Rightarrow \quad \omega_0 T = 2\pi \rightarrow T = \frac{2\pi}{\omega_0}$$

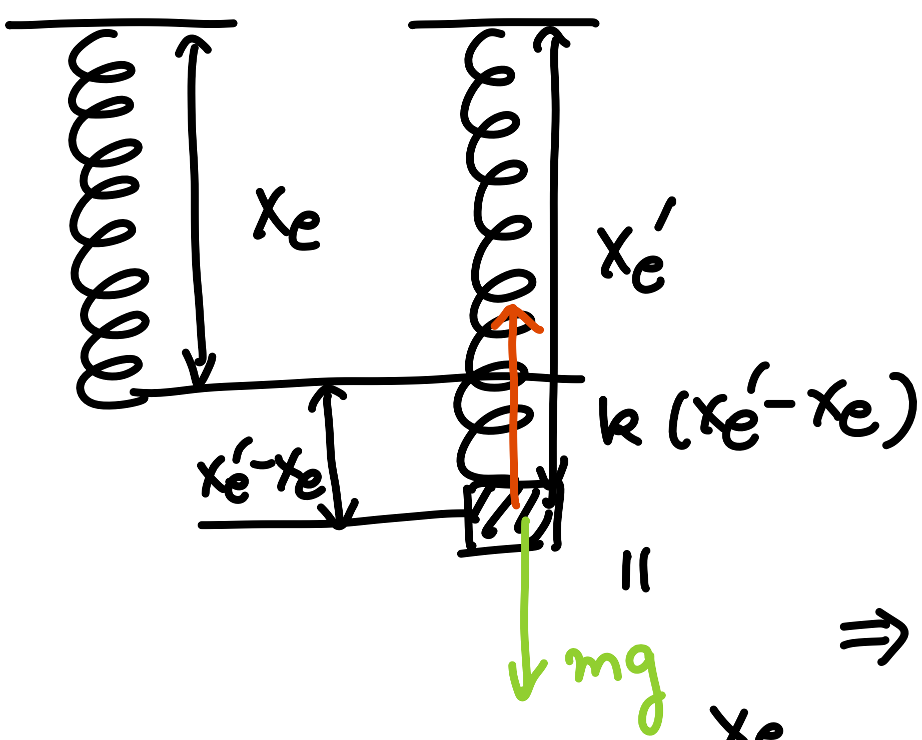
$$f = \frac{1}{T} = \frac{\omega_0}{2\pi}, \quad \omega_0 = 2\pi f$$

↑ 진동수 frequency

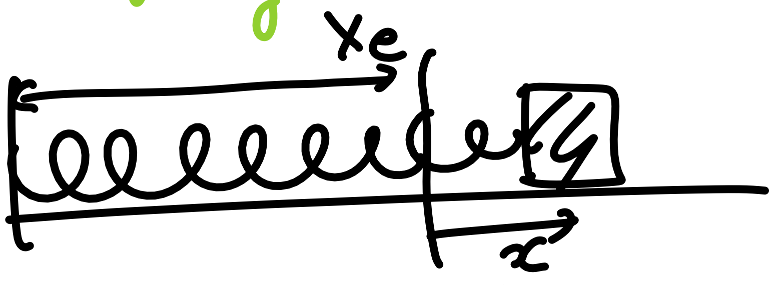
각진동수

$$x(t) = A\sin(\phi_0 + \omega_0 t)$$





$$\Rightarrow X_e' = X_e + \frac{mg}{k}$$

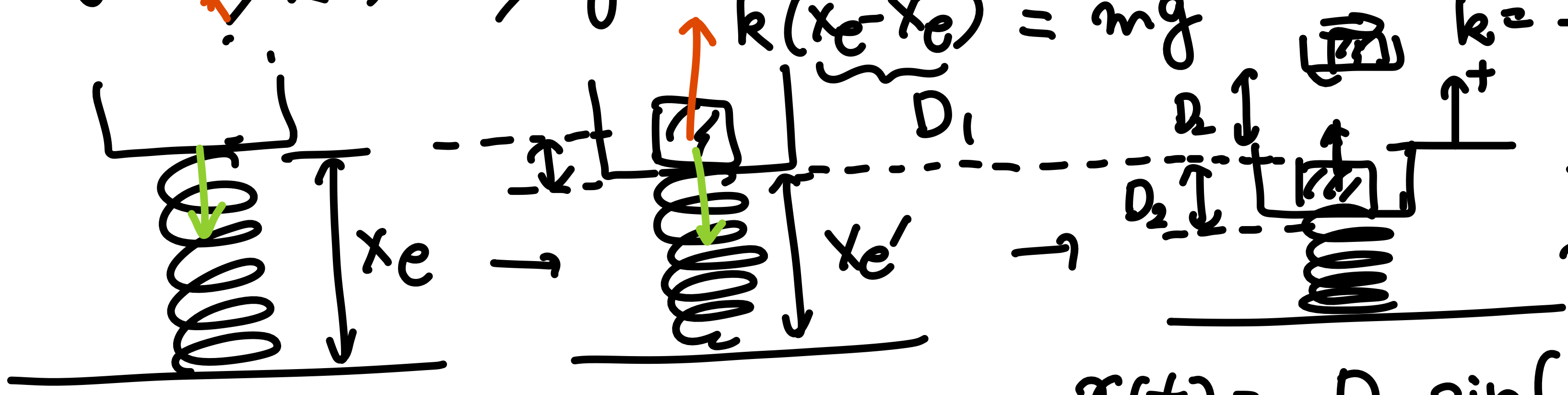


$$-k\left(x+\cancel{\frac{mg}{k}}\right)+\cancel{mg} = m\ddot{x} \Rightarrow m\ddot{x}+kx=0$$

(Ex)

$$k(\underbrace{X_e-X_e'}_{D_1}) = mg$$

$$k=\frac{mg}{D_1} \rightarrow \omega_0=\sqrt{\frac{k}{m}} = \sqrt{\frac{g}{D_1}}$$



$x(0) = -D_2$
$v(0) = 0$

$$x(t) = D_2 \sin\left(\omega_0 t - \frac{\pi}{2}\right)$$
$$= -D_2 \cos(\omega_0 t) = -D_2 \cos\left(\sqrt{\frac{g}{D_1}}t\right)$$

$$x = -D_2 \cos\left(\sqrt{\frac{g}{D_1}}\,t\right) \longrightarrow \dot{x} = D_2\sqrt{\frac{g}{D_1}}\sin\left(\sqrt{\frac{g}{D_1}}\,t\right)$$

(b)

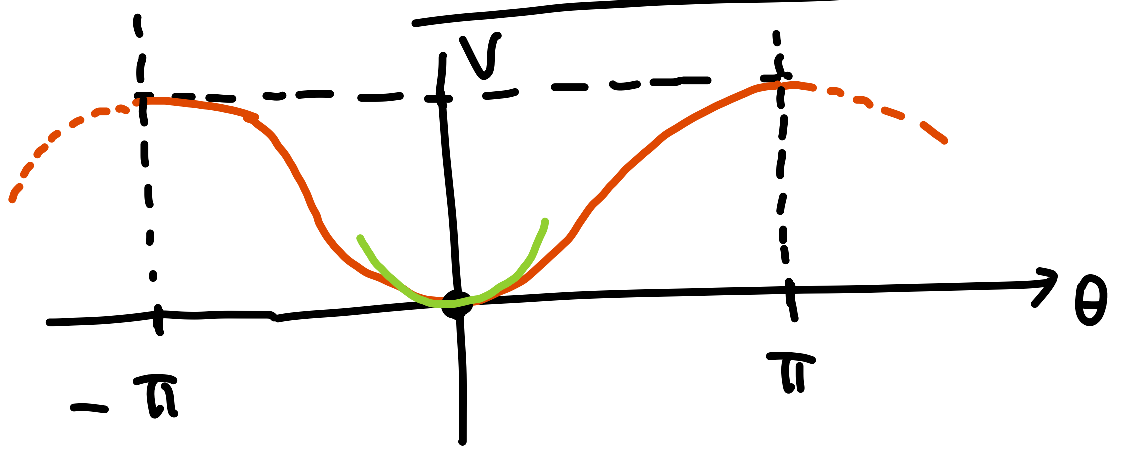
$$x=0 \longrightarrow \quad t_1 = \frac{\pi}{2}\sqrt{\frac{D_1}{g}} \qquad \underline{\dot{x}(t_1) = D_2\sqrt{\frac{g}{D_1}}}$$

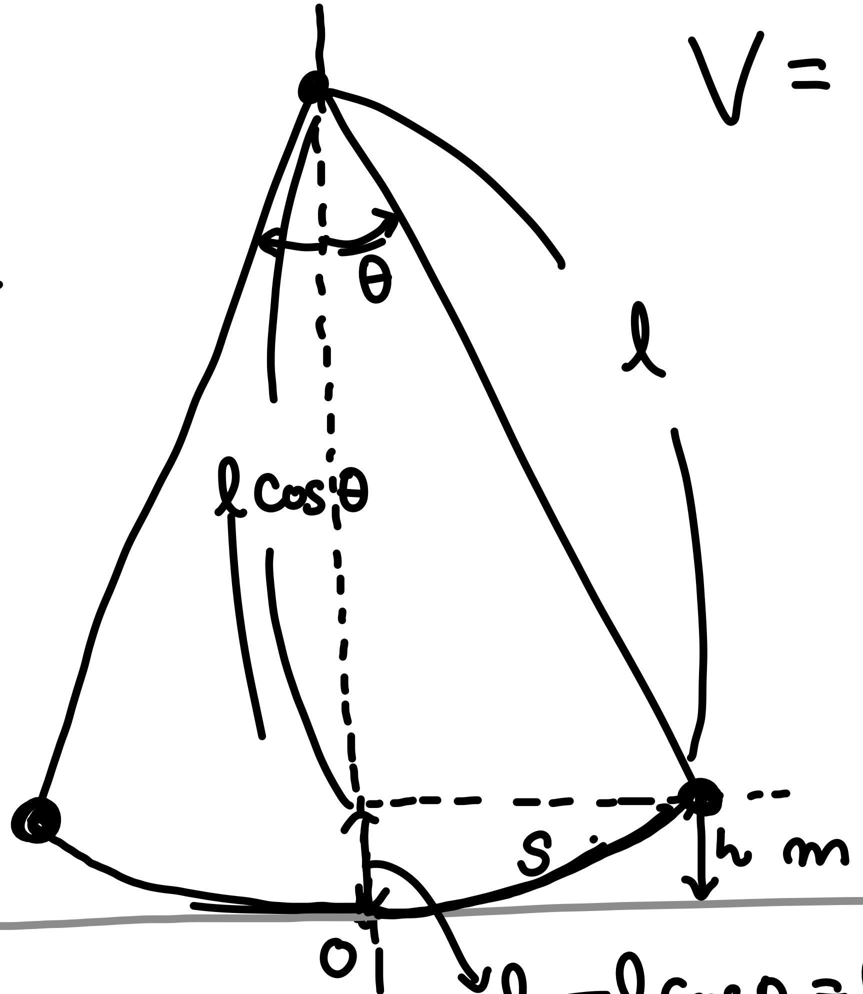
(c) $$\ddot{x} = D_2\frac{g}{D_1}\cos\left(\sqrt{\frac{g}{D_1}}\,\underset{2t_1}{\uparrow}t\right) = -\frac{D_2}{D_1}g \;/\!/$$

[Ex 3.2-2] 단진자

$$V = \frac{1}{2}kx^2$$

$$\underline{V = mgl(1-\cos\theta)}$$





$$l - l\cos\theta = l(1-\cos\theta)$$

$$\sin\theta \approx \theta$$

$$\cos\theta \cong 1 - \frac{\theta^2}{2} \Rightarrow V \cong \frac{1}{2}\underbrace{mgl}_{k}\theta^2$$

$\theta = 0$

$$ml\frac{d^2\theta}{dt^2} = -\frac{dV}{dS} = -\frac{1}{l}\frac{dV}{d\theta} \left(S = l\theta \quad \frac{dS}{dt} = l\frac{d\theta}{dt} \quad \frac{d^2S}{dt^2} = l\frac{d^2\theta}{dt^2} = a\right)$$

$$= -mg\theta \Rightarrow \boxed{\ddot{\theta} + \overset{\omega_0^2}{\left(\frac{g}{l}\right)}\theta = 0} \quad \text{(cf)} \quad \ddot{x} + \overset{\omega_0^2}{\left(\frac{k}{m}\right)}x = 0$$

$$\therefore \quad \theta(t) = \sqrt{\theta_0^2 + \frac{\Omega(0)^2}{\omega_0^2}}\ \sin(\omega_0 t + \phi_0)$$

$$\phi_0 = \tan^{-1}\left(\frac{\theta_0 \omega_0}{\Omega(0)}\right)$$

$$\Omega(0) = \dot{\theta}(0)$$

$$\omega_0^2 = \frac{g}{l}$$

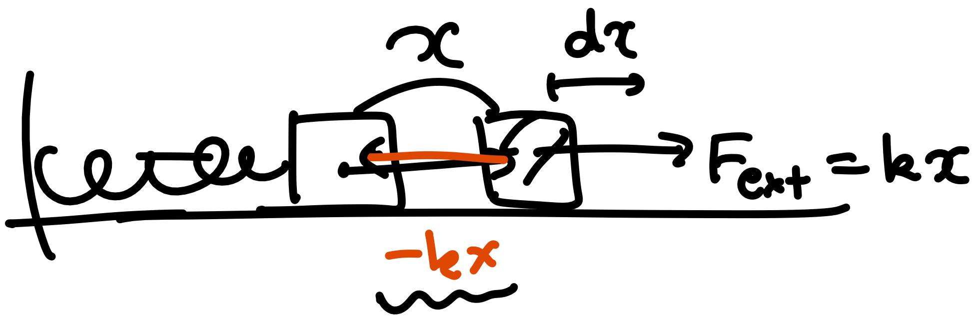
$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}} \leftarrow$$ m에 무관.

일반적 θ값에 대해서는 T는 m에 따라 달라짐

$\therefore\ \theta \ll 1$ 만 성립

3.3. 일 :

$$W = \int_0^x \underset{\text{외부힘}}{F_{ext}}\, dx$$



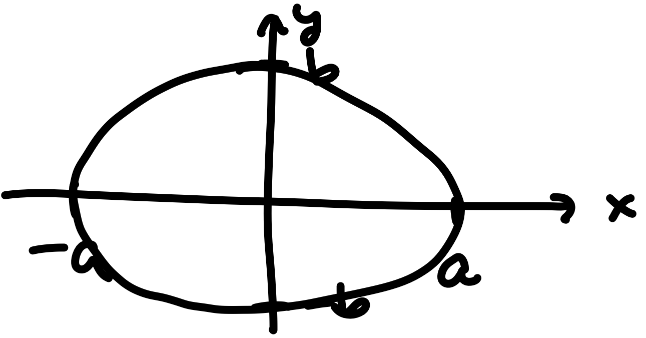
$$= \int_0^x kx\, dx = \frac{1}{2}kx^2 = V(x)$$

용수철의 힘 $\underline{-kx = -\frac{dV}{dx} = F}$

총 역학적 E $= T + V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 =$ 일정

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = \dot{x}(\underline{m\ddot{x} + kx}) = 0$$

(cf) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

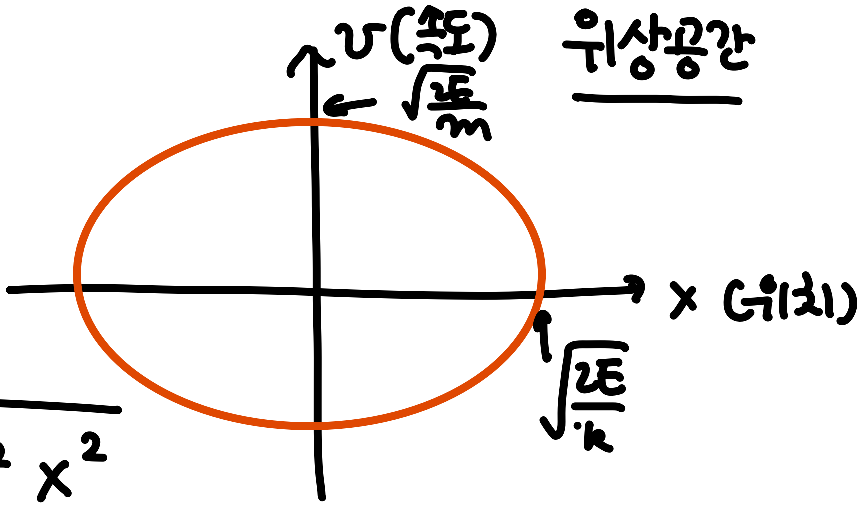


$$E=\frac{1}{2}m\dot{x}^2+\underbrace{\frac{1}{2}kx^2}_{V(x)} \longrightarrow \frac{v^2}{\left(\sqrt{\frac{2E}{m}}\right)^2}+\frac{x^2}{\left(\sqrt{\frac{2E}{k}}\right)^2}=1$$

$\uparrow$

$\downarrow$

$$\boxed{m\ddot{x}+kx=0}$$



$$\frac{dx}{dt}=\dot{x}=\pm\sqrt{\left(E-V(x)\right)\frac{2}{m}} \quad =\pm\sqrt{\frac{2E}{m}-\omega_0^2x^2}$$

$$\pm\int_{x_0}^{x}\frac{dx}{\sqrt{\frac{2E}{m}-\omega_0^2x^2}}=\int_0^t dt = t$$

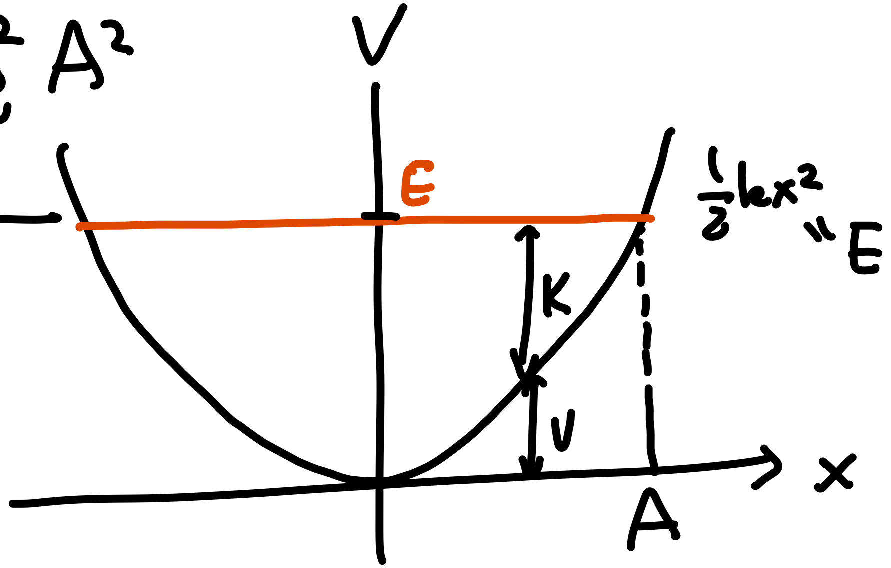
$$=\int_{x_0}^{x}\frac{dx}{\sqrt{\frac{2E}{m}}\sqrt{1-\underbrace{\frac{m\omega_0^2}{2E}}_{\frac{1}{A^2}}x^2}}=\underbrace{\sqrt{\frac{\cancel{m}}{2\cancel{E}}}\left(-\sqrt{\frac{2\cancel{E}}{\cancel{m}\omega_0^2}}\right)}\left[\cos^{-1}\left(\frac{x}{A}\right)-\cos^{-1}\left(\frac{x_0}{A}\right)\right]$$

(cf) $$\boxed{\int\frac{dx}{\sqrt{1-\underbrace{a^2x^2}_{\cos^2\theta}}}=\int\frac{\left(-\frac{1}{a}\right)\cancel{\sin\theta}\,d\theta}{\cancel{\sin\theta}}=-\frac{1}{a}\theta \qquad \underline{ax=\cos\theta}\rightarrow\theta=\cos^{-1}(ax) \qquad a\,dx=-\sin\theta\,d\theta}$$

$$\therefore\ t=\pm\frac{1}{\omega_0}\left[\cos^{-1}\left(\frac{x}{A}\right)-\cos^{-1}\left(\frac{x_0}{A}\right)\right]$$

$$\frac{x}{A}=\cos\left[\cos^{-1}\left(\frac{x}{A}\right)\right]=\cos\left[\underbrace{\cos^{-1}\left(\frac{x_0}{A}\right)}_{\pm\phi_0}\pm\omega_0t\right]=\cos(\omega_0t+\phi_0)$$

$$\frac{m\omega_0^2}{2E} = \frac{1}{A^2} \rightarrow E = \underbrace{\frac{1}{2}\underbrace{m\omega_0^2}_{k}}\, A^2$$



$$E = \frac{1}{2}kA^2$$

[Ex 3.3.2] 평균 운동, 위치, 총 에너지.

$$\underset{V}{\langle K \rangle} = \frac{1}{T_0}\int_0^{T_0} \underset{\uparrow V}{K(t)}\, dt = \frac{1}{T_0}\int_0^{T_0} \frac{1}{2} m \dot{x}^2 dt$$

$$x(t) = A\sin(\omega_0 t + \phi_0)$$

$$\dot{x}(t) = \omega_0 A\cos(\omega_0 t + \phi_0)$$

$$= \frac{1}{\overset{\frac{2\pi}{\omega_0}\nearrow}{T_0}}\int_{\phi_0 \to 0}^{2\pi+\phi_0 \to 2\pi} \frac{1}{2} m\, \omega_0^{\cancel{2}} A^2 \cos^2(\underbrace{\omega_0 t + \phi_0}_{\theta}) \frac{d\theta}{\cancel{\omega_0}} = \frac{1}{2}\frac{\cancel{\pi}}{\cancel{2\pi}} m\omega_0^2 A^2$$

$$d\theta = \omega_0 dt$$

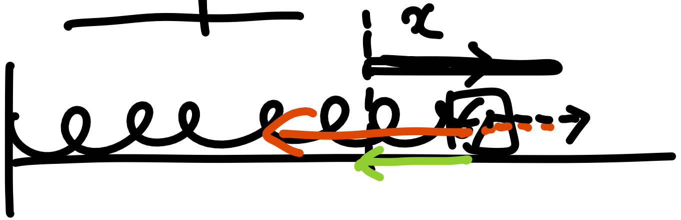
$$\int_0^{2\pi} \underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}\, d\theta = \pi$$

$$\int_0^{2\pi} \underbrace{\sin^2\theta}_{\frac{1-\cos 2\theta}{2}}\, d\theta = \pi$$

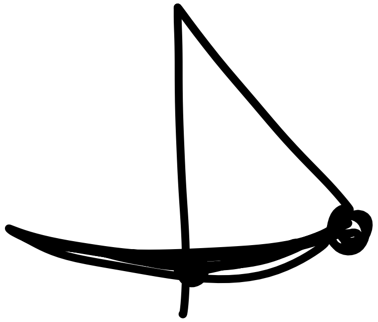
$$\langle E \rangle = \frac{1}{2} m\omega_0^2 A^2, \quad \langle K \rangle = \frac{1}{4} m\omega_0^2 A^2 = \frac{E}{2}$$

$$\langle V \rangle = \frac{1}{T_0}\int_0^{T_0} \frac{1}{2}\underbrace{m\omega_0^2}_{k} x^2 dt = \frac{1}{T_0}\int_0^{2\pi} \frac{1}{2} m\omega_0^2 A^2 \sin^2(\underbrace{\omega_0 t + \phi_0}_{\theta}) \frac{d\theta}{\omega_0} = \frac{1}{4} m\omega_0^2 A^2 = \frac{E}{2}$$

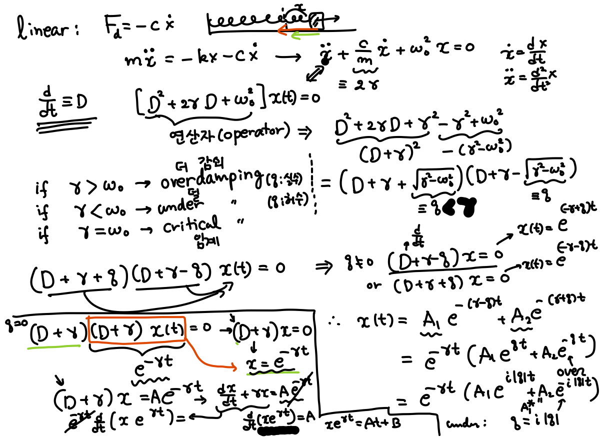
§3.4. Damped H.O.



linear $F_d = -c\underset{\sim}{\dot{x}}$



$$m\ddot{x} = -kx - c\dot{x}$$



linear: $F_d = -c\dot{x}$

$m\ddot{x} = -kx - c\dot{x} \longrightarrow \ddot{x} + \underbrace{\frac{c}{m}}_{\equiv 2\gamma}\dot{x} + \omega_0^2 x = 0$

$\dot{x} = \frac{dx}{dt}$

$\ddot{x} = \frac{d^2}{dt^2}x$

$\frac{d}{dt} \equiv D$

$\underbrace{[D^2 + 2\gamma D + \omega_0^2]}_{\text{연산자(operator)}}x(t) = 0 \Rightarrow \underbrace{D^2 + 2\gamma D + \gamma^2}_{(D+\gamma)^2}\underbrace{-\gamma^2 + \omega_0^2}_{-(\gamma^2-\omega_0^2)}$

$= (D + \gamma + \underbrace{\sqrt{\gamma^2-\omega_0^2}}_{\equiv q})(D + \gamma - \underbrace{\sqrt{\gamma^2-\omega_0^2}}_{\equiv q})$

if $\gamma > \omega_0$ → over(더)damping(감쇠) (q : 실수)

if $\gamma < \omega_0$ → under(덜) " (q : 허수)

if $\gamma = \omega_0$ → critical(임계) "

$(D+\gamma+q)(D+\gamma-q)x(t) = 0 \Rightarrow q \neq 0$ $(D+\gamma-q)x = 0 \to x(t) = e^{(-\gamma+q)t}$

or $(D+\gamma+q)x = 0 \to x(t) = e^{(-\gamma-q)t}$

$q=0$ $\underline{(D+\gamma)}\underbrace{\boxed{(D+\gamma)\ x(t)}}_{e^{-\gamma t}} = 0 \to (D+\gamma)x = 0 \to x = e^{-\gamma t}$

$(D+\gamma)x = Ae^{-\gamma t} \to \frac{dx}{dt} + \gamma x = A\cancel{e^{-\gamma t}}$

$\cancel{e^{-\gamma t}}\frac{d}{dt}(xe^{\gamma t}) = \leftarrow \frac{d}{dt}(xe^{\gamma t}) = A$

$xe^{\gamma t} = At + B$

$\therefore x(t) = \underset{\sim}{A_1}e^{-(\gamma-q)t} + \underset{\sim}{A_2}e^{-(\gamma+q)t}$

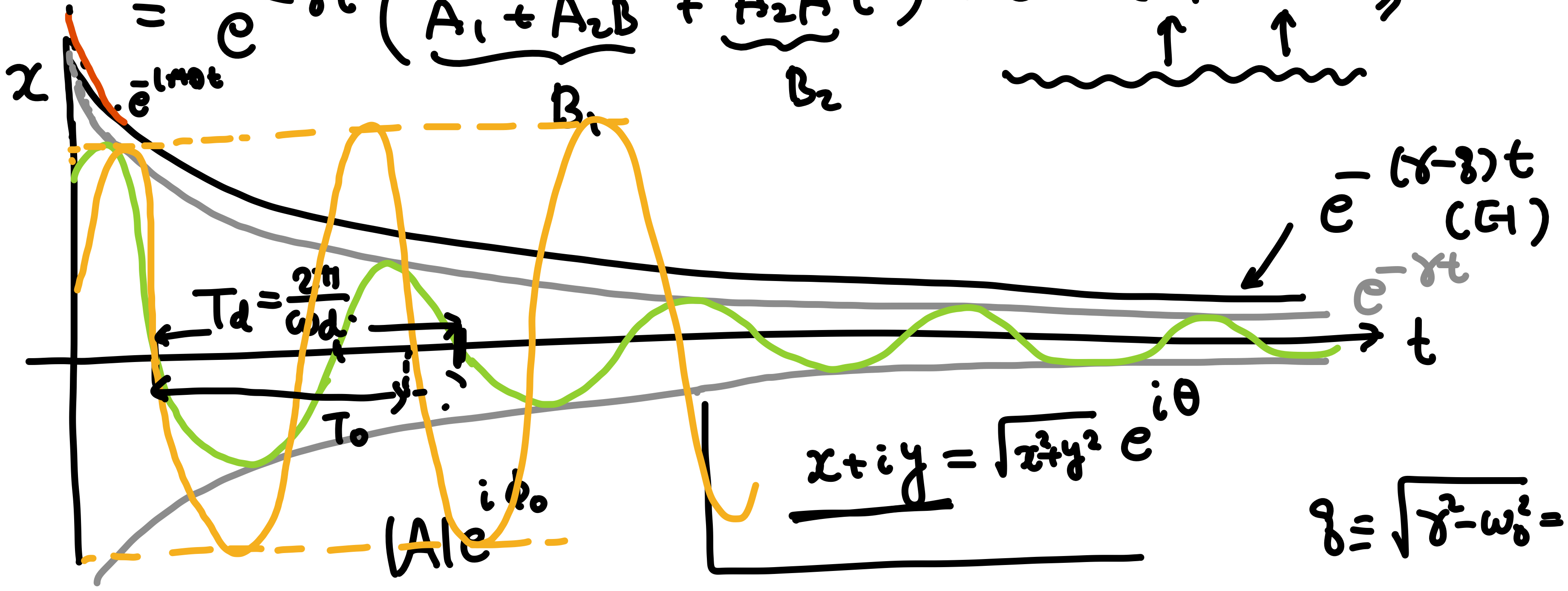
$= e^{-\gamma t}(A_1e^{qt} + A_2e^{-qt})$ over

$= e^{-\gamma t}(A_1e^{i|q|t} + A_2e^{-i|q|t})$ A_1^* "

under: $q = i|q|$

임계: $x = A_1 e^{-\gamma t} + A_2 (At+B) e^{-\gamma t}$

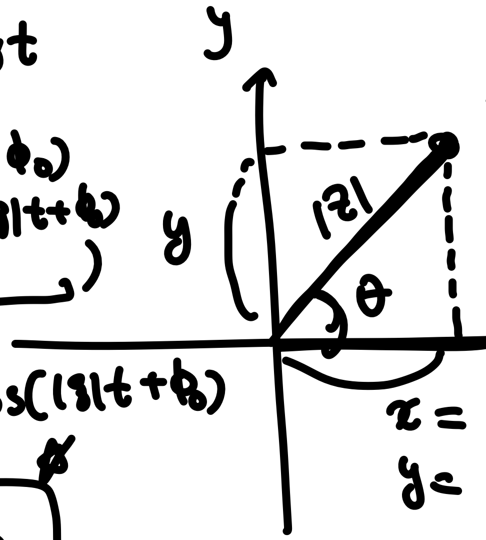
$$= e^{-\gamma t}\Big(\underbrace{A_1 + A_2 B}_{B_1} + \underbrace{A_2 A t}_{B_2}\Big) = e^{-\gamma t}\underbrace{(B_1 + B_2 t)}$$



$$\underline{x+iy} = \sqrt{x^2+y^2}\, e^{i\theta}$$

$$q = \sqrt{\gamma^2-\omega_0^2} = i\underbrace{\sqrt{\omega_0^2-\gamma^2}}_{|q|}$$

덜: $x = e^{-\gamma t}\Big(A_1 e^{i|q|t} + \underbrace{A_1^*}_{|A|e^{-i\phi_0}} e^{-i|q|t}\Big) = |A| e^{-\gamma t}\underbrace{\big(e^{i(|q|t+\phi_0)} + e^{-i(|q|t+\phi_0)}\big)}_{2\cos(|q|t+\phi_0)}$



$z = x+iy = |z|e^{i\theta}$

$|z| = \sqrt{x^2+y^2}$

$x = |z|\cos\theta$

$y = |z|\sin\theta$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$e^x \rightarrow e^{i\theta}$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$\pm)\ e^{-i\theta} = \cos\theta - i\sin\theta$$

$$z = |z|\cos\theta + i|z|\sin\theta$$
$$= |z|(\underbrace{\cos\theta + i\sin\theta}_{e^{i\theta}})$$
$$= |z|e^{i\theta}$$

Taylor

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \cdots$$

$$(a^x)^y = a^{xy}$$

$$e^{i\theta} = 1 + \underset{\sim}{i}\theta + \frac{(i\theta)^2}{2} + \cdots\cdots = \cos\theta + i\sin\theta \,/\!/$$

$$= \underbrace{1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \cdots + \frac{(-1)^n \theta^{2n}}{(2n)!} + \cdots}_{\cos\theta} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \cdots + \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} + \cdots\right)}_{\sin\theta}$$

(ex)

$$2^{\frac{\pi}{2}i} \doteq e^{\frac{i\pi}{2}\ln 2} = \left(\underset{\overset{\shortparallel}{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}}}{e^{\frac{i\pi}{2}}}\right)^{\ln 2} = i^{\ln 2}$$

$$= \cos\left(\frac{\pi}{2}\ln 2\right) + i\sin\left(\frac{\pi}{2}\ln 2\right)$$

$$2^i = e^{i\ln 2} = \cos(\ln 2) + i\sin(\ln 2)$$

$$x(t) = \underbrace{\overset{A}{\boxed{2|A|}}\, e^{-\gamma t}} \cos(\underbrace{|\delta|} t + \phi_0) \leftarrow \text{덜 감쇠}$$

$$= \underset{\sim}{\omega_d} = \sqrt{\omega_0^2 - \gamma^2} \quad (\omega_0 > \gamma)$$

$$T_d = \frac{2\pi}{\omega d} = \frac{2\pi}{\sqrt{\omega_0^2 - \gamma^2}} > \underset{\overset{\shortparallel}{T_0}}{\frac{2\pi}{\omega_0}}$$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2 \rightarrow \dot{E} = \frac{1}{\cancel{2}}m\dot{x}\ddot{x}\cdot\cancel{2} + \frac{1}{\cancel{2}}m\omega_0^2 x\dot{x}\cancel{2}$$

$$= m\dot{x}\big(\underbrace{\ddot{x} + \omega_0^2 x}_{\neq 0 \;=\; -2\gamma\dot{x}}\big) = -\overbrace{2\gamma m}^{c}\dot{x}^2 \qquad \gamma = \frac{c}{2m}$$

$$\therefore \frac{dE}{dt} = -c\dot{x}^2 < 0 \rightarrow E \text{ 는 시간의 감소함수}$$

$$x = \underbrace{Ae^{-\gamma t}}\sin(\omega_d t + \phi_0) \rightarrow \dot{x} = Ae^{-\gamma t}\left(\omega_d\cos(\underline{\omega_d t+\phi_0}) - \gamma\sin(\cdots)\right)$$

$\frac{dE}{dt} = -c\dot{x}^2 \longrightarrow$ 1 주기 $T_d = \frac{2\pi}{\omega_d}$ 당 줄어드는 E : $\Delta E = \int_0^{T_d}\left(\frac{dE}{dt}\right)dt$

$$\Delta E = \int_0^{T_d}(-c)A^2\underbrace{e^{-2\gamma t}}\left(\omega_d^2\cos^2(\underset{\omega_d t+\phi_0}{\uparrow}) + \gamma^2\sin^2(\ ) - 2\gamma\omega_d\cos(\ )\sin(\ )\right)$$

If $\gamma \ll 1$ $\approx -\frac{c}{2}A^2 e^{-2\gamma t}\underbrace{\frac{2\pi}{\omega_d}}_{T_d}(\underset{(\omega_0^2-\cancel{\gamma^2})}{\omega_d^2} + \gamma^2)$ $= -\frac{\overset{2m\gamma=\frac{m}{\tau}}{c}}{2}A^2\omega_0^2 T_d \underbrace{e^{-2\gamma t}}_{e^{-\frac{t}{\tau}}}$

$e^{-2\boxed{\gamma T_d}} \approx 1$ ($\gamma T_d \ll 1$)

$$\boxed{\Delta E = -\frac{m}{2\tau}A^2\omega_0^2 T_d e^{-\frac{t}{\tau}}}$$ $\left(2\gamma = \frac{1}{\tau}\right)$

$$\left(\int_0^{T_d}\cos^2(\underbrace{\omega_d t+\phi_0}_{\theta})\,dt = \int_0^{2\pi}\underset{\frac{1+\cos 2\theta}{2}}{\cos^2\theta}\,\frac{d\theta}{\omega_d} = \frac{\pi}{\omega_d}\right)$$

$d\theta = \omega_d\,dt$

$$Q = 2\pi\frac{E(t)}{\Delta E} = 2\pi\frac{\cancel{\tfrac{1}{2}}m\cancel{\omega_0^2}\cancel{A^2}\cancel{e^{-2\gamma t}}}{\cancel{\tfrac{1}{2}}\frac{\cancel{m}}{\tau}\cancel{\omega_0^2}\cancel{A^2}T_d\cancel{e^{-\frac{t}{\tau}}}} = \frac{\tau}{T_d}\cdot 2\pi = \omega_d\tau = \frac{\omega_d}{2\gamma}$$

[Ex 3. 4.1] 예제1 → $\gamma = \frac{c}{2m} = \omega_0$ $T_0 = 1s = \frac{2\pi}{\omega_0}$

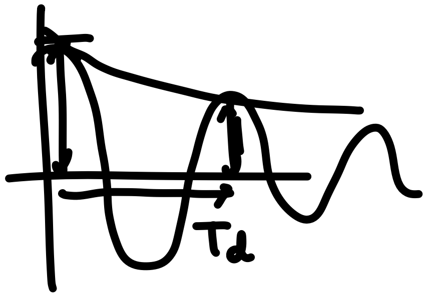
$$x(t) = e^{-\overset{=2\pi}{\gamma} t}(B_1 + B_2 t) = e^{-2\pi t}(\underbrace{B_1}_{x_0} + B_2 t) = x_0 e^{-2\pi t}(1+2\pi t)$$

$x(0) = x_0$, $\dot{x}(0) = 0$ → $x(0) = B_1 = x_0$

$\dot{x}(t) = e^{-2\pi t}\left(B_2 - 2\pi(B_1 + B_2 t)\right)$ → $\dot{x}(0) = B_2 - 2\pi B_1 = 0$

$\therefore B_2 = 2\pi x_0$

$\therefore x(t=1) = x_0 e^{-2\pi}(1+2\pi)$



$Ae^{-\gamma t}$

[Ex 3. 4. 2] $\omega_d = \frac{\omega_0}{2} \rightarrow T_d = \frac{2\pi}{\omega_d} = \frac{4\pi}{\omega_0}$

↓

$\omega_d^2 = \omega_0^2 - \gamma^2 \rightarrow \gamma^2 = \omega_0^2 - \omega_d^2 = \frac{3}{4}\omega_0^2$

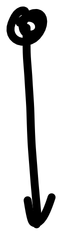
$\therefore \gamma = \frac{\sqrt{3}}{2}\omega_0$

$$\frac{\cancel{A}e^{-\gamma T_d}}{\cancel{A}e^{-\gamma 0}} = e^{-\gamma T_d}$$

$$= e^{-\frac{\sqrt{3}}{2}\cancel{\omega_0}\frac{4\pi}{\cancel{\omega_0}}}$$

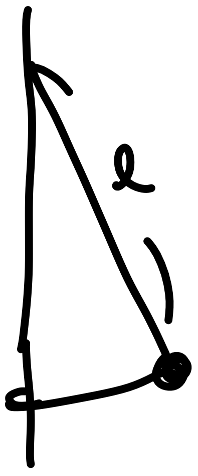
$$= e^{-2\pi\sqrt{3}}$$

[Ex 3. 4.3]



$v_t = \frac{mg}{c_1}$

흥기

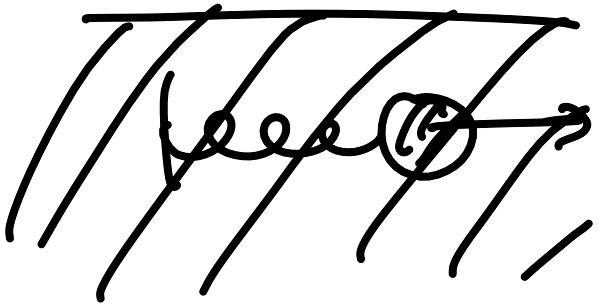


$\gamma = \frac{c_1}{2m} = \frac{\frac{mg}{v_t}}{2m} = \frac{g}{2v_t} = 0.163(s^{-1})$

↑ 30m/s

$\omega_0 = \sqrt{\frac{g}{l}} \longrightarrow T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{\frac{g}{l} - \gamma^2}}$

[Ex 3.4.4]



viscosity $\eta = 10^{-3}$ N s/m²
of water

$\gamma = \frac{1}{2\tau}$

$C = 6\pi\eta r = 5\times10^{-5}$ N s/m

$r = 0.00265$ m

$$\frac{A}{A_0} = e^{-\gamma t} = e^{-\frac{t}{2\tau}} = \frac{1}{2} \longrightarrow t = 2\tau \ln 2$$

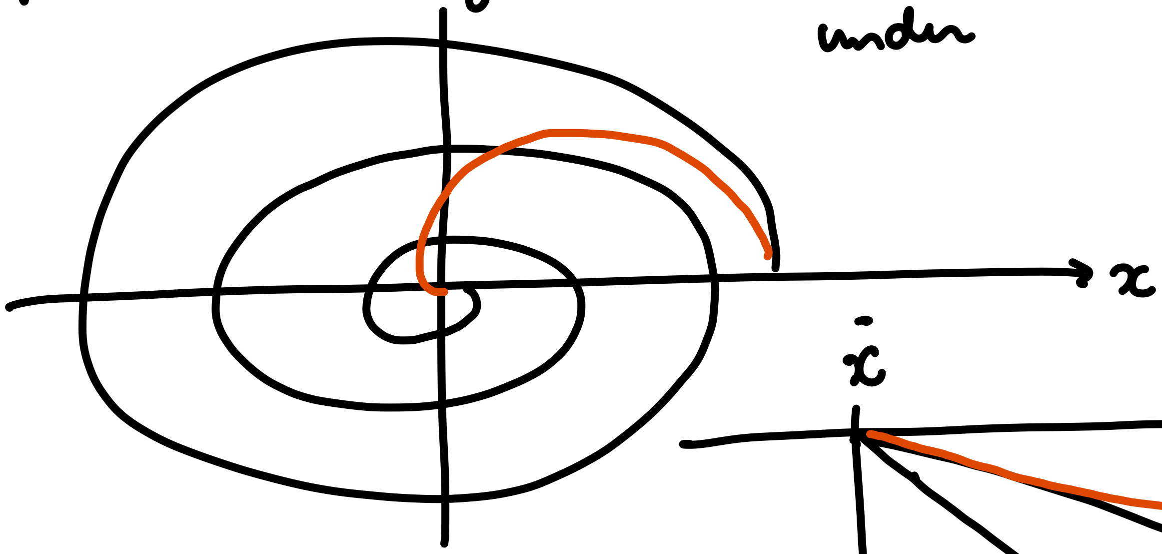
$T_d = \frac{2\pi}{\omega_d}$

$$n = \frac{t}{T_d} = \frac{2\tau \ln 2}{\frac{2\pi}{\omega_d}} = \frac{\omega_d \tau \ln 2}{\pi}$$

$$= Q\frac{\ln 2}{\pi}$$

---

damped H.O. 의 위상공간

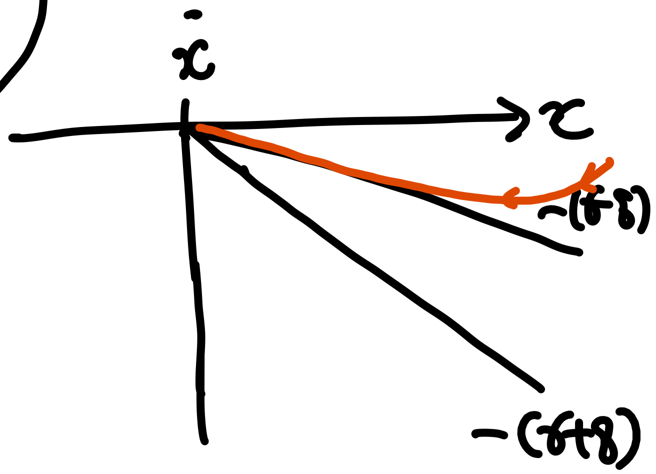


under

over

$$x(t) = \underbrace{A_1 e^{-(\gamma-\delta)t} + A_2 e^{-(\gamma+\delta)t}}_{e^{-\gamma t}(A_1 e^{\delta t} + A_2 e^{-\delta t})}$$

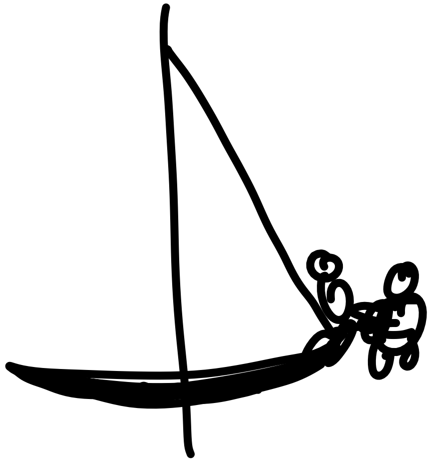
$$\dot{x}(t) = -\gamma x(t)$$
$$+ \delta e^{-\gamma t}(A_1 e^{\delta t} - A_2 e^{-\delta t})$$



$$\dot{x} + (\gamma-\delta)x = (\gamma-\delta)x_0 e^{-(\gamma+\delta)t}$$

$$\dot{x} + (\gamma+\delta)x = (\gamma+\delta)x_0 e^{-(\gamma-\delta)t}$$

3.6. Forced H.O.



resonance
공명

$$m\ddot{x} = -kx - c\dot{x} + \underbrace{F_{ext}(t)}_{F_0 \cos \omega t}$$

특수해 x_p

steady-state 해

일반해 $x = x_p + \underbrace{A e^{-\gamma t} \sin(\omega_d t + \phi_0)}_{\underset{x_0(t)}{\shortparallel}}$

$$m\ddot{x}_p + m\ddot{x}_0 = -kx_p - kx_0 - c\dot{x}_p - c\dot{x}_0 + F_{ext}$$

$$\underbrace{m\ddot{x}_p + kx_p + c\dot{x}_p - F_{ext}}_{\underset{0}{\shortparallel}} + \underbrace{m\ddot{x}_0 + c\dot{x}_0 + kx_0}_{\underset{0}{\shortparallel}} = 0$$

t가 커지면

x_p는 작아지지 않지만 x_0는 → 0. ⇒ 특수해만 중요해짐

특수해 : $(c=0)$

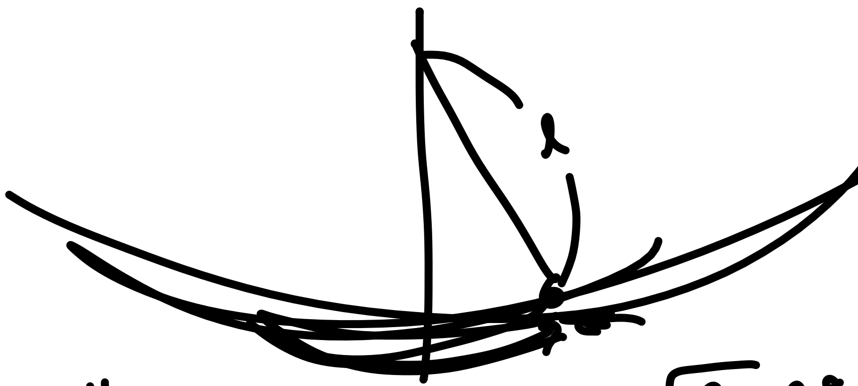
$x = A \cos \omega t$ 라고 가정 → $\dot{x} = -\omega A \sin \omega t$

$$\ddot{x} = -\omega^2 A \cos \omega t$$

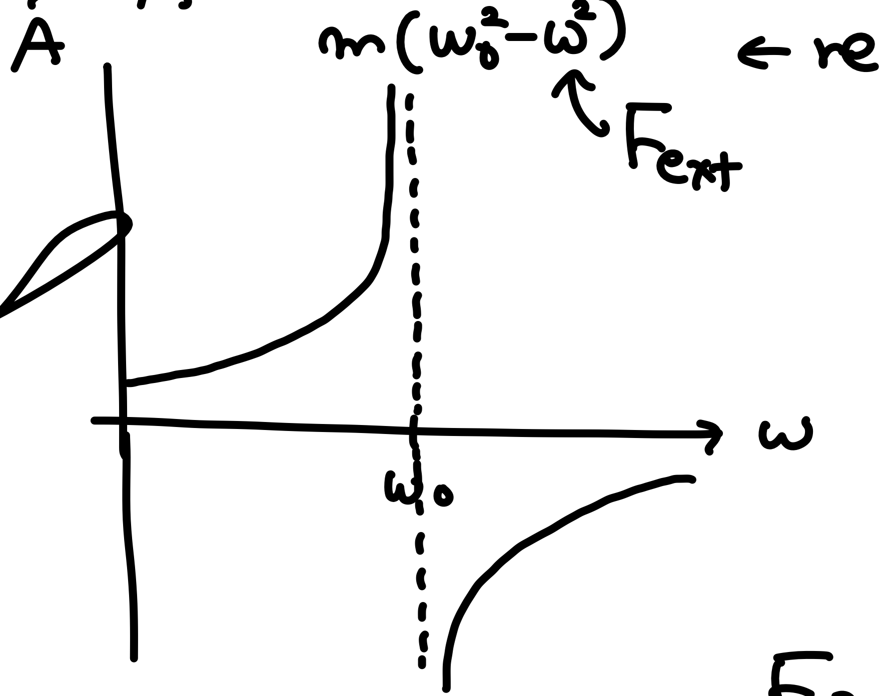
$$m\ddot{x} + kx = F_0 \cos \omega t \rightarrow (-m\omega^2 A + kA)\cos \omega t = F_0 \cos \omega t$$

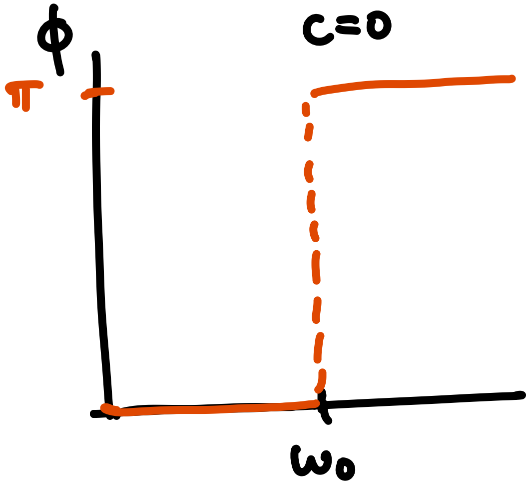
$$\therefore A = \frac{F_0}{k - m\omega^2}$$

$$k = m\omega_0^2 \rightarrow A = \frac{F_0}{m(\omega_0^2-\omega^2)}$$ ← resonance



$$T = 2\pi\sqrt{\frac{g \leftarrow 9.8}{l}}$$





$$\overset{\text{실수부}}{\downarrow}$$
$$\mathrm{Re}(x+iy) = x$$
$$\mathrm{Im}(x+iy) = y$$
$$\underset{\text{허수부}}{\uparrow}$$

$$x_p(t) = A\cos\omega t = \frac{F_0}{m(\omega_0^2-\omega^2)}\cos\omega t, \quad \omega<\omega_0$$

$$= \frac{F_0}{m(\omega^2-\omega_0^2)}\cos(\omega t+\underline{\underline{\overset{\downarrow}{\pi}}}) \quad \omega>\omega_0$$

$$F_{ext} = \overset{\substack{\text{실수부}\\ \downarrow}}{\mathrm{Re}}\left[F_0 \underbrace{e^{i\omega t}}_{\cos\omega t + i\sin\omega t}\right]$$

$$m\ddot{x} + c\dot{x} + kx = F_0 e^{i\omega t} \rightarrow \text{해를 } x(t) = \overset{\substack{\text{양의 실수}\\ \downarrow}}{A} e^{i(\omega t-\overset{\substack{\text{실수}\\ \downarrow}}{\phi})}$$

라고 가정

$$\dot{x} = i\omega A e^{i(\omega t-\phi)}$$

$$\ddot{x} = -\omega^2 A e^{i(\omega t-\phi)}$$

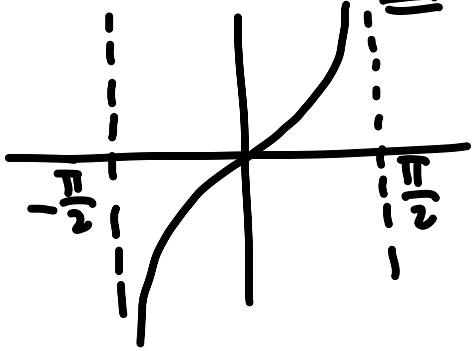
$$(-m\omega^2 + ic\omega + k)Ae^{i(\omega t-\phi)} = F_0 e^{i\omega t}$$

$$\therefore \quad Ae^{-i\phi} = \frac{F_0}{-m\omega^2+ic\omega+k} \;//$$

$$A e^{-i\phi} = \frac{F_0}{-m\omega^2+ic\omega+\underset{\overset{\shortparallel}{m\omega_0^2}}{k}} \longrightarrow A\left(m(\omega_0^2-\omega^2)+ic\omega\right) = F_0 e^{i\phi} = F_0(\cos\phi+i\sin\phi)$$

$$\begin{cases} m\,\underline{\underline{A}}\,(\omega_0^2-\omega^2) = F_0\cos\underline{\underline{\phi}} \\ c\omega\,\underline{\underline{A}} = F_0\sin\underline{\underline{\phi}} \end{cases}$$

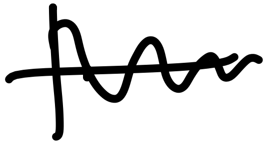
$$\therefore \tan\phi = \frac{c\omega}{m(\omega_0^2-\omega^2)} \rightarrow \omega > \omega_0$$



$$A^2\left(c^2\omega^2 + m^2(\omega_0^2-\omega^2)^2\right) = F_0^{\,2}$$

$$\therefore A(\omega) = \frac{F_0/m}{\underbrace{\sqrt{(\omega_0^2-\omega^2)^2+\underbrace{\frac{c^2}{m^2}}_{4\gamma^2}\omega^2}}_{D}}$$

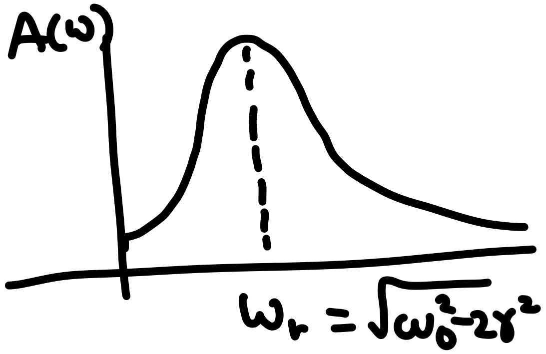
$$\frac{c}{2m} = \gamma$$



$$\frac{d}{d\omega^2}(D(\omega))^2 = 4\gamma^2+2(\omega^2-\omega_0^2) = 0 \longrightarrow \omega_r^2 = \omega_0^2-2\gamma^2 = \omega_d^2-\gamma^2$$

↖ resonance

$$\underline{\omega_0^2 > 2\gamma^2}$$



$$\omega_0^2 \underline{\underline{\leq}} 2\gamma^2$$

$$: \quad \omega_0 = \sqrt{2}\gamma$$

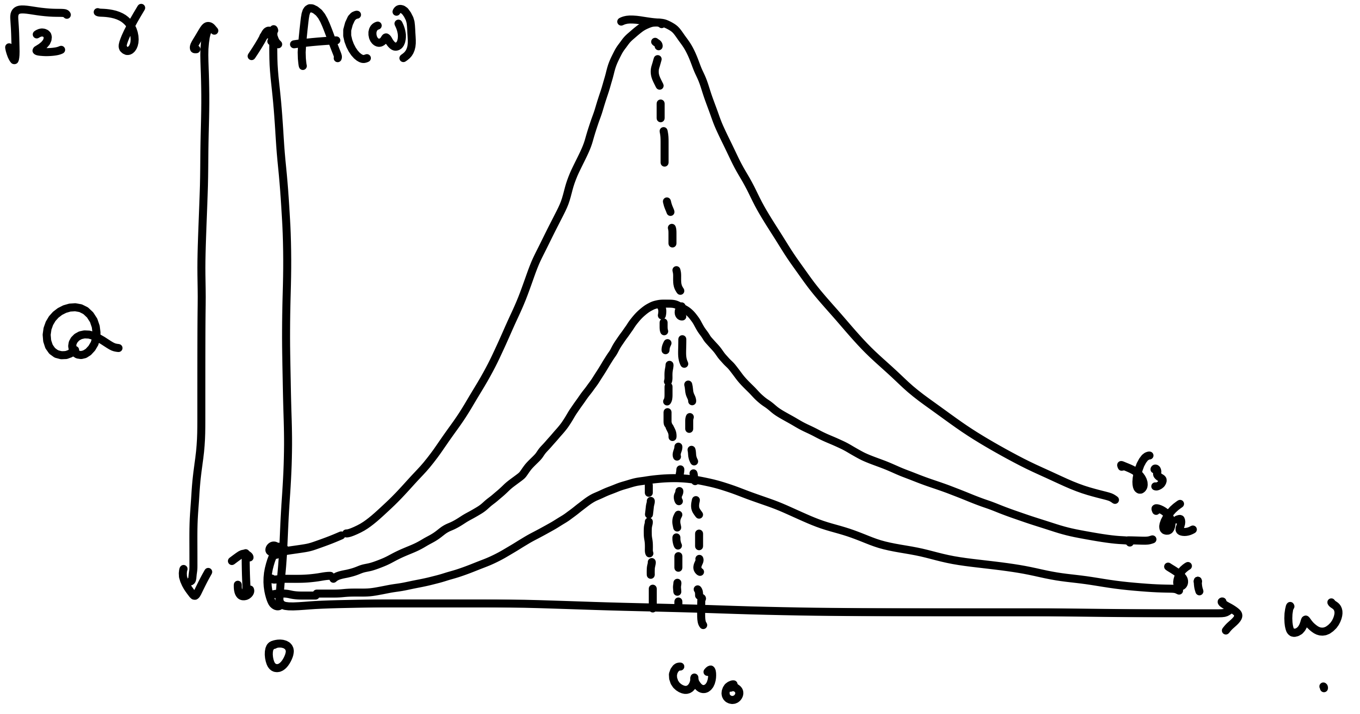
$$A(\omega) = \frac{F_0/m}{\sqrt{\omega^4+\omega_0^4}}$$

$$\sqrt{\omega^4+\omega_0^4}$$

$$\shortparallel$$

$$D = \sqrt{\omega^4 - 2\cancel{\omega_0^2}\omega^2 + \omega_0^4 + 2\underbrace{\cancel{\omega_0^2}}_{4\gamma^2}\omega^2}$$

$\omega_0 > \sqrt{2}\,\gamma$



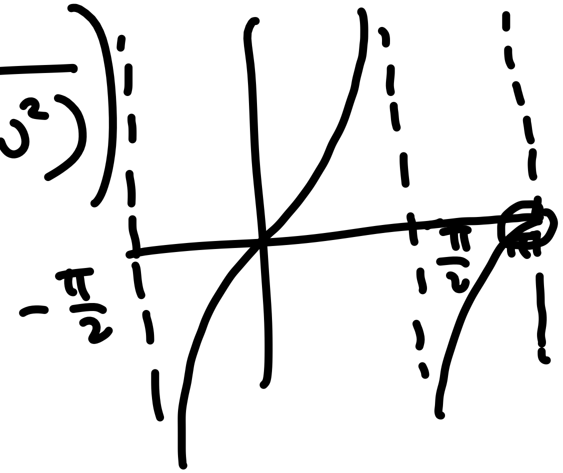
$$\omega_r^2 = \omega_0^2 - 2\gamma^2$$

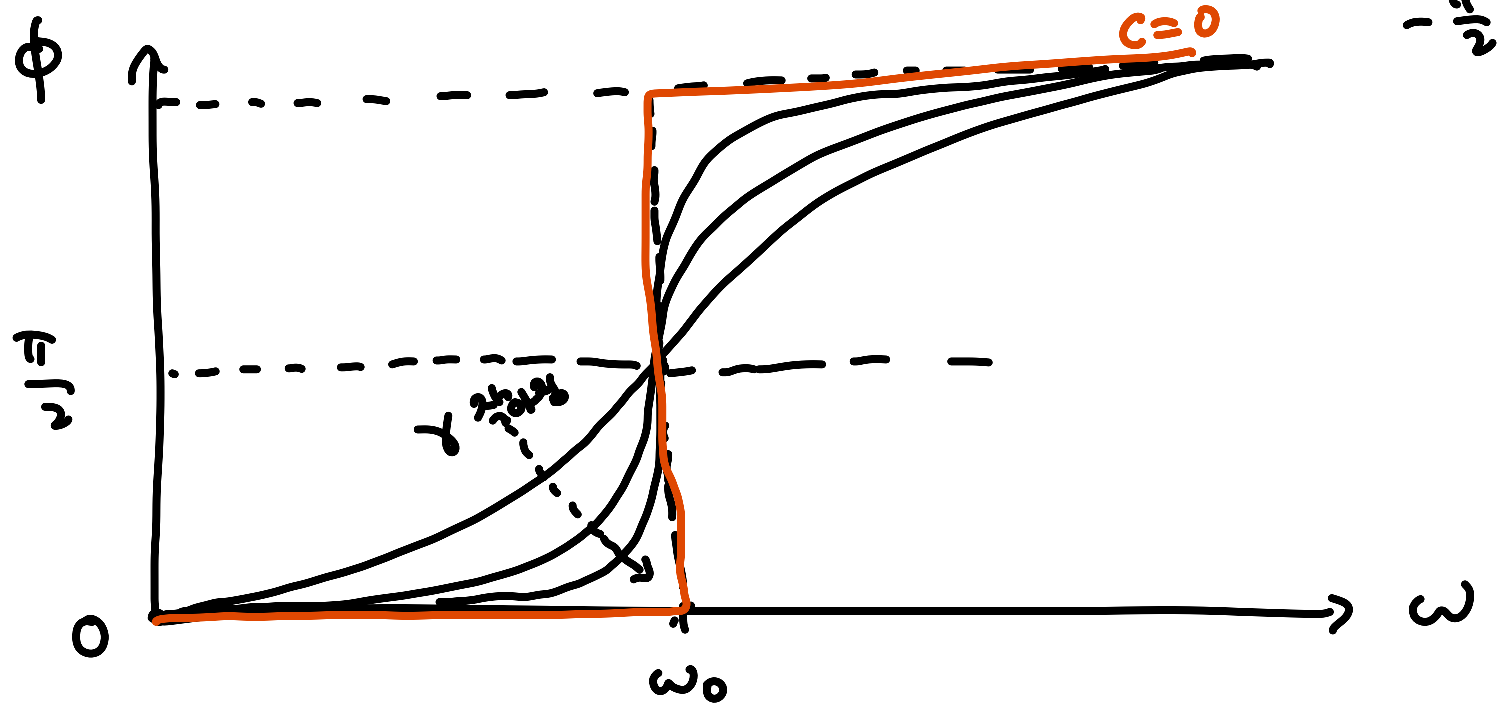
$$\frac{c}{2m} = \gamma$$

$$\gamma_1 > \gamma_2 > \gamma_3 .$$

$\bar{0}$

$$\tan\phi = \frac{c\,\omega}{m(\omega_0^2-\omega^2)} \longrightarrow \phi = \tan^{-1}\left(\frac{2\gamma\,\omega}{(\omega_0^2-\omega^2)}\right)$$





$$A(\omega) = \frac{F_0/m}{\sqrt{\underbrace{(\omega_0^2-\omega^2)^2}_{(2\gamma^2)^2} + \underset{\downarrow}{4\gamma^2\omega^2}}} \rightarrow A_{max} = A(\omega_r)$$

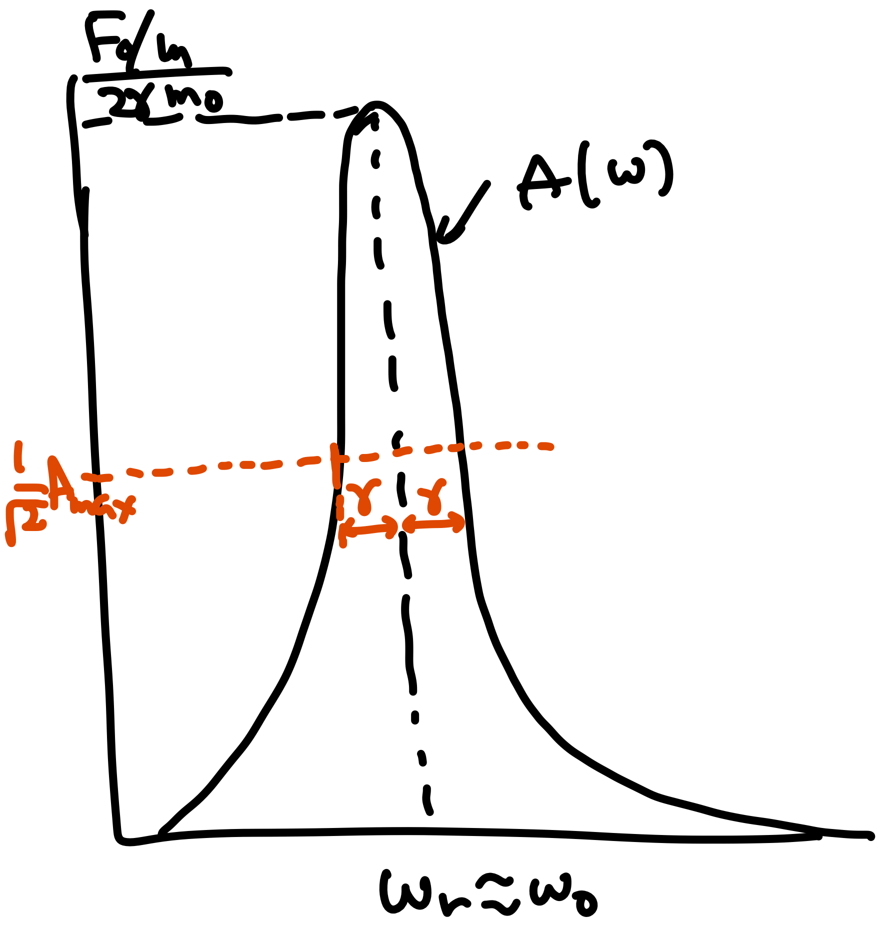
$$\omega_r^2 = \omega_0^2 - 2\gamma^2$$

$$(2\gamma^2)^2 \qquad 4\gamma^2(\omega_0^2-2\gamma^2)$$

$$\overset{\shortparallel}{4\gamma^4} + 4\gamma^2\omega_0^2 - 8\gamma^4$$

$$\boxed{A_{max} = \frac{F_0/m}{2\gamma\sqrt{\omega_0^2-\gamma^2}}}$$

weak damping: $\underbrace{\gamma \ll \omega_0}$ $\qquad \omega_0^2 - \gamma^2 \approx \omega_0^2 \quad \therefore A_{max} \approx \frac{F_0/m}{2\gamma\omega_0}$



$$\omega_0^2 - \omega^2 = (\underbrace{\omega_0 - \omega}_{-\Delta\omega})(\omega_0 + \omega) = -\Delta\omega(2\omega_0 + \Delta\omega)$$

$$=: -\Delta\omega \cdot 2\omega_0$$

$$\omega = \omega_0 + \Delta\omega \qquad \Delta\omega \ll \omega_0 \rightarrow \Delta\omega^2 \ll 1$$

$$4\gamma^2\omega^2 = \underline{\underline{4}}\gamma^2(\underbrace{\omega_0+\Delta\omega}_{\omega_0^2 + \cancel{2\Delta\omega\,\omega_0} + \cancel{\Delta\omega^2}})^2 \approx 4\gamma^2\omega_0^2$$

$$D = \sqrt{4\gamma^2\omega_0^2 + \underset{\overset{\shortparallel}{(\Delta\omega)^2 4\omega_0^2}}{(\Delta\omega\cdot 2\omega_0)^2}} = 2\omega_0\sqrt{\underset{\overset{\shortparallel}{(\omega-\omega_0)^2}}{(\Delta\omega)^2} + \gamma^2}$$

$$\therefore A(\omega) \approx \frac{F_0/m}{2\omega_0\sqrt{(\omega-\omega_0)^2+\gamma^2}} \rightarrow \text{if } \omega = \omega_0 \pm \gamma$$

$$\therefore A(\omega) \approx \frac{F_0/m}{2\omega_0\sqrt{(\omega-\omega_0)^2+\gamma^2}} \rightarrow \text{if } \omega = \omega_0 \pm \gamma \qquad A(\omega_0 \pm \gamma) = \frac{F_0/m}{2\omega_0\sqrt{2\gamma^2}} = \frac{1}{\sqrt{2}} \underbrace{\frac{F_0/m}{2\omega_0\gamma}}_{A_{max}}$$

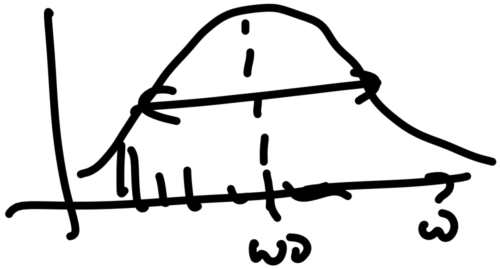
$$\therefore A^2(\underbrace{\omega_0 \pm \gamma}) = \frac{1}{2} A_{max}^2$$

$$\omega_r^2 = \omega_0^2 - 2\gamma^2, \quad \omega_d^2 = \omega_0^2 - \gamma^2$$

$$Q = \frac{\omega_d}{2\gamma} \approx \underbrace{\frac{\omega_0}{2\gamma}}$$

$$2\gamma = \Delta\omega = \frac{\omega_0}{Q}$$

$$\frac{\Delta\omega}{\omega_0} = \frac{\Delta f}{f_0} = \frac{1}{Q} \qquad \omega = 2\pi f$$



$$\phi = \tan^{-1}\left(\frac{2\gamma\omega}{(\omega_0^2-\omega^2)}\right)$$

$$\omega_0 > \omega \rightarrow 0 < \phi < \frac{\pi}{2}$$

$$\omega > \omega_0 \rightarrow \frac{\pi}{2} < \phi < \pi$$

$$\omega = \omega_0 \rightarrow \phi = \frac{\pi}{2}$$

$$\omega \rightarrow 0 \text{ limit}: \; A(\omega\rightarrow 0) = \frac{F_0/m}{\omega_0^2} = \frac{F_0}{m\omega_0^2} = \frac{F_0}{k} \; ; \quad \frac{A_{max}}{A(\omega\rightarrow 0)} = \frac{\frac{\cancel{F_0}/\cancel{m}}{2\cancel{\omega_0}\gamma}}{\cancel{F_0}/\cancel{m}\omega_0^{\cancel{2}}} = \frac{\omega_0}{2\gamma} = Q$$

$$F_{ext} = F_0\cos(\omega t) = F_0 \longrightarrow x = \frac{F_0}{k} \overset{\checkmark}{\longleftrightarrow} x = A\cos(\omega t)$$

$$\big\downarrow \omega\rightarrow 0$$

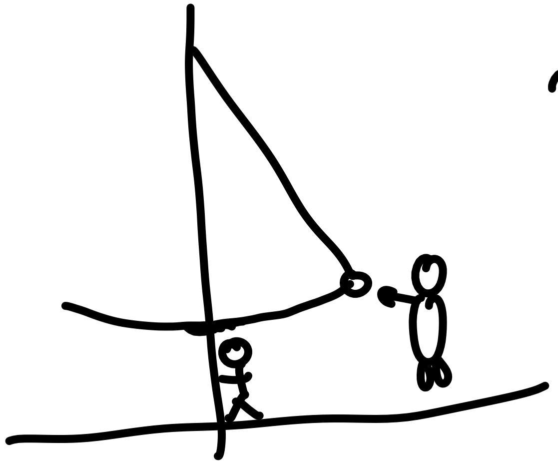
$$\frac{F_0}{k}$$

$$F = F_0 e^{i\omega t} \longrightarrow x = A(\omega)\, e^{i(\omega t - \phi)}$$

실수부 : $F = F_0 \cos(\omega t) \longrightarrow$ steady-state 해 $x = \underline{\underline{A(\omega)}} \cos(\omega t - \underline{\underline{\phi}})$

$$\therefore \tan\phi = \frac{c\omega}{m(\omega_0^2 - \omega^2)} \quad \boxed{A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}}}$$

at resonance $\omega = \omega_r \underset{\text{weak}}{\approx} \omega_0 \longrightarrow \phi(\omega = \omega_0) = \frac{\pi}{2}$

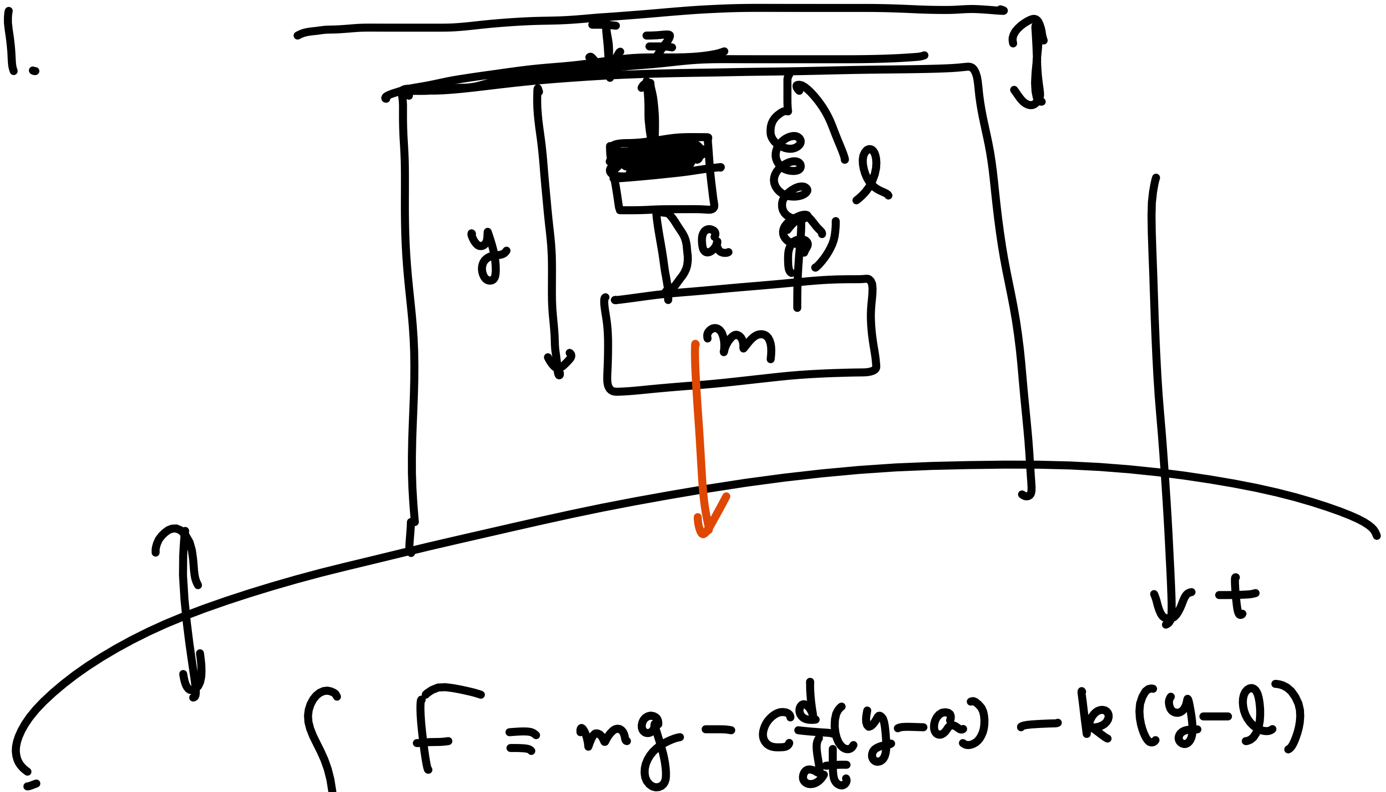


$$x_{res} = A_{max} \cos\left(\omega_0 t - \frac{\pi}{2}\right) = A_{max} \sin(\omega_0 t)$$

$$\dot{x}_{res} = \omega_0 A_{max} \cos(\omega_0 t) \propto F = F_0 \cos(\omega_0 t)$$

$$P = \frac{dW}{dt} = F \cdot \boxed{v}$$

[Ex 3.6.1] 지진계.



$$F = mg - c\frac{d}{dt}(y-a) - k(y-l)$$

$$= mg - c\dot{y} - ky + kl = m(\ddot{y}+\ddot{z})$$

$$\underbrace{mg}_{} - ky + \underbrace{kl}_{} \equiv -kx$$

$$\dot{y} = \dot{x} \quad \ddot{y} = \ddot{x}$$

$$-c\dot{x} - kx = m\ddot{x} + m\ddot{z}$$

$$m\ddot{x} + c\dot{x} + kx = -m\ddot{z}$$

$$z = De^{i\omega t} \rightarrow \ddot{z} = -\omega^2 De^{i\omega t}$$

$$\therefore \quad \ddot{x} + 2\gamma\dot{x} + \underset{\frac{k}{m}=\omega_0^2}{\omega_0^2} x = \underbrace{\omega^2 D}_{F_0} e^{i\omega t} \rightarrow \underline{A(\omega) = \frac{\omega^2 D/m}{\sqrt{(\omega^2-\omega_0^2)^2+4\gamma^2\omega^2}}}$$

[Ex 3.6.2] $\gamma = \frac{\gamma_{crit}}{10} = \frac{\omega_0}{10} \rightarrow \omega_r^2 = \omega_0^2 - 2\gamma^2$

$= \omega_0^2 - \frac{2}{100}\omega_0^2 = \frac{98}{100}\omega_0^2$

$\rightarrow \omega_r = \frac{7\sqrt{2}}{10}\omega_0$

$\xi = 0 = \gamma - \omega_0$

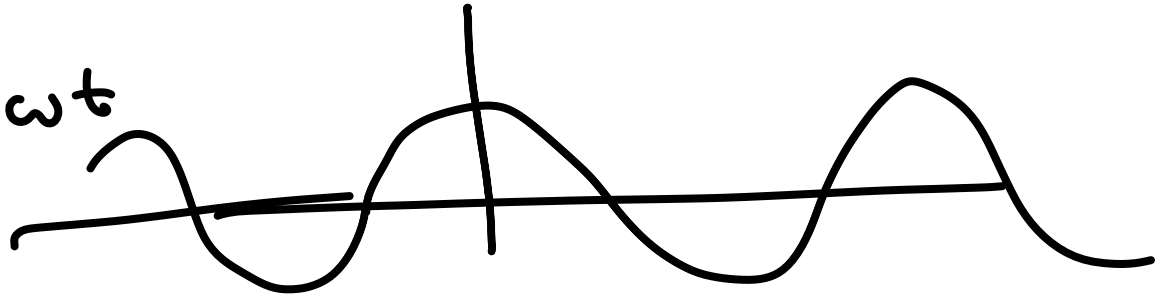
$$Q = \frac{\omega_0}{2\gamma} = \frac{\cancel{\omega_0}}{2 \cdot \frac{\cancel{\omega_0}}{10}} = 5$$

$$\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right) = \tan^{-1}\left(\frac{2\frac{\omega_0}{10}\frac{\omega_0}{2}}{\omega_0^2 - \frac{\omega_0^2}{4}}\right) = \tan^{-1}\left(\frac{\frac{1}{10}}{\frac{3}{4}}\right) = 7.6°$$

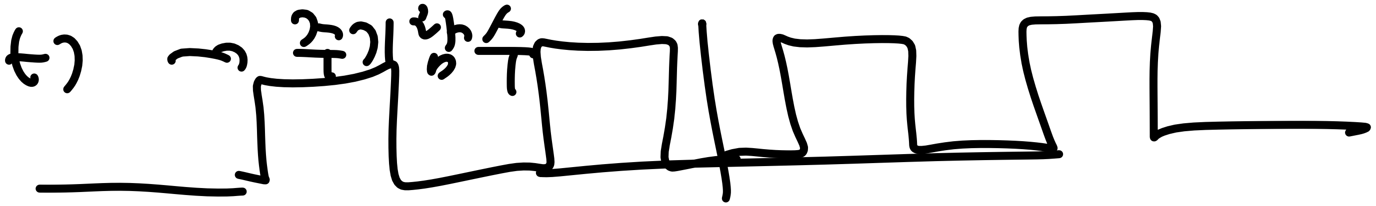
$\omega = \frac{\omega_0}{2}$

$$A\left(\omega = \frac{\omega_0}{2}\right) = \frac{F_0/m}{\sqrt{\left(\frac{\omega_0^2}{4} - \omega_0^2\right)^2 + 4\frac{\omega_0^2}{100}\frac{\omega_0^2}{4}}} = \frac{F_0}{m\omega_0^2}\frac{1}{\sqrt{\frac{9}{16} + \frac{1}{100}}}$$

3.9. $F(t) = F_0 \cos \omega t$

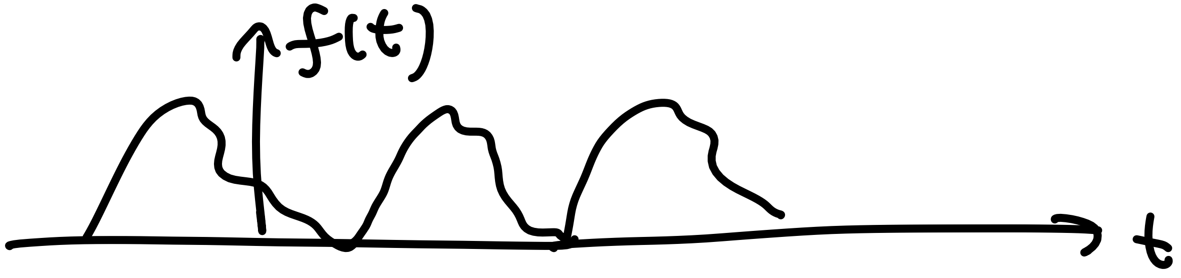


$F(t+T) = F(t) \rightarrow$ 주기함수



Fourier series

$f(t) = f(t+T)$

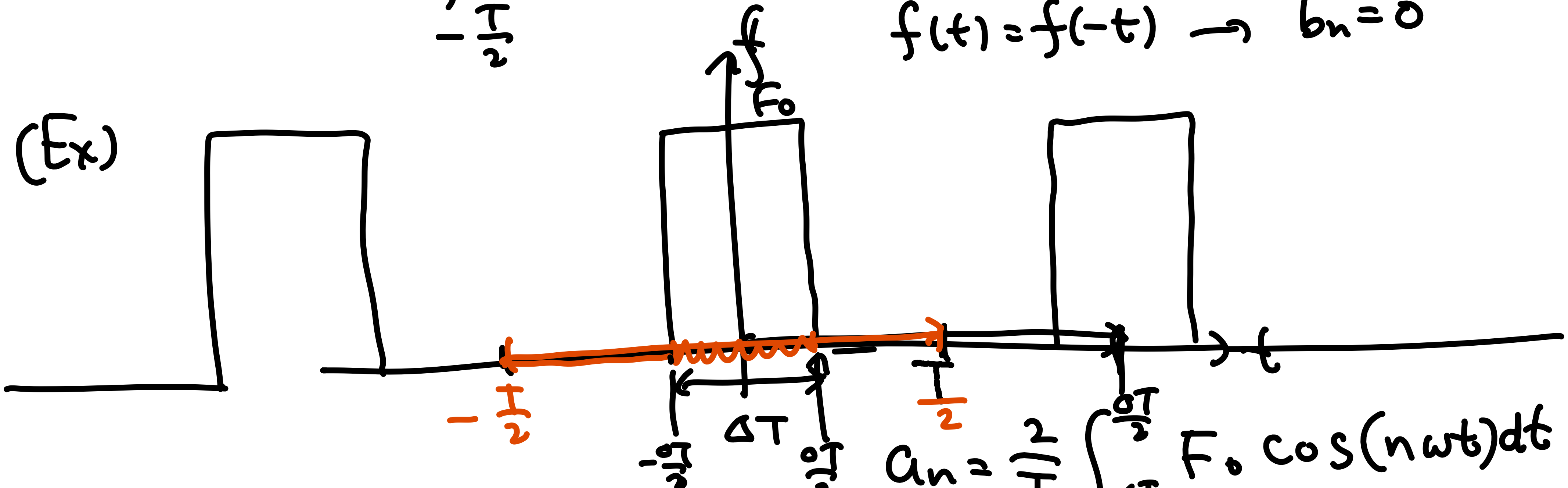


$$\underset{\sim}{f(t)} = \frac{1}{2}\underline{\underline{a_0}} + \sum_{n=1}^{\infty}\left[\underline{\underline{a_n^{(\omega)}}}\cos(n\omega t) + \underline{\underline{b_n^{(\omega)}}}\sin(n\omega t)\right]$$

$$\begin{cases} a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos(n\omega t)\,dt & n=0,1,\cdots \\ b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(n\omega t)\,dt & n=1,2,\cdots \end{cases}$$

$f(t) = f(-t) \rightarrow b_n = 0$

(Ex)



$$a_n = \frac{2}{T}\int_{\frac{\Delta T}{2}}^{\frac{\Delta T}{2}} F_0\cos(n\omega t)\,dt = \frac{2F_0}{T}\frac{1}{n\omega}\left.\sin(n\omega t)\right|_{-\frac{\Delta T}{2}}^{\frac{\Delta T}{2}}$$

$$\Rightarrow a_n = F_0 \frac{2 \sin\left(n\pi \frac{\Delta T}{T}\right)}{n\pi} \qquad a_0 = F_0 \frac{2\Delta T}{T}$$

$$f(t) = \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos(2\omega t) + \cdots$$

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = \frac{f(t)}{m} = \frac{a_0}{2m} + \sum_{n=1}^{\infty} \frac{a_n}{m} \cos(n\omega t)$$

$$\Rightarrow \quad \ddot{x}_n + 2\gamma \dot{x}_n + \omega_0^2 x_n = \underbrace{\frac{a_n}{m}}_{F_0} \cos(\underbrace{n\omega}_{\omega} t)$$

$$x_n = A_n \cos(n\omega t - \phi_n)$$

$$\underline{x = \sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \underset{\uparrow}{A_n} \cos\left(n\omega t - \underset{\uparrow}{\phi_n}\right)}$$

4 장. 3차원 운동

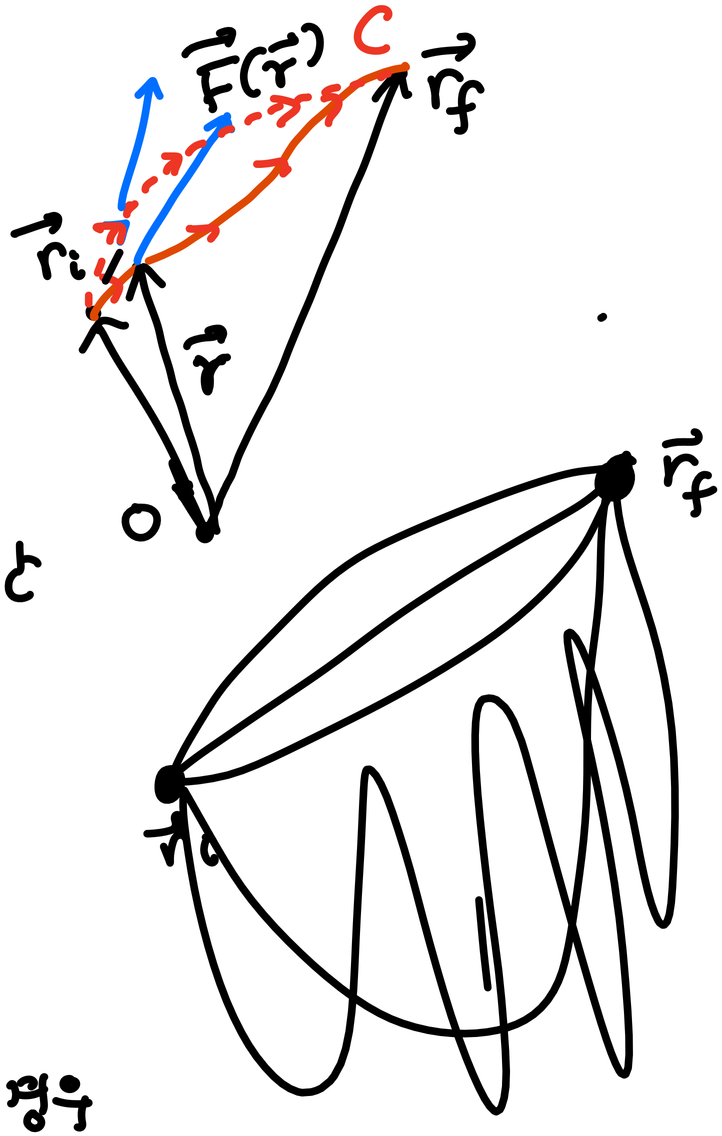
vector $\vec{F}$, $\vec{v}$

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d\vec{p}}{dt} \qquad \vec{p} = m\vec{v}$$

$$\vec{F}\cdot\vec{v} = \frac{d\vec{p}}{dt}\cdot\vec{v} = m\frac{d\vec{v}}{dt}\cdot\vec{v} = \frac{1}{2}m\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{d}{dt}\Big(\underbrace{\frac{1}{2}mv^2}_{T}\Big)$$

$$\vec{F}\cdot\frac{d\vec{r}}{dt} = \frac{dT}{dt}$$

$$\int_{C\ \vec{r}_i}^{\vec{r}_f}\vec{F}\cdot d\vec{r} = \int_{T_i}^{T_f} dT = \underbrace{T_f - T_i}_{} = \Delta T \ :$$



$\boxed{-V(\vec{r}_f)+V(\vec{r}_i)}$ 오로지 양끝점 만

경로에 따라 달라짐.

" 어떤 힘 일때 $\underbrace{\text{경로에 무관한가?}}_{=}$ "

양끝점의 위치에만 달린 경우

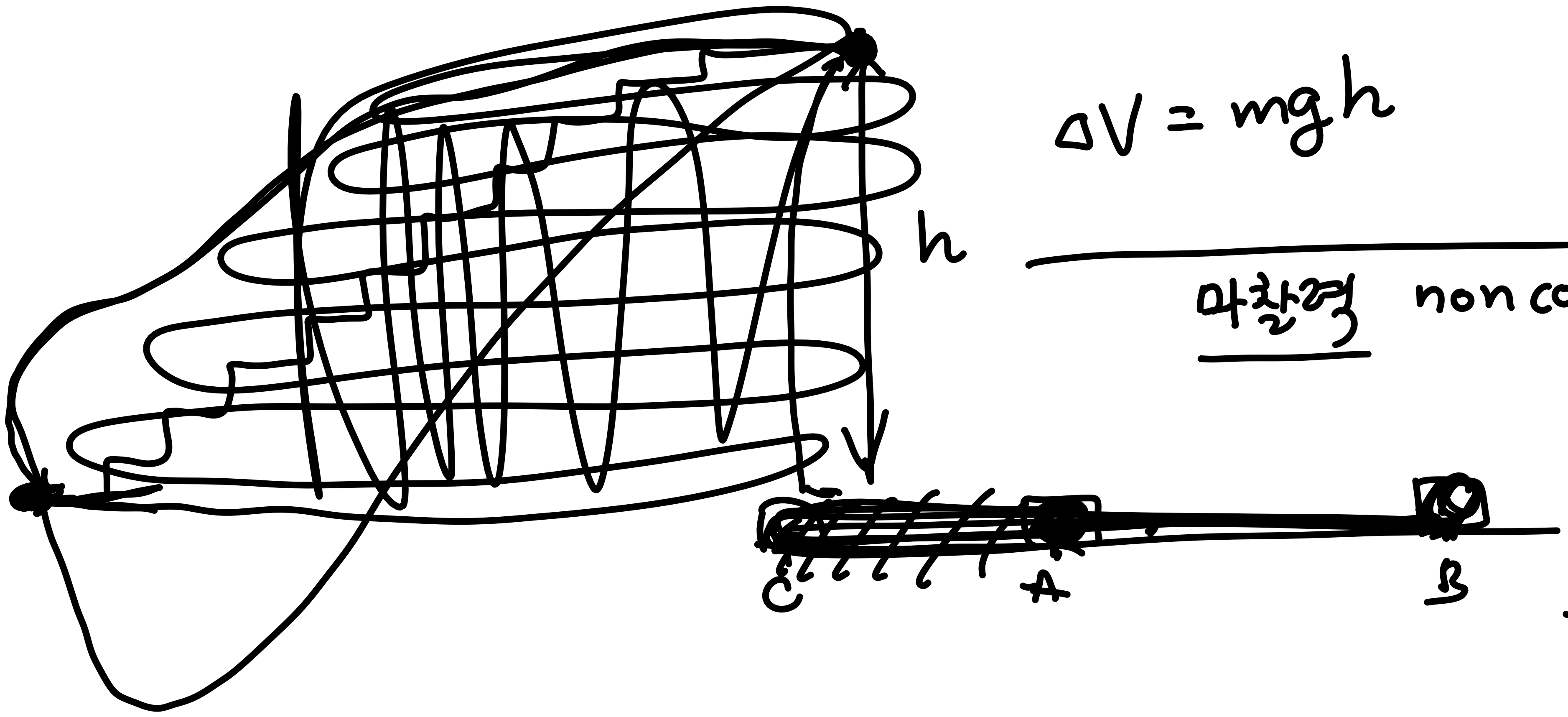
$$-V(\vec{r_f}) + V(\vec{r_i}) = \Delta T = T_f - T_i$$

$$\Rightarrow \quad T(\vec{r_f}) + \underbrace{V}_{\uparrow}(\vec{r_f}) = T(\vec{r_i}) + V(\vec{r_i}) = E$$

potatial energy (위치E)

$$W = \int_{\vec{r_i}}^{\vec{r_f}} \vec{F} \cdot d\vec{r} = \text{경로에 무관} = -V(\vec{r_f}) + V(\vec{r_i}) \rightarrow \text{에너지가 보존}$$

→ <u>보존력</u> (conservative force)



$\Delta V = mgh$

<u>마찰력</u> nonconservative

"del" $\vec{\nabla}$

직교좌표계 (x, y, z)

$$\vec{\nabla} = \vec{i}\,\underset{\sim}{\frac{\partial}{\partial x}} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$: 미분 연산자 (differential operator) → vector

$\frac{d}{dx}$, $\frac{d}{dt}$, . . . ; scalar

gradient ; scalar 함수 → vector 함수

(ex) $f(x, y, z) = x^2 + y^2 + z^2$; $\overbrace{\vec{\nabla} f} = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$

"scalar" 함수

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\hookrightarrow \vec{\nabla} f = (2x, 2y, 2z)$$

Divergence ; vector 함수 → scalar 함수

$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\vec{A} = (A_x(x,y,z), A_y(x,y,z), A_z(x,y,z))$$

(ex) $\vec{A} = (x, y, z) \rightarrow \vec{\nabla} \cdot \vec{A} = 3$

$\vec{A} = (xy, yz, zx) \rightarrow \vec{\nabla} \cdot \vec{A} = y + z + x$

__Curl__ : vector 함수 → vector 함수 $\vec{A}\times\vec{B}$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \qquad \vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

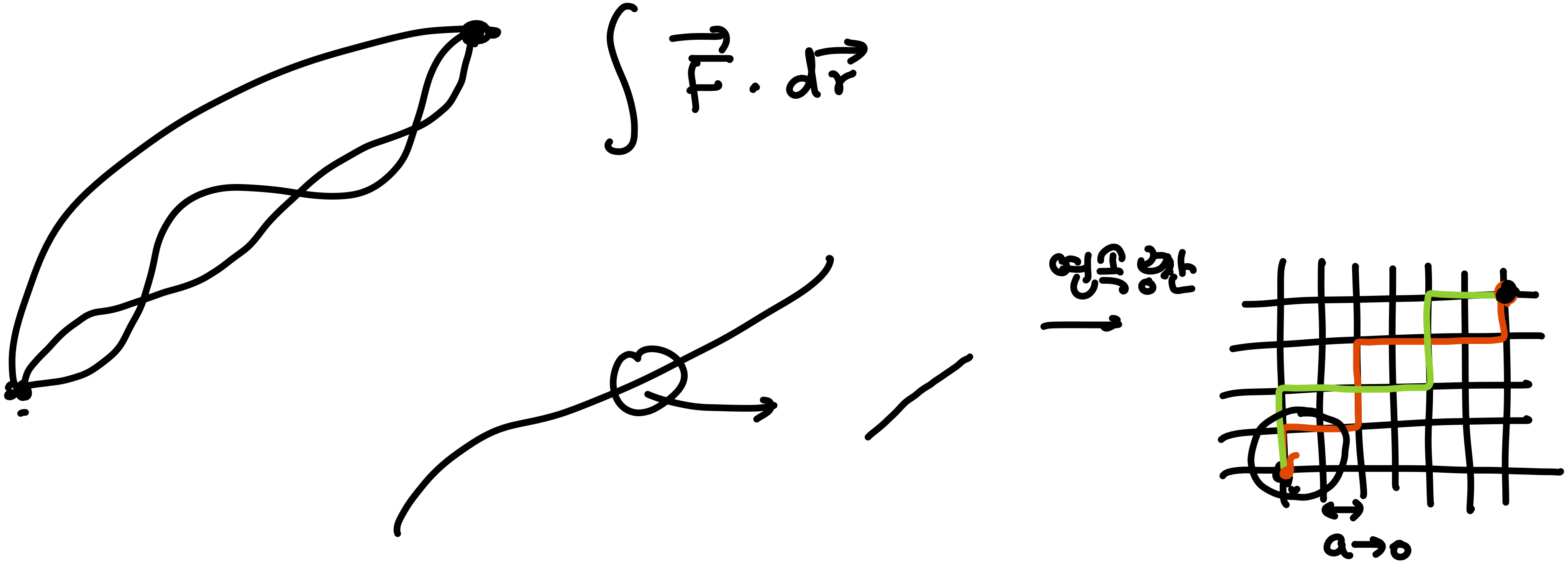
$$= \vec{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \vec{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \vec{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

(ex) $\vec{A} = (x, y, z)$; $\vec{\nabla}\times\vec{A} = 0$

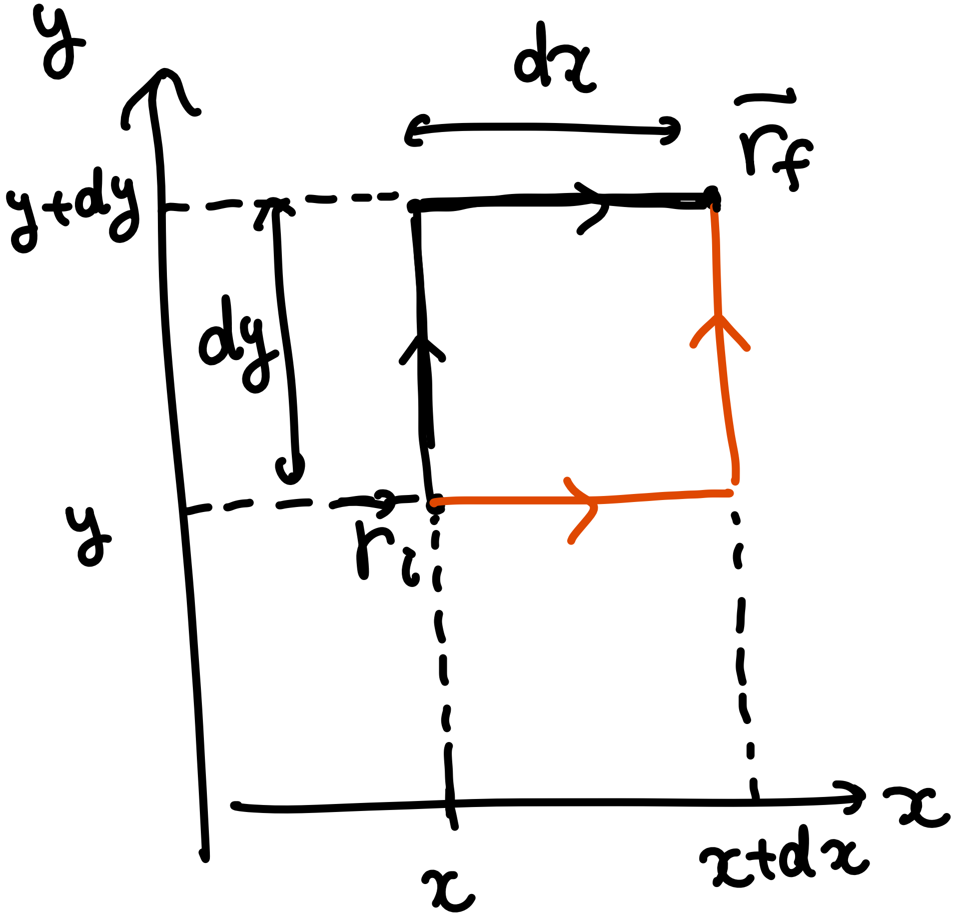
$\vec{A} = (xy, yz, zx)$; $\vec{\nabla}\times\vec{A} = (-y, -z, -x)$

$$\left(\vec{\nabla}\cdot\vec{\nabla} = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

$$\vec{\nabla}\times\vec{\nabla} = 0 \qquad (\vec{A}\times\vec{A} = 0)$$



$$\int \vec{F}\cdot d\vec{r}$$



$$\int_{\uparrow\rightarrow} \vec{F}\cdot d\vec{r} - \int_{\rightarrow\uparrow} \vec{F}\cdot d\vec{r}$$

$$= F_x^{(y+dy)}\,dx + F_y^{(x)}\,dy$$

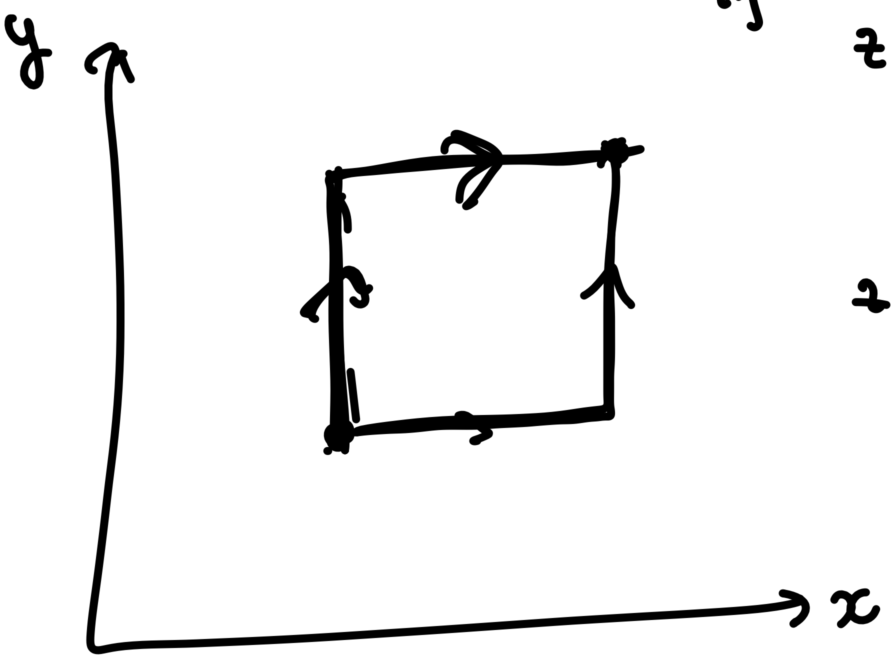
$$-\left(F_x^{(y)}\,dx + F_y^{(x+dx)}\,dy\right)$$

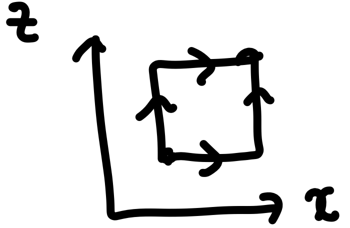
$$= \left[F_x(y+dy) - F_x(y)\right]dx \leftarrow \frac{\partial F_x}{\partial y}dy$$

$$+ \left[F_y(x) - F_y(x+dx)\right]dy \rightarrow -\frac{\partial F_y}{\partial x}dx$$

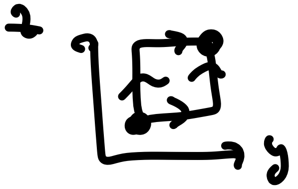
$$\Rightarrow \left(\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{dx}\right) dx\,dy = \int_{\uparrow\rightarrow} \vec{F}\cdot d\vec{r} - \int_{\rightarrow\uparrow} \vec{F}\cdot d\vec{r}$$

if $= 0$ $\Rightarrow \frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} = 0$





$\Rightarrow \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} = 0$



$\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = 0$

$\frac{\partial}{\partial x}$ vs. $\frac{d}{dx}$

$\uparrow$ 편미분 $\qquad \uparrow$ 미분

$\underline{\frac{\partial}{\partial x} f(x, \underbrace{y, z}_{\text{상수로 간주}})}$

$$\boxed{\text{미분} \quad \frac{d}{dx} f(x)}$$

$$\frac{d f(x,y,z)}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial z}\frac{dz}{dx}$$

(ex) $f(x,y,z) = x^2 y z$

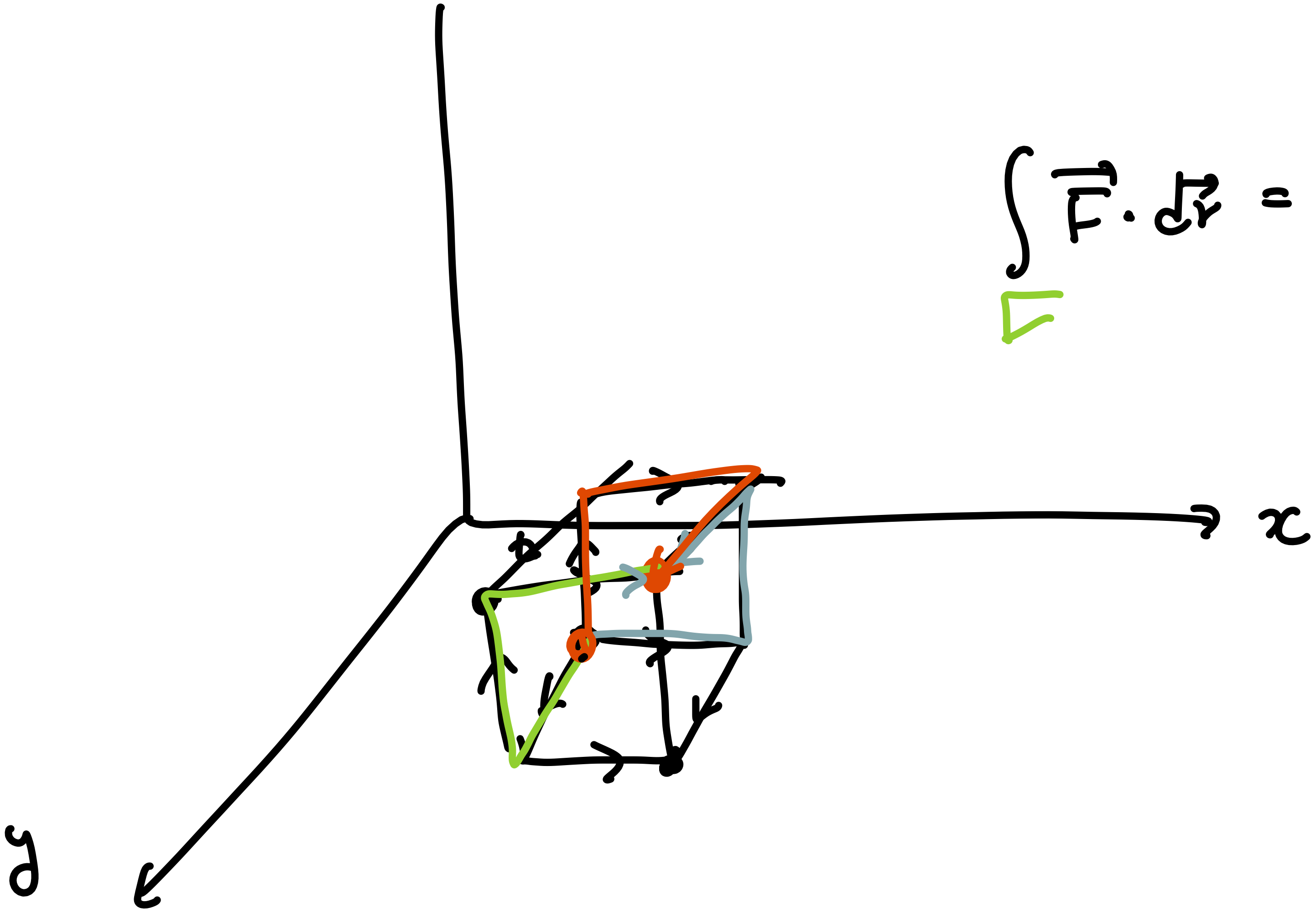
$\frac{\partial f}{\partial x} = 2xyz, \quad \frac{\partial f}{\partial y} = x^2 z \quad \ldots$

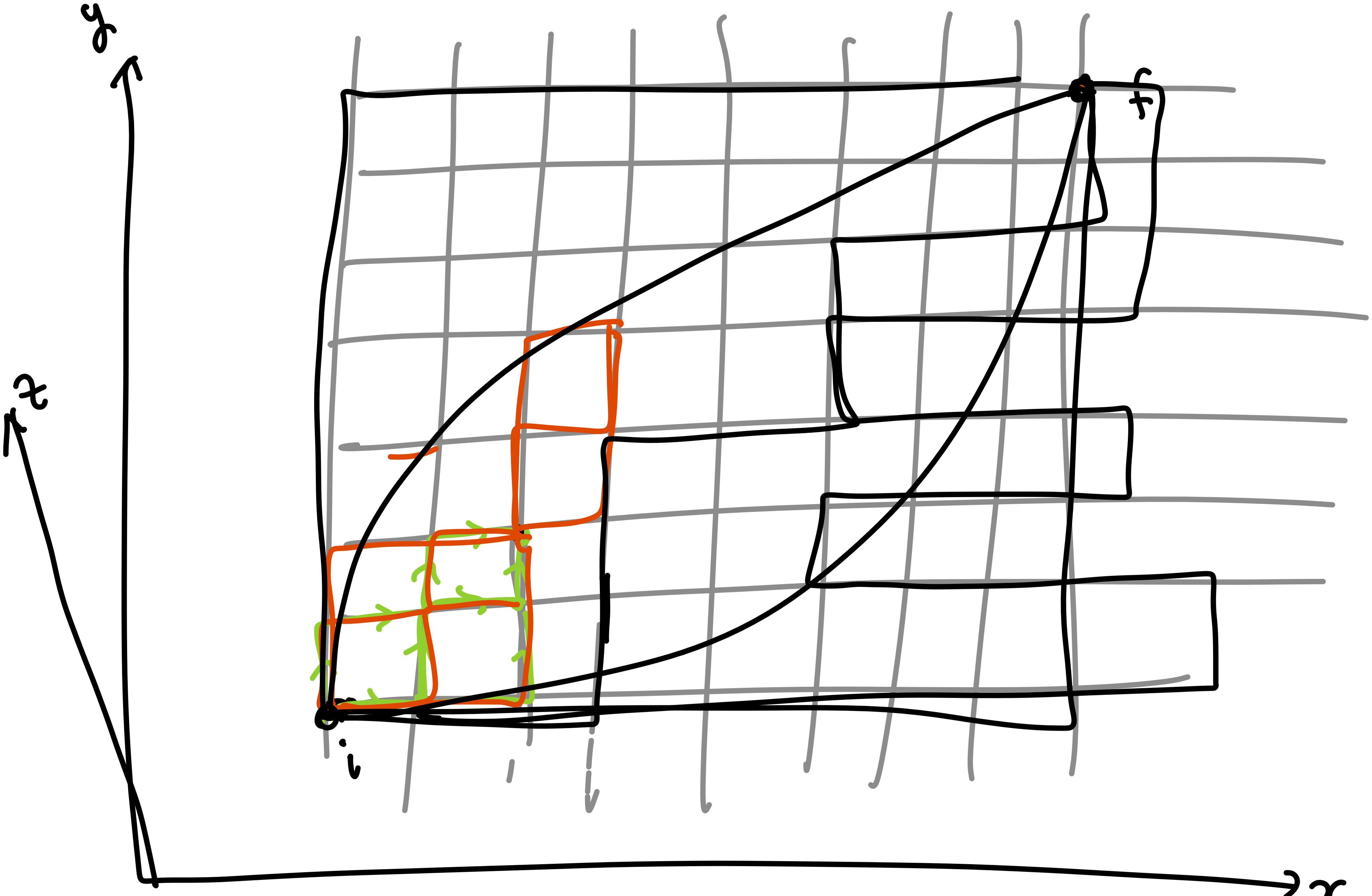
$V(\vec{r}) = V(x,y,z)$ $\quad$ $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

---

if $\vec{\nabla} \times \vec{F} = 0 \quad \longrightarrow \quad \frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} = \cdots = 0$

$\int \vec{F} \cdot d\vec{r} = \int \vec{F} \cdot d\vec{r}$





$$\vec{\nabla} \times \vec{F} = 0$$

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0$$

$\vec{\nabla}\times\vec{F}=0 \Longleftrightarrow \int_{\vec{r}_i}^{\vec{r}_f}\vec{F}\cdot d\vec{r}$ = 경로에 무관

↑

conservative

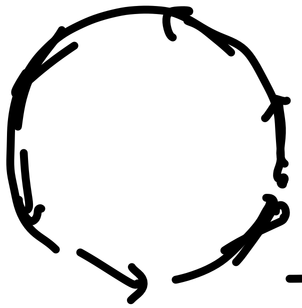
$\Updownarrow$

$\vec{F}$: 보존력

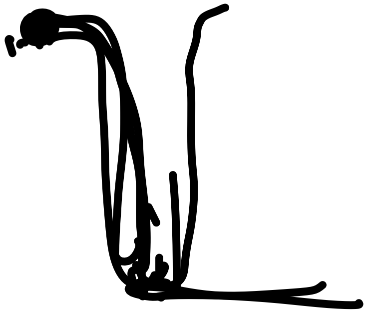
↓

$T+V$ = 보존.

curl



$\vec{\nabla}\times\vec{F}\neq 0$



If

$$\vec{\nabla}\times\vec{F}=0 \;\rightleftarrows\; \Longleftrightarrow \; \overset{\text{force}}{\overset{\downarrow}{\vec{F}}} = \underbrace{\vec{\nabla}}_{\text{gradient}}(-V) = -\vec{\nabla}\overset{\text{위치E.}}{\overset{\downarrow}{V}}$$

$\vec{\nabla}\cdot\vec{\nabla}$

$\underbrace{\vec{\nabla}\times\vec{\nabla}}_{0}f=0$

$$\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}\cdot d\vec{r} = -\int_{\vec{r}_i}^{\vec{r}_f} \vec{\nabla} V \cdot d\vec{r} = -\left(V(\vec{r}_f) - V(\vec{r}_i)\right)$$

$$\vec{F} = -\vec{\nabla} V \quad \left(\boxed{\vec{\nabla}\times\vec{F} = 0}\right)$$

$$\vec{\nabla} V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$$

$$d\vec{r} = (dx, dy, dz)$$

$$\int_{\vec{r}_i}^{\vec{r}_f} \vec{\nabla} V \cdot d\vec{r} = \int \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

$$= V(\vec{r}_f) - V(\vec{r}_i)$$

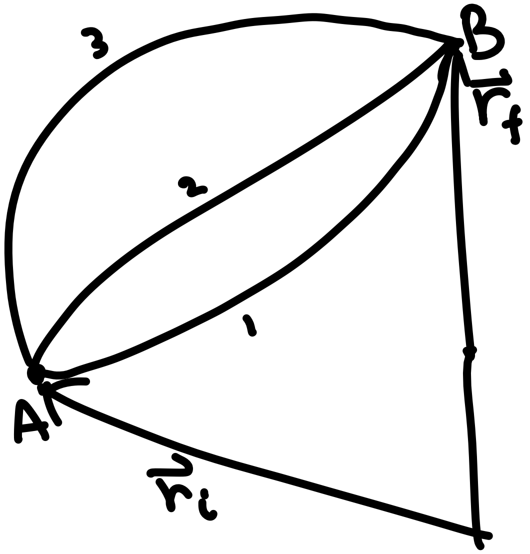
$$\Rightarrow W = \int \vec{F}\cdot d\vec{r} = -V(\vec{r}_f) + V(\vec{r}_i)$$

$$= \Delta T = T(\vec{r}_f) - T(\vec{r}_i)$$

$$\vec{F} = -\vec{\nabla} V \quad (\vec{\nabla}\times\vec{F} = 0)$$

$\Rightarrow$ $T + V =$ 보존.

목요일 : 수업시간 변경
오후 6:30 ~ 7:45



$$W = \int_A^B \vec{F}\cdot d\vec{r} = \text{path에 무관}$$

$$\updownarrow$$

$$\vec{\nabla}\times\vec{F} = 0 \qquad \text{conservative force}$$

보존력

$$\updownarrow$$

$$\vec{F} = -\vec{\nabla}V \qquad (\because\ \vec{\nabla}\times\vec{\nabla} = 0)$$

$$W = -\int_A^B \vec{\nabla}V\cdot d\vec{r} = -V(\vec{r}_f) + V(\vec{r}_i)$$

$$\shortparallel$$

$$\int m\frac{d\vec{v}}{dt}\cdot d\vec{r} = \int m\vec{v}\cdot d\vec{v} = \underbrace{\frac{1}{2}mv^2(\vec{r}_f)}_{T} - \underbrace{\frac{1}{2}mv^2(\vec{r}}_{T}$$

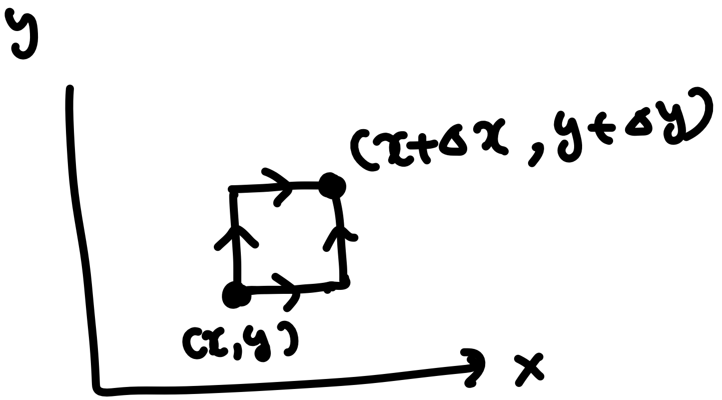
$$\Rightarrow\ T + V = \text{일정} = E$$

$$\int_{\square}\vec{F}\cdot d\vec{r}$$

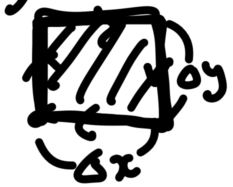
$$\shortparallel$$

$$\left(\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{dx}\right)\Delta x\,\Delta y = \int_{\uparrow\rightarrow}\vec{F}\cdot d\vec{r} - \underbrace{\int_{\rightarrow\uparrow}\vec{F}\cdot d\vec{r}}_{+\int_{\downarrow\leftarrow}\vec{F}\cdot d\vec{r}}$$

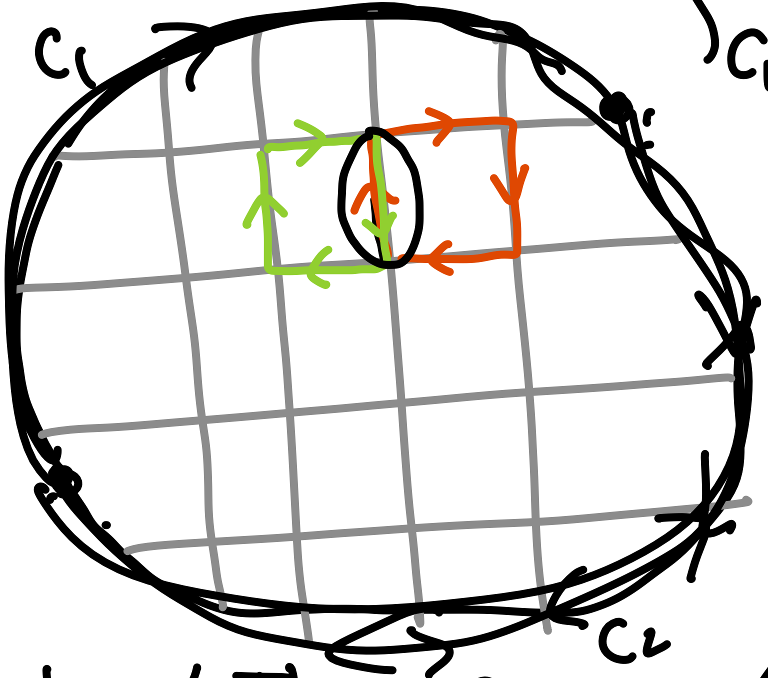
$$\left(\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right)$$



$$\oint \vec{F}\cdot d\vec{r} = (\vec{\nabla}\times\vec{F})_z \underbrace{\Delta x \Delta y}_{da}$$

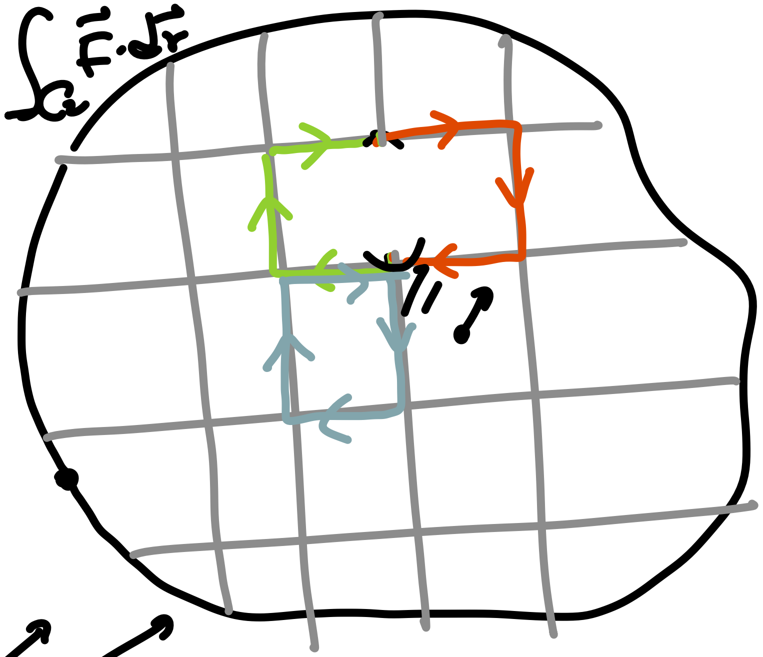


$$\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x}$$



$$\int_{C_1}\vec{F}\cdot d\vec{r} + \int_{C_2}\vec{F}\cdot d\vec{r}$$

=



Stokes' Theorem

$$\int_{C_1}\vec{F}\cdot d\vec{r} - \int_{C_2}\vec{F}\cdot d\vec{r} = \oint_C \vec{F}\cdot d\vec{r}$$

↑

선적분

$$= \int (\vec{\nabla}\times\vec{F})\cdot d\vec{a}$$

〃

$da\,\hat{z}$

면적분

= 0

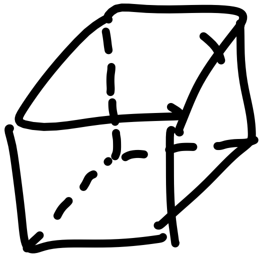
if $\vec{\nabla} \times \vec{F} = 0 \longrightarrow \oint_C \vec{F} \cdot d\vec{r} = 0 \rightarrow \int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$

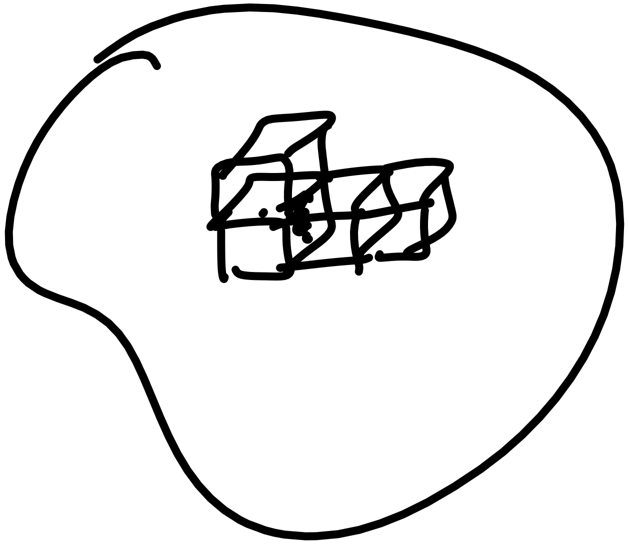
if $\vec{\nabla} \times \vec{F} \neq 0$

Gauss 정리 : 면적분 ⇔ 부피적분

$$\oint_C \quad \wr \quad \oint_S \vec{F} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{F} \, dv \qquad \vec{\nabla} \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

폐곡면





[Ex 4.2.1] $V(\vec{r}) = V_0 - \frac{1}{2}k\delta^2 e^{-r^2/\delta^2}$

$$\vec{F} = -\vec{\nabla}V \quad \left(-\frac{dV}{dr}\frac{\partial r}{\partial x}, -\frac{dV}{dr}\frac{\partial r}{\partial y}, -\frac{dV}{dr}\frac{\partial r}{\partial z}\right)$$

$$r = \sqrt{x^2+y^2+z^2} \qquad \frac{\partial r}{\partial x} = \cancel{\frac{1}{2}}\frac{\cancel{2}x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$\vec{F} = -k\cancel{r}\, e^{-\frac{r^2}{\delta^2}} \underbrace{(x, y, z)}_{\vec{r}} \cancel{\frac{1}{r}} = -k\vec{r}\, e^{-\frac{r^2}{\delta^2}}$$

[Ex 4.2.2] $\underbrace{\vec{r}=0 \quad v=v_0}$ $\rightarrow$ $\vec{r} = e_r\Delta \ (\Delta \ll \delta)$ $v = ?$

$$E = \frac{1}{2}mv(\vec{r})^2 + V(\vec{r}) = \frac{1}{2}mv_0^3 + V_0 - \frac{1}{2}k\delta^2$$

$$= \frac{1}{2}mv^2 + V(\vec{e}_r\Delta) = \frac{1}{2}mv^2 + V_0 - \frac{1}{2}k\delta^2 \underbrace{e^{-\boxed{\frac{\Delta^2}{\delta^2}}^{\ll 1}}}_{\simeq 1 - \frac{\Delta^2}{\delta^2}}$$

$$e^x \underset{x\ll 1}{\simeq} \underline{1+x} + \cancel{\frac{x^2}{2}}$$

$$\overbrace{\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}k\cancel{\delta^2}\left(\frac{\Delta^2}{\cancel{\delta^2}}\right)}$$

$$v^2 = v_0^2 - \frac{k}{m}\Delta^2$$

[Ex 4.2.3] $\vec{F} = (xy, zx, yz) \longrightarrow$

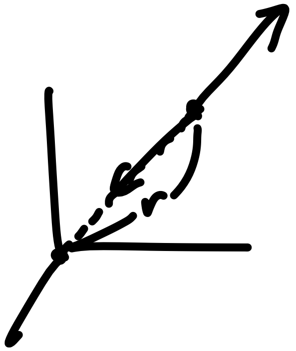
[Ex 4.2.4] $\vec{F} = \vec{i}(ax+by^2) + \vec{j}(cxy)$: conservative

$\vec{\nabla}\times\vec{F} = 0$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax+by^2 & cxy & 0 \end{vmatrix} = \vec{i}(0) + \vec{j}(0) + \vec{k}(cy - 2by) = 0$$

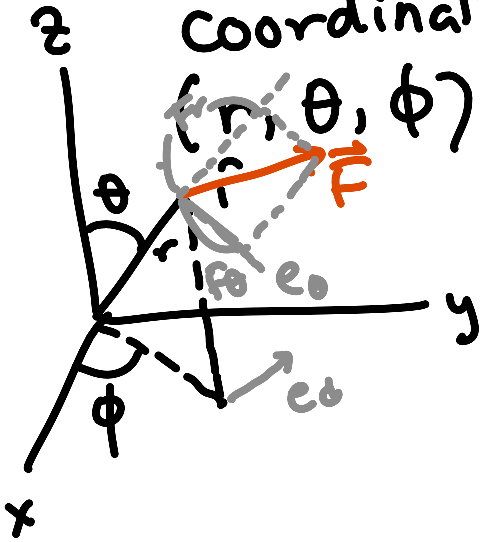
$\to c = 2b$ //

$$\vec{\nabla}\times\vec{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{e}_r & \vec{e}_\theta r & \vec{e}_\phi r\sin\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}$$



spherical coordinate

(r, θ, ϕ)



$\vec{F} = (F_r, F_\theta, F_\phi)$

(Ex) $\vec{F} = -\underline{\vec{e}_r}\,\overset{f(r)}{\boxed{\frac{k}{r^2}}}$ $\nearrow$ 중심력

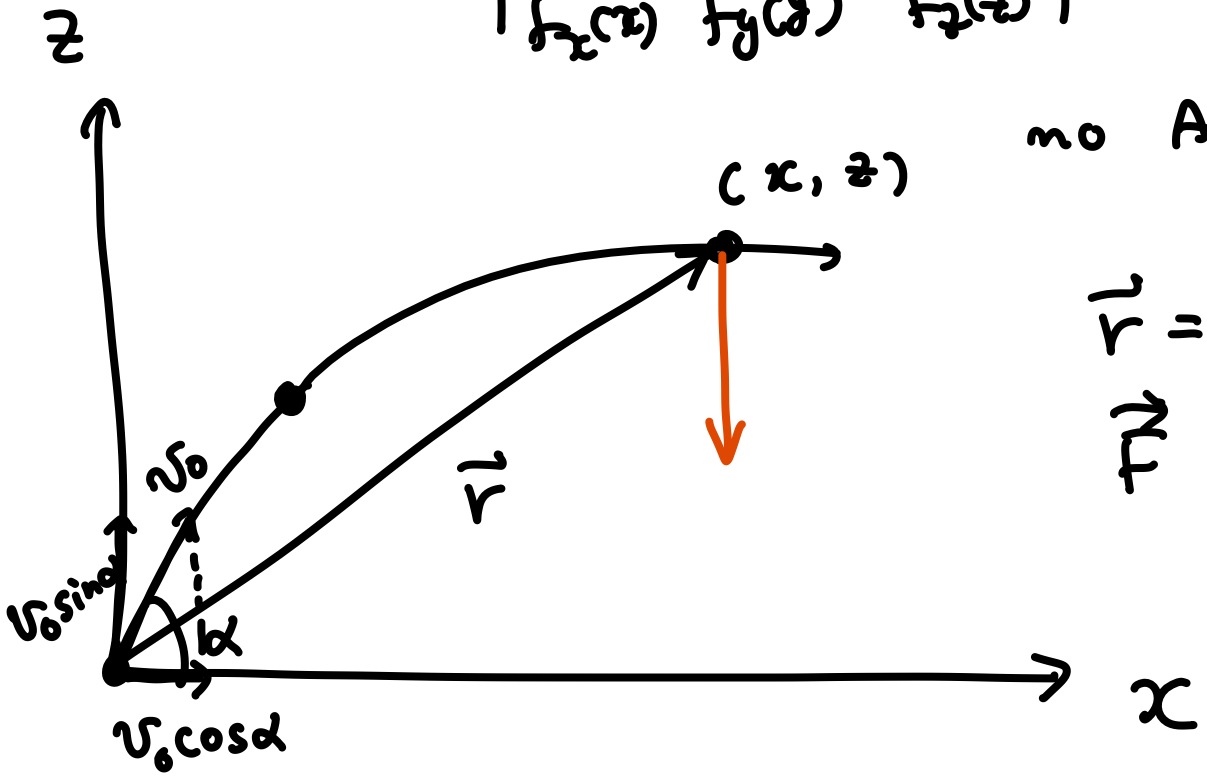
$\to F_\theta = F_\phi = 0$

$F_r = -\frac{k}{r^2}$

$\to \vec{\nabla}\times\vec{F} = 0$ $\Rightarrow$ $\vec{F}$: conservative

4.3 특수하게 $\vec{F} = \vec{i}\,F_x(x) + \vec{j}\,F_y(y) + \vec{k}\,F_z(z)$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x(x) & F_y(y) & F_z(z) \end{vmatrix} = 0 \longrightarrow \text{보존력.}$$



no Air resistance

$\vec{r} = \vec{i}\,x + \vec{k}\,z \rightarrow \dot{\vec{r}} = \vec{v} = i\,\dot{x} + \vec{k}\,\dot{z}$

$$\vec{F} = \vec{k}(-mg) = -\vec{\nabla} V$$
$$= -\vec{k}\frac{\partial V}{\partial z}$$
$$\rightarrow V = mgz$$

$$\frac{1}{2}m(\underbrace{\dot{x}^2 + \dot{z}^2}_{v^2}) + mgz = E = \text{일정.} = \frac{1}{2}m v_0^2 + 0$$

$$F_x = 0 = m\frac{dv_x}{dt}$$

$$\Rightarrow v^2 = v_0^2 - 2gz$$

$$\Rightarrow \underline{v_x} = v_0 \cos\alpha$$

$$\Rightarrow \vec{V} = \vec{v_0} - \vec{k}gt \Leftarrow \underline{v_z} = v_0 \sin\alpha - gt \leftarrow F_z = -mg = m\frac{dv_z}{dt} \rightarrow v_z = -gt + c$$

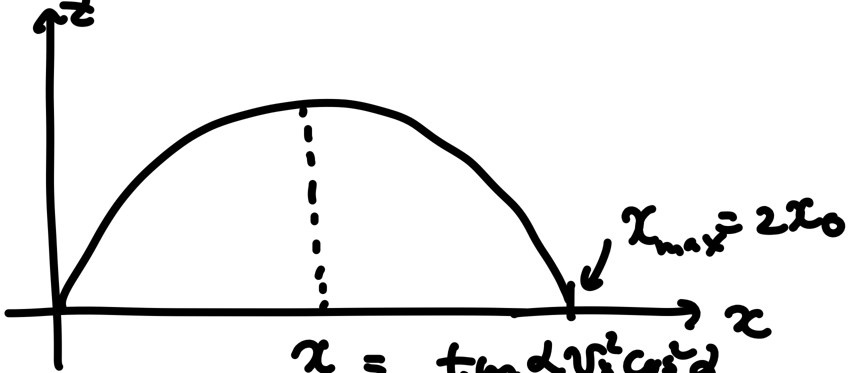
$$\frac{d\vec{r}}{dt} = \vec{v} = \vec{v}_0 - \vec{k} g t$$

$$\vec{r} = \cancel{\vec{r}_0} + \vec{v}_0 t - \frac{\vec{k}}{2} g t^2 \longrightarrow \vec{r}_0 = 0$$

$$= \underbrace{\vec{i} v_0 \cos\alpha t}_{x} + \vec{k} \big(\underbrace{v_0 \sin\alpha t - \tfrac{1}{2} g t^2}_{z} \big)$$

$$t = \frac{x}{v_0 \cos\alpha} .$$

$$z = \tan\alpha \, x - \frac{1}{2} \frac{g}{v_0^2 \cos^2\alpha} x^2$$



$$x_0 = \frac{\tan\alpha \, v_0^2 \cos^2\alpha}{g}$$

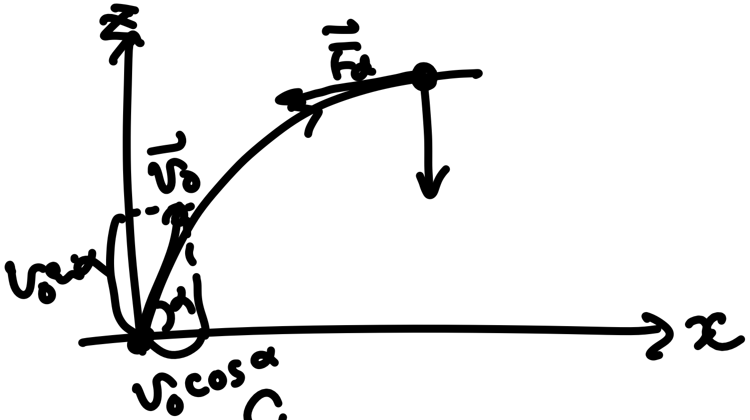
$$= \frac{v_0^2 \cos\alpha \sin\alpha}{g}$$

$$= \frac{v_0^2 \sin(2\alpha)}{2g}$$

$$z = 0 \longrightarrow T = \frac{2 v_0 \sin\alpha}{g}$$

with air resistance (linear)

$$\vec{F}_d = -c_1\vec{v}$$



$$\frac{c_1}{m} \equiv \gamma$$

$$\vec{F} = -\vec{k}mg - c_1\vec{v} = m\frac{d\vec{v}}{dt}$$

$$\frac{d\vec{v}}{dt} = -\vec{k}g - \gamma\vec{v}$$

$$\frac{d}{dt}v_x = -\gamma v_x\,, \quad \frac{d}{dt}v_y = -\gamma v_y \longrightarrow v_y = \overset{0}{\overset{\shortparallel}{v_{y0}}}e^{-\gamma t} = 0$$

$$\longrightarrow v_x = v_{x0}e^{-\gamma t} = \underset{\sim\sim\sim}{v_0\cos\alpha\, e^{-\gamma t}}$$

$$= \underset{\sim\sim}{\frac{dx}{dt}}$$

$$\frac{d}{dt}v_z = -\gamma v_z - g\,,$$

$$\overset{\shortparallel}{\frac{dV_z}{dt}} = -\gamma\Big(\underbrace{v_z + \frac{g}{\gamma}}_{\overset{\shortparallel}{V_z}}\Big) \longrightarrow \frac{dV_z}{dt} = -\gamma V_z$$

$$\frac{dV_z}{dt} = \frac{dv_z}{dt}$$

$$\rightarrow \underset{v_z + \frac{g}{\gamma}}{\overset{}{\underset{\shortparallel}{V_z}}} = \underbrace{V_z(0)}_{\underbrace{v_z(0)}_{v_0\sin\alpha} + \frac{g}{\gamma}} e^{-\gamma t}$$

$$\therefore\ v_z = -\frac{g}{\gamma} + \left(v_0\sin\alpha + \frac{g}{\gamma}\right)e^{-\gamma t}$$

$$\frac{dx}{dt} = v_0 \cos\alpha \; e^{-\gamma t}$$

$$\rightarrow \; x = -\frac{v_0\cos\alpha}{\gamma} e^{-\gamma t} + \underset{\overset{\shortparallel}{\frac{v_0\cos\alpha}{\gamma}}}{C} \quad \longrightarrow x(t=0)=0$$

$$= \frac{v_0\cos\alpha}{\gamma}\left(1-e^{-\gamma t}\right) \; '' \qquad\qquad \frac{dy}{dt} = v_y = 0 \longrightarrow y = 0$$

$$v_z = -\frac{g}{\gamma} + \left(v_0\sin\alpha + \frac{g}{\gamma}\right)e^{-\gamma t} = \frac{dz}{dt}$$

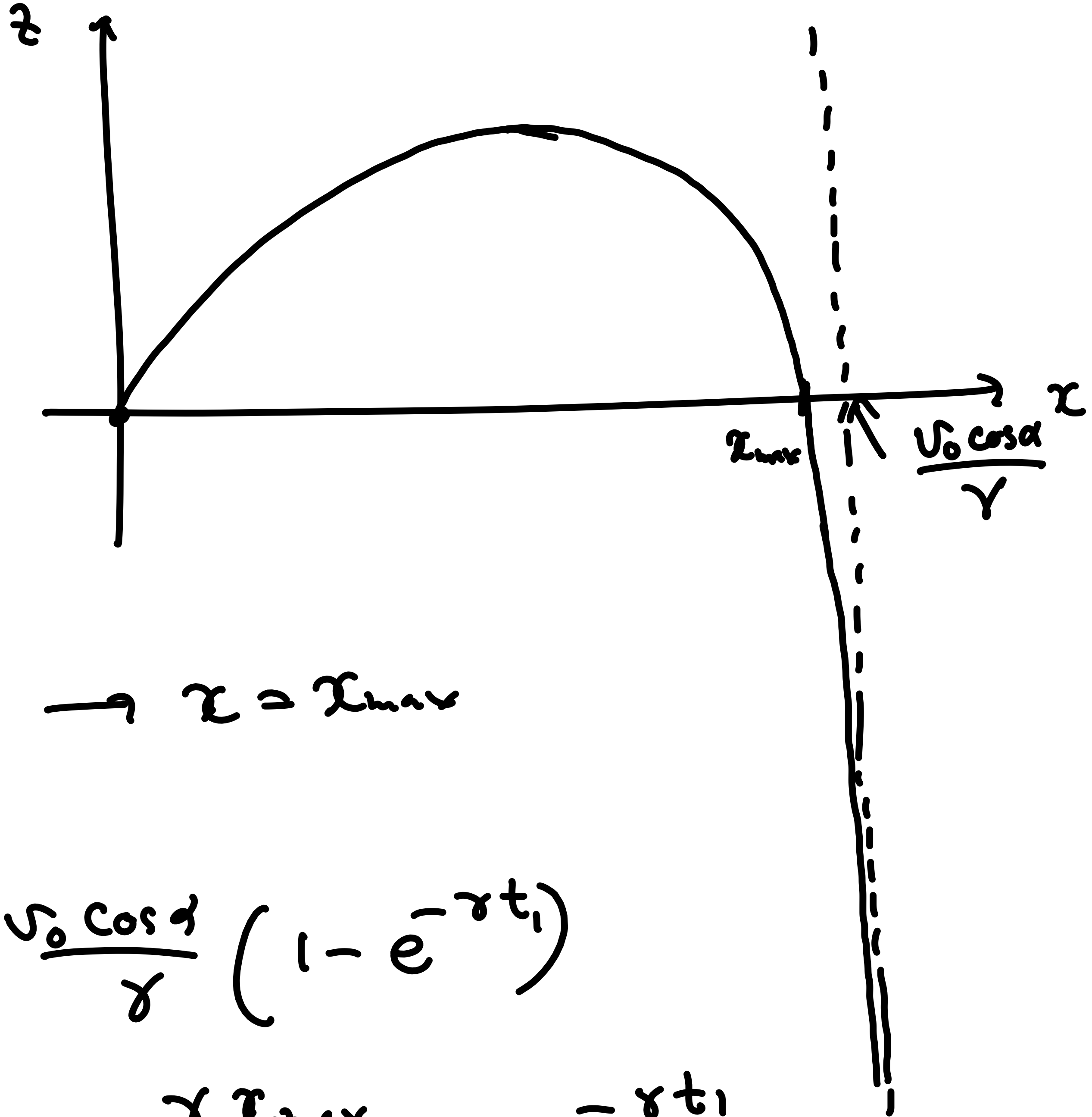
$$z = C - \frac{g}{\gamma}t - \underbrace{\frac{1}{\gamma}\left(v_0\sin\alpha + \frac{g}{\gamma}\right)}_{''} e^{-\gamma t}$$

$$z(t=0) = 0 = C - \qquad '' \qquad \longrightarrow C = \qquad ''$$

$$\therefore \; z = -\underset{\sim\sim}{\frac{g}{\gamma}t} + \frac{1}{\underset{!}{\gamma}}\left(\underset{\sim\sim}{v_0\sin\alpha} + \underset{=}{\frac{g}{\gamma}}\right)\left(\underline{1-e^{-\gamma t}}\right) \; ''$$

$$\vec{r} = -\vec{k}\frac{g}{\gamma}t + \vec{k}\frac{g}{\gamma^2}\left(1-e^{-\gamma t}\right) + \frac{\vec{v_0}}{\gamma}\left(1-e^{-\gamma t}\right)$$

$$t \rightarrow \infty \qquad x_{max} = \frac{v_0\cos\alpha}{\gamma}$$



$$z = 0 \longrightarrow x = x_{max}$$

$$x_{max} = \frac{v_0 \cos\alpha}{\gamma}\left(1 - e^{-\gamma t_1}\right)$$

$$\rightarrow 1 - \frac{\gamma x_{max}}{v_0 \cos\alpha} = e^{-\gamma t_1} \longrightarrow t_1 = -\frac{1}{\gamma}\ln\Big(1 - \underbrace{\frac{\gamma x_{max}}{v_0 \cos\alpha}}_{\overset{\shortparallel}{u}}\Big)$$

$$z = -\frac{g}{\gamma}t_1 + \underbrace{\frac{1}{\gamma}\left(v_0 \sin\alpha + \frac{g}{\gamma}\right)}_{\equiv a}\underbrace{\left(1 - e^{-\gamma t_1}\right)}_{u} = 0 ,$$

$$\Rightarrow + \frac{g}{\gamma^2}\ln(1-u) + au = 0$$

$$\frac{g}{\gamma^2}\ln(1-u) + au = 0$$

$$\ln(1-u) \approx -u - \frac{u^2}{2} - \frac{u^3}{3} + \cdots$$

$u \ll 1$

$$u\left[\frac{g}{\gamma^2}\left(-1 - \frac{u}{2} - \frac{u^2}{3} + \cdots\right) + a\right] = 0$$

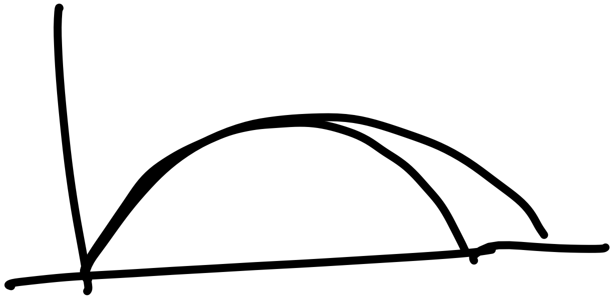
$\cancel{u=0}$

$$\underbrace{\frac{g}{\gamma^2}\left(-1 - \frac{u}{2} - \frac{u^2}{3} + \cdots\right) + a}_{=\,0}$$

$$\leftarrow \quad \frac{g}{2\gamma^2}u = a - \frac{g}{\gamma^2}$$

$$u = \frac{\gamma x_{max}}{v_0\cos\alpha} = \frac{2\gamma^2}{g}\left(\frac{1}{\gamma}\left(v_0\sin\alpha + \cancel{\frac{g}{\gamma}}\right) - \cancel{\frac{g}{\gamma^2}}\right)$$

$$\therefore \quad x_{max} = \frac{v_0\cos\alpha}{\cancel{\gamma}}\,\frac{2\cancel{\gamma^2}}{g}\left(\frac{v_0\sin\alpha}{\cancel{\gamma}} + \cdots\right)$$



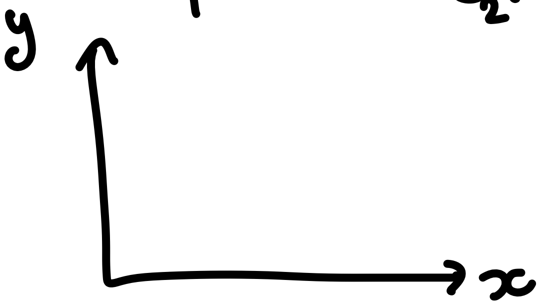
$$= \frac{2v_0^2\sin\alpha\cos\alpha}{g} + \cdots$$

$$= \frac{v_0^2\sin 2\alpha}{2} + \cdots$$

$$-\frac{8v_0\cos\alpha\, v_0^2\sin^2\alpha}{3g^2}\gamma + \cdots$$

[Ex 4.3.1] $$\vec{F}(\vec{v}) = -\vec{v}\,\underbrace{\left(c_1 + c_2|\vec{v}|\right)}_{\approx\, c_1 + c_2 v_0} \propto -\underline{(c_1 + c_2 v_0)}\,\vec{v} = m\frac{d\vec{v}}{dt}$$

[Ex 4.3.2] $$\vec{F} = -c_2|\vec{v}|\,\vec{v} = m\frac{d\vec{v}}{dt}$$

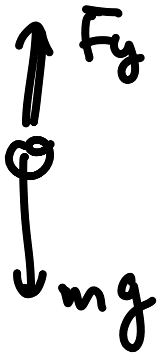


$$\vec{v} = (v_x, v_y)$$

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2}$$

$$\begin{cases} \underline{m\dfrac{dv_x}{dt} = -c_2\sqrt{v_x^2+v_y^2}\;v_x} \\ \underline{m\dfrac{dv_y}{dt} = -c_2\sqrt{v_x^2+v_y^2}\;v_y - mg} \end{cases}$$

(cf) $$F_x = -c_2 v_x^{\cancel{2}} = m\frac{dv_x}{dt} = m\frac{dv_x}{dx}\underbrace{\frac{dx}{dt}}_{\cancel{v_x}}$$



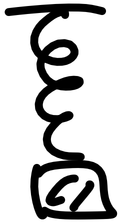
$$F_y = +c_2 v_y^2 - mg$$

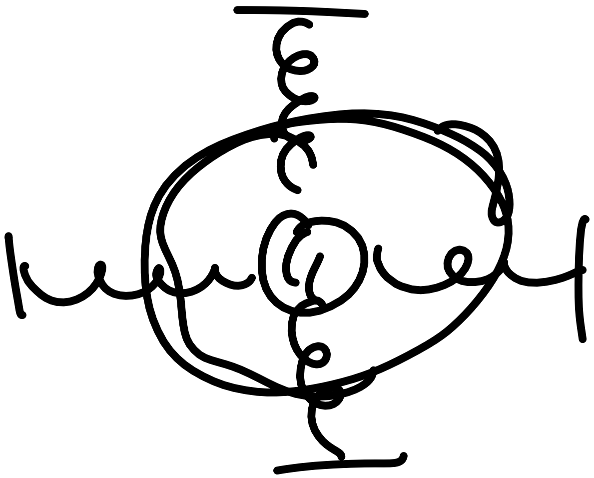
$$\Rightarrow \frac{dv_x}{dx} = -\frac{c_2}{m}v_x$$

$$\downarrow$$

$$\underline{v_x = v_{x0}\,e^{-\frac{c_2}{m}x}}$$

§ 4.4. 2, 3 차원의 단진자.

(1차원  $F=-kx$, )



$$\vec F = -k\vec r$$

$$m\frac{d^2\vec r}{dt^2} = \vec F = -k\vec r \Rightarrow \begin{array}{l} m\ddot x = -kx \\ m\ddot y = -ky, \quad \omega = \sqrt{\frac{k}{m}} \end{array}$$

$$\Rightarrow x = A\,\underset{\sim\sim\sim\sim\sim}{\cos(\omega t+\alpha)}$$

$$y = B\cos(\omega t+\beta)$$

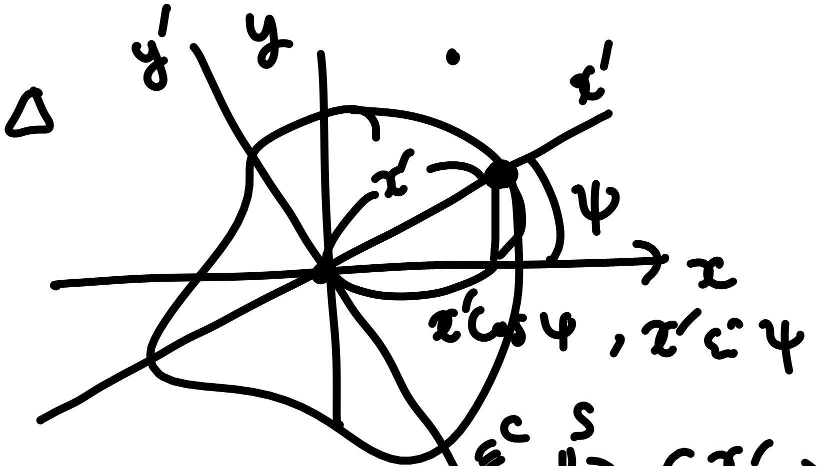
$$y = B\cos\Big(\omega t+\alpha+\overbrace{\beta-\alpha}^{\overset{\Delta}{\shortparallel}}\Big)$$

$$= B\Big(\underbrace{\cos(\omega t+\alpha)}_{\frac{x}{A}}\cos\Delta - \underbrace{\sin(\omega t+\alpha)}_{\sqrt{1-\frac{x^2}{A^2}}}\sin\Delta\Big) = B\Big(\frac{x}{A}\cos\Delta - \sqrt{1-\frac{x^2}{A^2}}\sin\Delta\Big)$$

$$\underset{\shortparallel\ \frac{y^2}{B^2}+\frac{x^2}{A^2}\cos^2\Delta-2\frac{xy}{AB}\cos\Delta}{\Big(\frac{y}{B}-\frac{x}{A}\cos\Delta\Big)^2} = \Big(-\sqrt{1-\frac{x^2}{A^2}}\sin\Delta\Big)^2 = \Big(1-\frac{x^2}{A^2}\Big)\sin^2\Delta$$

$$\Rightarrow \frac{y^2}{B^2}+\frac{x^2}{A^2}-2\frac{xy}{AB}\cos\Delta = \sin^2\Delta$$

$$\frac{y^2}{B^2}+\frac{x^2}{A^2}-2\frac{xy}{AB}\cos\Delta = \sin^2\Delta$$



$$\cos\psi \equiv c$$
$$\sin\psi \equiv s$$

$$x = cx' - sy'$$
$$y = sx' + cy'$$

$$\leftarrow \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\overset{\equiv c}{\cos\psi} & -\overset{\overset{s}{\|}}{\sin\psi}\\ \sin\psi & \cos\psi\end{pmatrix}\begin{pmatrix}x'\\y'\end{pmatrix}$$

$$\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}c & s\\ -s & c\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$\frac{1}{B^2}\left(s^2\underset{\sim}{\overset{\downarrow}{x'}}{}^2 + c^2\underset{\sim}{\overset{\downarrow}{y'}}{}^2 + 2sc\,\underset{\sim}{x'y'}\right)$$

$$+\frac{1}{A^2}\left(c^2\underset{\sim}{\overset{\downarrow}{x'}}{}^2 + s^2\underset{\sim}{\overset{\downarrow}{y'}}{}^2 - 2sc\,\underset{\sim}{x'y'}\right)$$

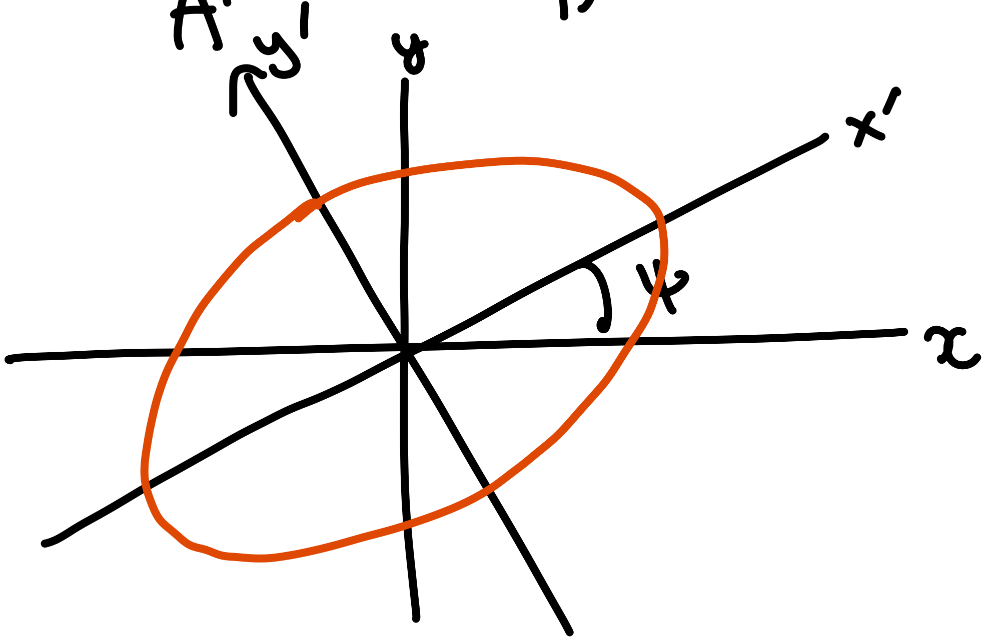
$$-2\frac{\cos\Delta}{AB}\left(cs\,\underset{\sim}{\overset{\downarrow}{x'}}{}^2 - cs\,\overset{\downarrow}{y'}{}^2 + (c^2-s^2)\,\underset{\sim}{x'y'}\right) = \sin^2\Delta$$

$$\Rightarrow x'y'\left[\frac{\boxed{2sc}}{B^2} \overset{\sin(2\psi)=}{-} \frac{\boxed{2sc}}{A^2} - 2\frac{\cos\Delta}{AB}\underbrace{(c^2-s^2)}_{\cos 2\psi}\right] = 0$$

$$\sin 2\psi\left(\frac{1}{B^2}-\frac{1}{A^2}\right) = 2\frac{\cos\Delta}{AB}\cos 2\psi$$

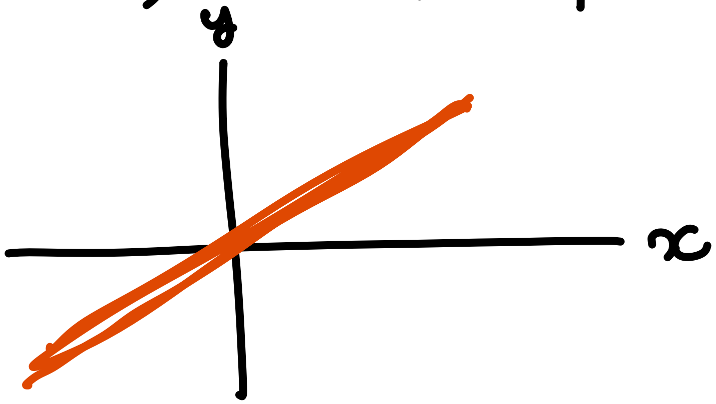
$$\Rightarrow \tan 2\psi = \frac{2\cos\Delta}{AB}\Big/\left(\frac{1}{B^2}-\frac{1}{A^2}\right)^{=\frac{A^2-B^2}{A^2B^2}} = \frac{2AB\cos\Delta}{A^2-B^2}$$

$$\rightarrow \quad \frac{x'^2}{A'^2} + \frac{y'^2}{B'^2} = \sin^2\Delta \qquad \frac{1}{A'^2} = \dots \,,\ \frac{1}{B'^2} = \dots$$



If $\Delta = 0$

$$\frac{x}{A} = \cos(\omega t + \alpha) = \cos(\omega t + \beta) = \frac{y}{B}$$



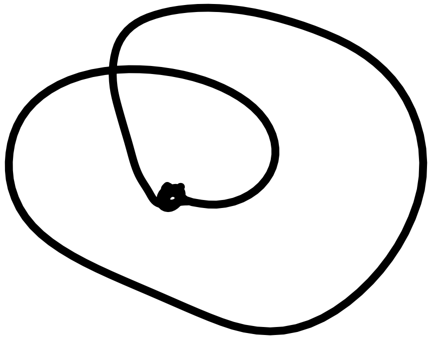
$\vec{F} = -\boxed{k}\vec{r}$ "isotropic" 등방성

↓

$\vec{F} = (-k_1 x, -k_2 y, -k_3 z)$ "non-isotropic"

$m\ddot{x} = -k_1 x \longrightarrow x = A\cos(\omega_1 t + \alpha)$ $\omega_1 = \sqrt{\frac{k_1}{m}} \rightarrow T_1 = \frac{2\pi}{\omega_1}$

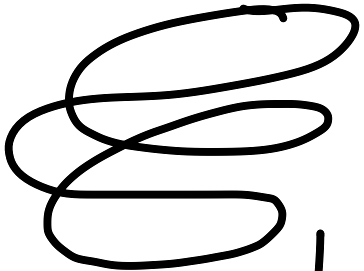
$m\ddot{y} = -k_2 y \longrightarrow y = B\cos(\omega_2 t + \beta)$ $\omega_2 = \sqrt{\frac{k_2}{m}} \rightarrow T_2 = \frac{2\pi}{\omega_2}$



자연수

↓

$n_1 T_1 = n_2 T_2 \rightarrow \frac{\omega_2}{\omega_1} = \frac{T_1}{T_2} = \frac{n_2}{n_1} =$ 유리수



[redacted] $\sqrt{2} \neq \frac{n_2}{n_1}$

Lissajous 곡선

$\vec{F} = -\vec{\nabla}V \rightarrow V = \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2 + \frac{1}{2}k_3 z^2$

$$\begin{pmatrix} \text{if } k_1 = k_2 = k_3 = k \\ = \frac{1}{2}kr^2 \end{pmatrix}$$

[Ex 4.4.1] $V = \frac{1}{2}(kx^2 + 4ky^2) \rightarrow \vec{F} = -\vec{\nabla} V = (-kx, -4ky) = m(\ddot{x}, \ddot{y})$

$m\ddot{x} + kx = 0 \longrightarrow x = A\cos(\omega t + \alpha)$

$m\ddot{y} + 4ky = 0 \Rightarrow y = B\cos(2\omega t + \beta)$

$\begin{cases} \omega = \sqrt{\frac{k}{m}} \\ \omega_2 = \sqrt{\frac{4k}{m}} = 2\sqrt{\frac{k}{m}} = 2\omega \end{cases}$

$t=0 \quad x = a, \; y = 0 \quad \dot{x} = 0 \quad \dot{y} = v_0$

$y(0) = B\cos\beta = 0 \Rightarrow \cos\beta = 0 \qquad \beta = \frac{\pi}{2}$

$y = B\cos(2\omega t + \frac{\pi}{2}) = \underline{-B\sin(2\omega t)} \rightarrow \dot{y} = -2\omega B\cos(2\omega t)$

$\dot{y}(0) = -2\omega B = v_0$

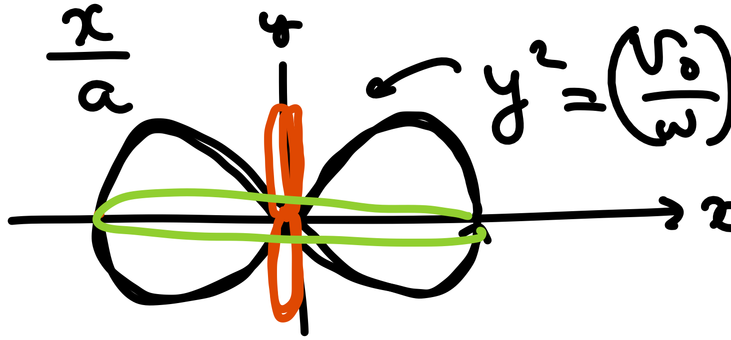
$\therefore B = -\frac{v_0}{2\omega}$

$$\therefore \boxed{y(t) = \frac{v_0}{2\omega}\sin(2\omega t)}$$

$\dot{x} = -\omega A\sin(\omega t + \alpha) \rightarrow \dot{x}(0) = -\omega A\sin\alpha = 0 \rightarrow \sin\alpha = 0 \rightarrow \alpha = 0$

$x = A\cos\omega t \rightarrow x(0) = A = a \longrightarrow \therefore x = a\cos\omega t$

$$y = \frac{v_0}{2\omega}\, 2\, \underbrace{\sin\omega t}_{\sqrt{1-\frac{x^2}{a^2}}}\; \underbrace{\cos\omega t}_{\frac{x}{a}} = \frac{v_0}{\omega}\sqrt{1-\frac{x^2}{a^2}}\cdot\frac{x}{a}$$



$y^2 = \left(\frac{v_0}{\omega}\right)^2 \frac{x^2}{a^2}\left(1 - \frac{x^2}{a^2}\right)$

$\frac{2\pi}{\omega} = T_1 = \textcircled{2}\, \underbrace{T_2} \rightarrow \frac{2\pi}{2\omega}$

§4.5. Electro-magnetic 힘. $\vec{E}, \vec{B}$

$\vec{E} = -\vec{\nabla}\Phi \longrightarrow \vec{\nabla}\times\vec{E} = 0 \Longleftrightarrow \int \vec{E}\cdot d\vec{r}$ = path에 무관

↑ 전위

↑

conservative

↓

$$\oint \vec{E}\cdot d\vec{r} = 0$$

$$\vec{F} = q\vec{E} = m\frac{d^2\vec{r}}{dt^2}$$

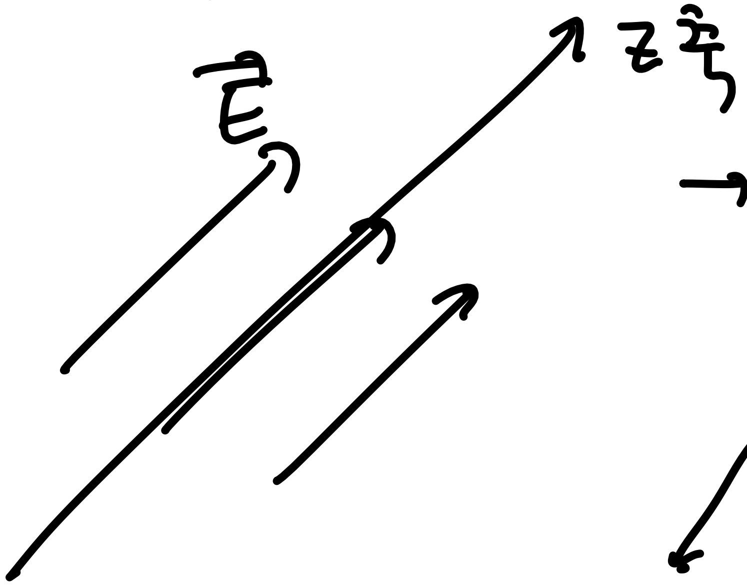
if $\vec{E}$ = constant, uniform

(시간에 무관) (장소에 무관)

$\longrightarrow m\ddot{x} = qE_x$ = const

$m\ddot{y} = qE_y$

$m\ddot{z} = qE_z$



$\rightarrow E_x = E_y = 0 \quad E_z = E$

$m\ddot{x} = 0 = m\ddot{y}$ → 가속도=0 등속

$m\ddot{z} = qE \rightarrow \ddot{z} = \frac{q}{m}E$ = 일정한 가속도

$x = x_0 + v_{x0}t$

$y = y_0 + v_{y0}t$

$z = z_0 + v_{z0}t + \frac{1}{2}\frac{qE}{m}t^2$

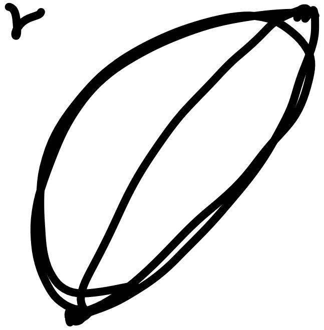
$\vec{\nabla}\times\vec{B} \neq 0$, $\vec{\nabla}\cdot\vec{B} = 0$ ← 자하 없음

$\int \vec{B}\cdot d\vec{r}$ 그 경로에 따라 달라짐

$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$

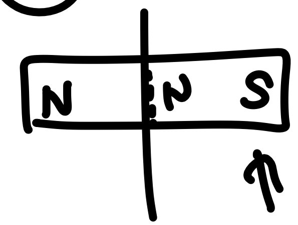
+, −

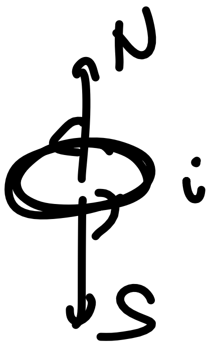
ρ : 전하 있기

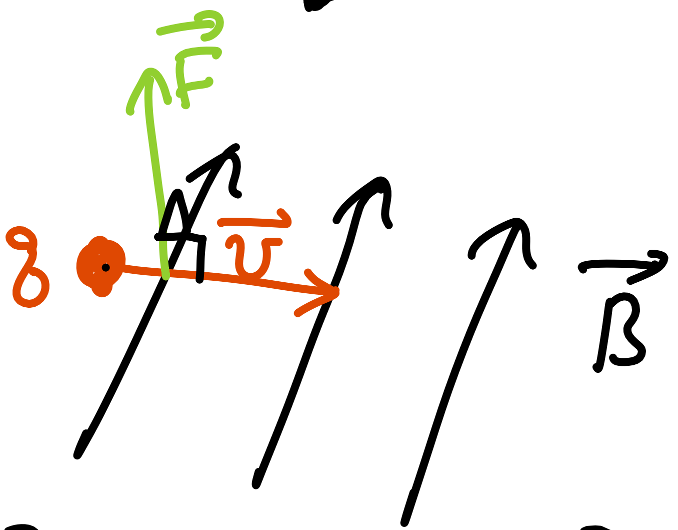


$\oint \vec{B}\cdot d\vec{r} \neq 0$

Ⓝ, S







$\vec{F} = q(\vec{v}\times\vec{B})$

Lorentz 힘

$\vec{F} \perp \vec{v}$ ← 원운동.

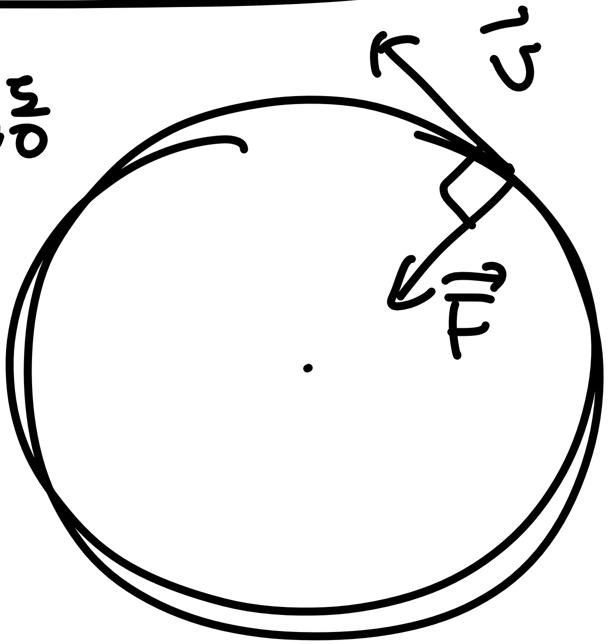
$$\vec{F}\cdot\vec{v} = q\,\vec{v}\cdot(\vec{v}\times\vec{B})$$

$$\overset{\shortparallel}{\vec{B}\cdot(\vec{v}\times\vec{v})}$$

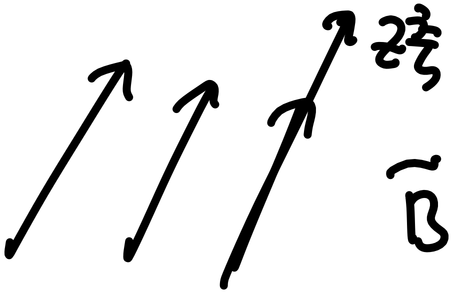
$$\overset{\shortparallel}{0}$$

$= 0$

(cf) 원운동



$$m\frac{d^2\vec{r}}{dt^2} = q(\vec{v}\times\vec{B})$$



$\vec{B}=(0,0,B)$

$$m(\ddot{x},\ddot{y},\ddot{z}) = q\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ v_x & v_y & v_z\\ 0 & 0 & B\end{vmatrix} = q(Bv_y, -Bv_x, 0)$$

$$\left.\begin{aligned}\ddot{x} &= \frac{qB}{m}\dot{y}\\ \ddot{y} &= -\frac{qB}{m}\dot{x}\end{aligned}\right\}$$ integrate

$$\dot{x} = \underbrace{\boxed{\frac{qB}{m}}}_{}\!\!{}^{=\omega_B}\, y + c_1$$

$$\dot{y} = \underbrace{-\frac{qB}{m}x + c_2}$$

$$\to \ddot{x} = \frac{qB}{m}\boxed{\dot{y}} = -\omega_B^2 x + C_2 = -\omega_B^2\underbrace{\left(x - \frac{C_2}{\omega_B^2}\right)}_{\equiv x'}$$

$$\ddot{z} = 0 \to z = z_0 + v_{z0}t$$

$$\to \ddot{x}' = -\omega_B^2 x' \Rightarrow x' = A\cos(\omega_B t+\alpha)$$

$$\boxed{x = A\cos(\omega_B t+\alpha) + C}$$

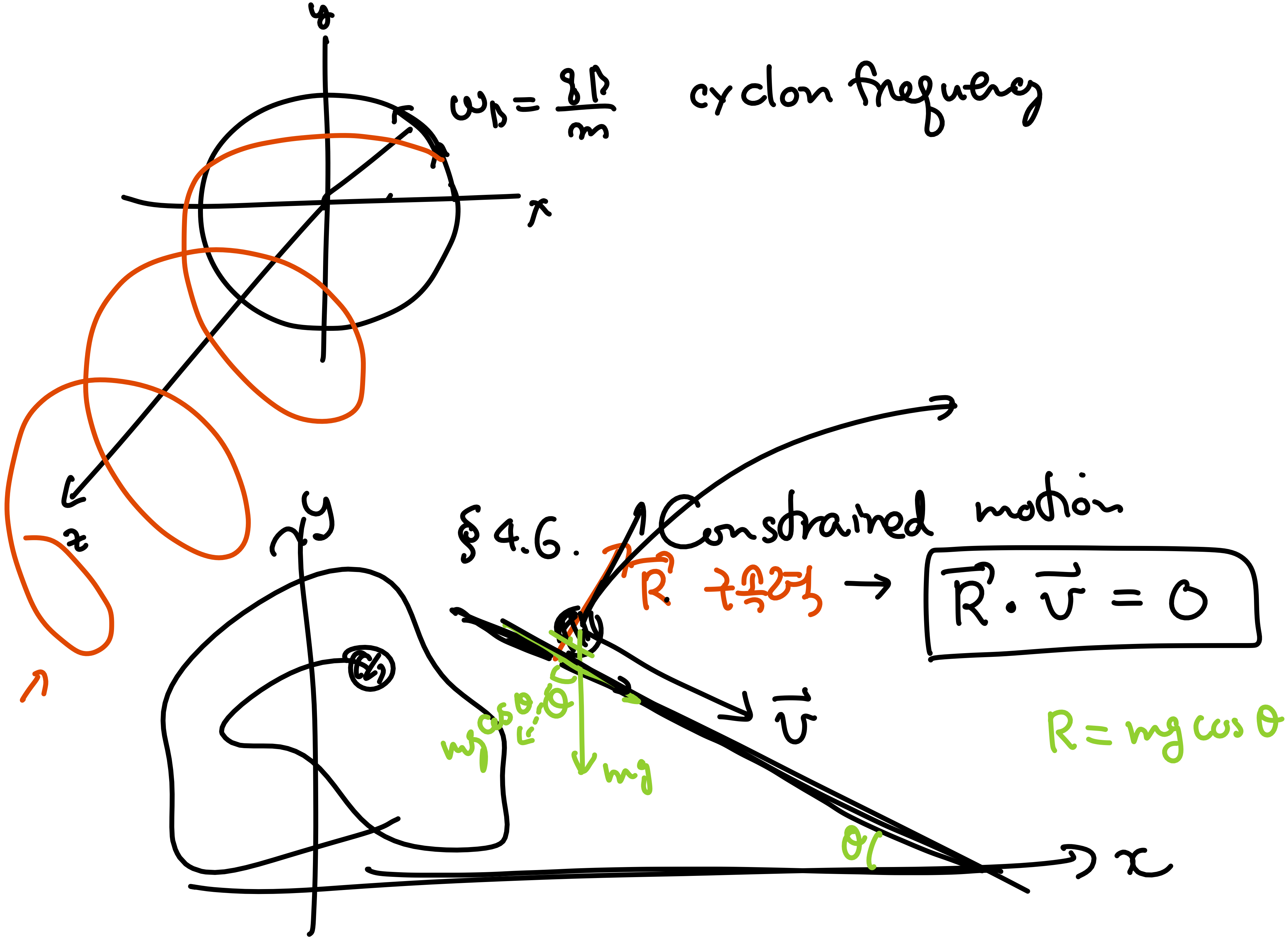
$$\dot{x} = -\omega_B A\sin(\omega_B t+\alpha)$$

$$= \omega_B y + c_1$$

$$\to \boxed{y = -A\sin(\omega_B t+\alpha) - \overset{C'}{\boxed{\frac{c_1}{\omega_B}}}}$$

$$(x-C)^2 + (y+C')^2 = A^2$$

$\omega_B = \frac{qB}{m}$ cyclon frequency

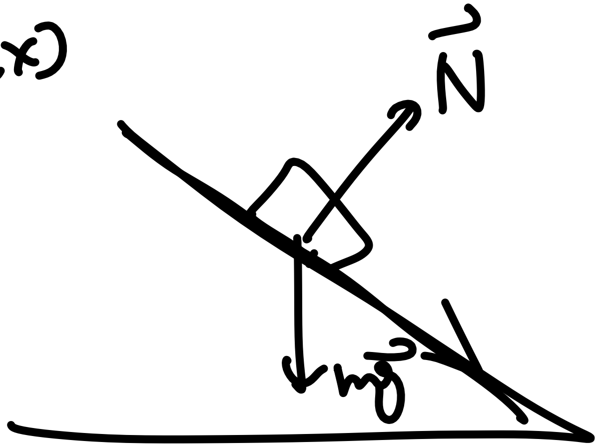


§ 4.6. Constrained motion

$\vec{R}$ 구속력 → $\boxed{\vec{R} \cdot \vec{v} = 0}$

$R = mg\cos\theta$

(ex)



$$\vec{F}_{\text{합}} = m\vec{g} + \vec{N}$$

$$\vec{F}_{\text{합}} = \underbrace{\vec{F}}_{} + \vec{R} = m\frac{d^2\vec{r}}{dt^2}$$

$$= m\frac{d\vec{v}}{dt}$$

$$\frac{d}{dt}\Big(\underbrace{\frac{1}{2}mv^2}_{T}\Big) = m\,\vec{v}\cdot\frac{d\vec{v}}{dt} = \vec{v}\cdot\vec{F} + \underbrace{\vec{v}\cdot\vec{R}}_{\overset{\shortparallel}{0}} = \vec{v}\cdot\vec{F} = \frac{d}{dt}V$$

$$\vec{v}\cdot\underbrace{\vec{F}}_{-\vec{\nabla}V} = -\vec{v}\cdot\vec{\nabla}V = -\underbrace{\frac{d}{dt}V(\vec{r}(t))}_{}$$

$$\frac{d}{dt}V(x(t),y(t),z(t))$$

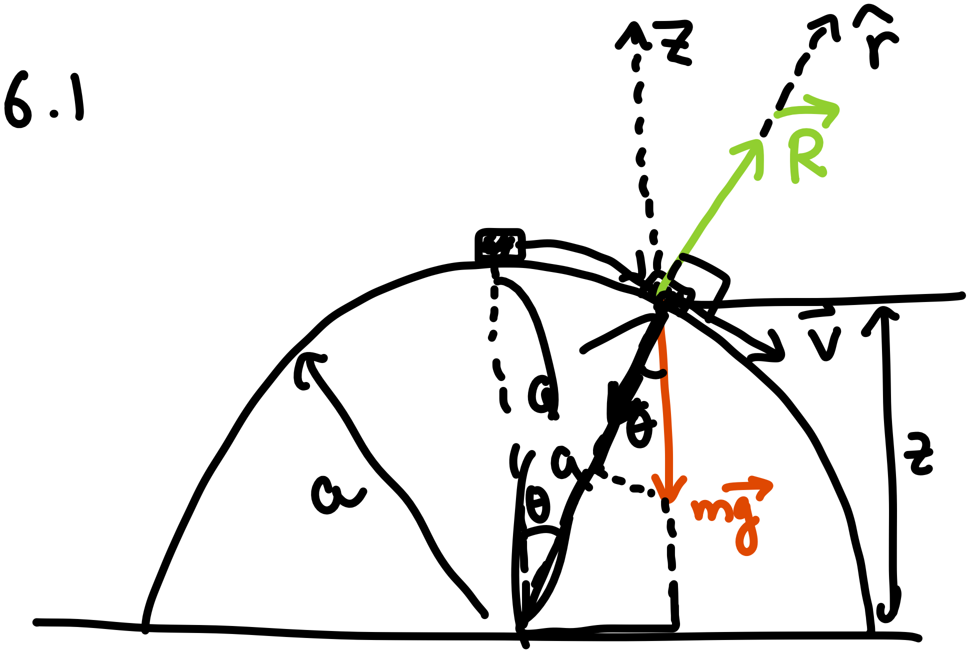
$$= \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z}$$

$$= \dot{\vec{r}}\cdot\vec{\nabla}V$$

$$\to \frac{d}{dt}T = -\frac{dV}{dt} \to \frac{d}{dt}(T+V) = 0$$

$$\therefore T+V = \text{const.}$$

Ex 4.6.1



$$\underline{m\frac{d\vec v}{dt} = m\vec g + \vec R}$$

$\vec v$ 와 내적

$$m\vec v\cdot\frac{d\vec v}{dt} = \frac{1}{2}\frac{d}{dt}(mv^2) = \underbrace{m\vec g\cdot\vec v}_{\vec g = -g\vec k} + \underset{0}{\overset{}{\vec v\cdot\vec R}} \qquad \vec v = \frac{d\vec r}{dt}$$

$$\vec g\cdot\vec v = -g\,\vec k\cdot\frac{d\vec r}{dt} = -g\frac{d}{dt}(z)$$

$$\vec k\cdot\vec r = z$$

$$\therefore \frac{d}{dt}\left(\frac{1}{2}mv^2 + mgz\right) = 0 \longrightarrow \frac{1}{2}mv^2 + mgz = \text{constant} = mga$$

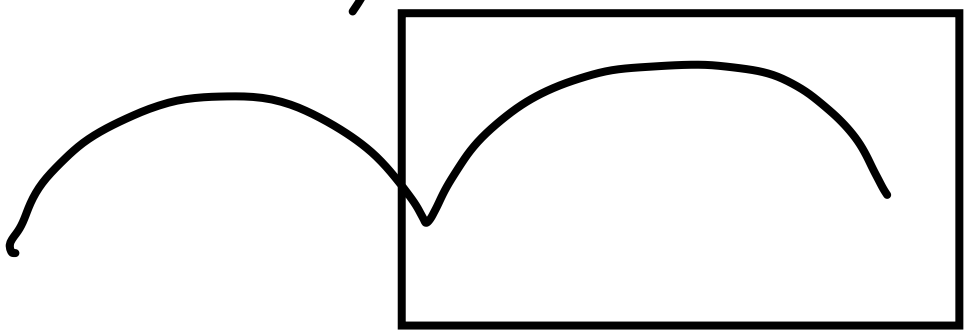
$$\therefore v^2 = 2g(a-z)$$

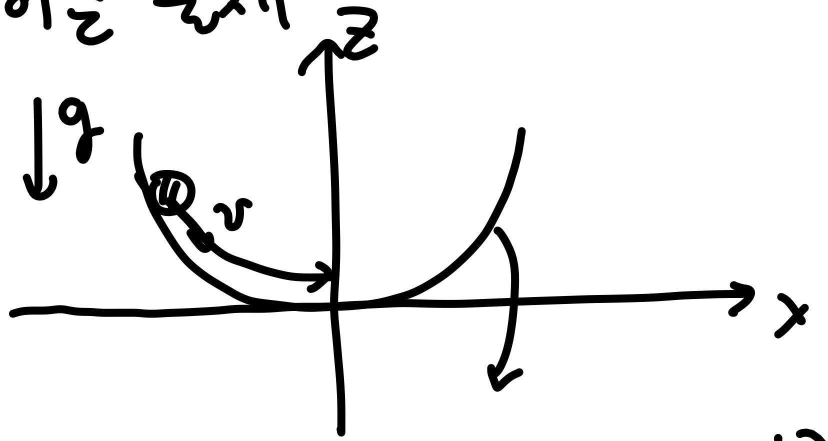
$$m\frac{d\vec v}{dt}\Big|_{\hat r} = \overset{\text{구심가속도}}{-m\frac{v^2}{a}} = -mg\cos\theta + R$$

$$\therefore R = m\left(g\underset{\frac{z}{a}}{\cos\theta} - \frac{v^2}{a}\right) = \frac{m}{a}(gz - v^2) = \frac{m}{a}(3gz - 2ga)$$

구속력 $R=0 \longrightarrow 3z = 2a$ or $z = \frac{2}{3}a$

[Ex 4.6.2] Cycloid 위에서 움직이는 물체





$$x = A(2\phi + \sin 2\phi)$$
$$z = A(1 - \cos 2\phi)$$

$$E = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) + mgz$$

$$= \frac{m}{2}(2A\dot{\phi})^2\left[\overbrace{(1+\cos 2\phi)^2 + \sin^2 2\phi}^{2+2\cos 2\phi}\right]$$
$$+ mgA(1 - \cos 2\phi)$$

$$\dot{x} = A(2\dot{\phi} + 2\dot{\phi}\cos 2\phi)$$
$$= 2A\dot{\phi}(1+\cos 2\phi)$$
$$\dot{z} = 2A\dot{\phi}\sin 2\phi$$

$$1+\cos 2\phi = 2\cos^2\phi$$
$$1-\cos 2\phi = 2\sin^2\phi$$

$$= \underbrace{8mA^2\dot{\phi}^2\cos^2\phi} + 2mgA\sin^2\phi$$

$$4A\sin\phi \equiv s$$
$$\dot{s} = 4A\dot{\phi}\cos\phi$$

$$= \frac{1}{2}m\dot{s}^2 + \underbrace{2mgA\left(\frac{s}{4A}\right)^2}$$
$$+ \frac{1}{2}\underbrace{\boxed{\frac{mg}{4A}}}_{k}{}'' s^2$$

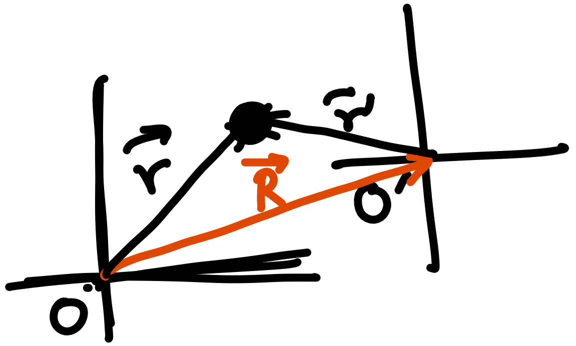
$$\omega = \sqrt{\frac{\boxed{k}}{m}}$$

$$\rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\cancel{m}}{\cancel{m}g/4A}} = 4\pi\sqrt{\frac{A}{g}}$$

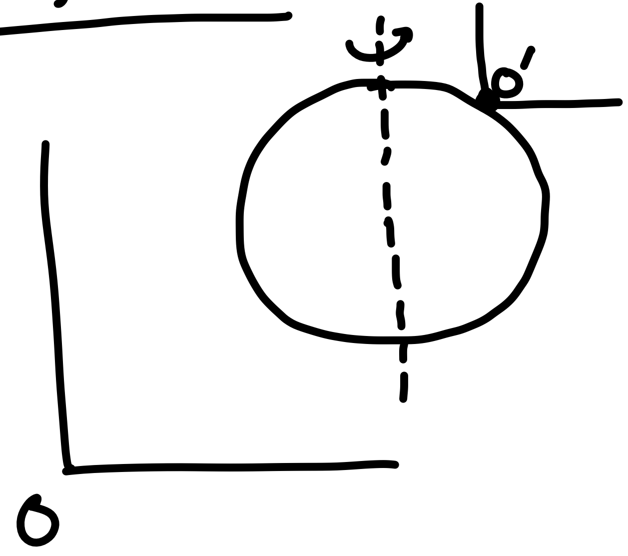
5장. 비관성 좌표계

≡ ↓ 상대적으로 등속이 아닌 속도로 움직이는 좌표계 → 가속 or 회전

└→ $\vec{F} = m\vec{a}$



상대적으로 등속 $\quad \vec{a} = \vec{a}'$



$$\vec{r} = \vec{R} + \vec{r}'$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{r}'}{dt} = \vec{V}_0 + \vec{v}'$$

[가정: 양 좌표계의 시간은 동일하다.]

$$\vec{a} = \underbrace{\frac{d\vec{V}_0}{dt}}_{= 0 \ (\text{등속})} + \vec{a}' \rightarrow \vec{a} = \vec{a}'$$

$$\downarrow$$

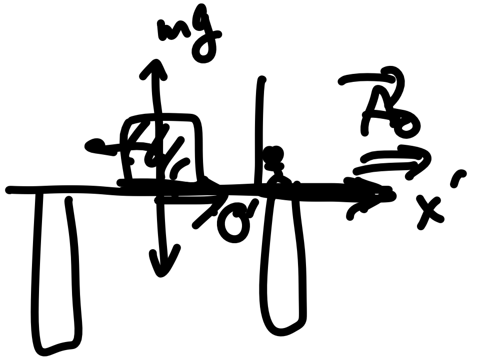
$$m\vec{a} = m\vec{a}' = \vec{F}$$

if O'이 등속이 아니면 (비관성 좌표계); $\vec{a} = \vec{A}_0 + \vec{a}' \quad \left(\vec{A}_0 = \frac{d\vec{V}_0}{dt}\right)$

$$m\vec{a} = \vec{F} = m\vec{A}_0 + m\vec{a}'$$

$$\underline{\underline{m\vec{a}' = \vec{F}'}} = \underline{\underline{\vec{F}}} \underbrace{- m\vec{A}_0}_{} \rightarrow \text{관성력 (가상의 힘)}$$

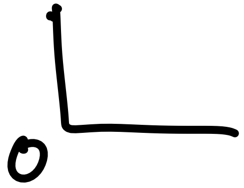
O의 관점
실제 힘.



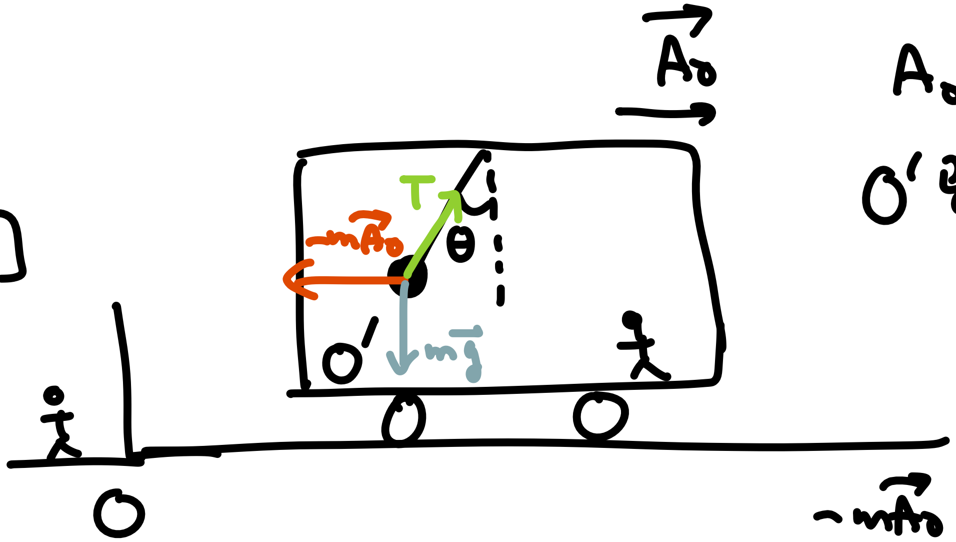
$$\vec{F}' = -m\underbrace{\vec{A}_0}_{A_0\vec{i}'} + \underbrace{m\vec{g} + \mu_s mg\vec{i}'}_{\vec{F}}$$

$$\mu_s mg - mA_0 \geq 0 \rightarrow \text{~~미끄~~}$$

$$A_0 > \mu_s g$$

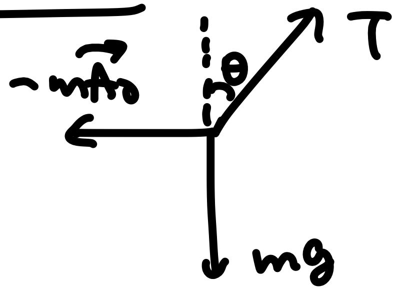


[Ex 5.1.2]



O' 관점

$$m\vec{a}' = \vec{F}' = \vec{F} - m\vec{A}_0$$
$$m\vec{a}' = 0$$
$$\vec{F} = \vec{T} + m\vec{g}$$



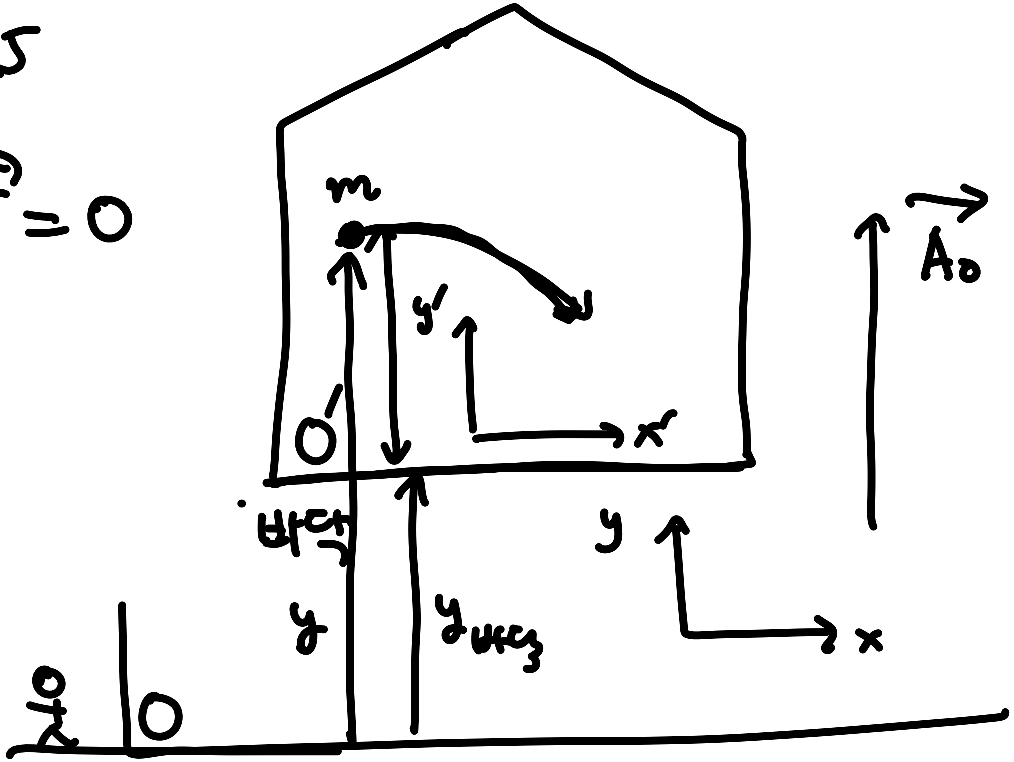
$$T\sin\theta = m|\vec{A}_0|$$
$$T\cos\theta = mg$$
$$\therefore \tan\theta = \frac{|\vec{A}_0|}{g}$$

O 관점.

$$m\vec{a} = \vec{F} = \vec{T} + m\vec{g}$$
$$\vec{a} = \vec{A}_0$$

Ex 5

$\vec{F} = 0$



O' 관점

$$\vec{F}' = -m\vec{A}_0 = m\vec{a}'$$

$$\vec{a}' = \frac{d^2\vec{r}'}{dt^2} = (\ddot{x}', \ddot{y}')$$

$$m\ddot{x}' = 0$$

$$m\ddot{y}' = -mA_0$$

$$x' = x'_0 + \dot{x}'_0 t$$

$$y' = y'_0 + \dot{y}'_0 t - \frac{1}{2}\overset{g}{\widetilde{A_0}} t^2$$

O 관점

$$\vec{F} = 0 = m\overbrace{(\ddot{x}, \ddot{y})}^{\vec{a}}$$

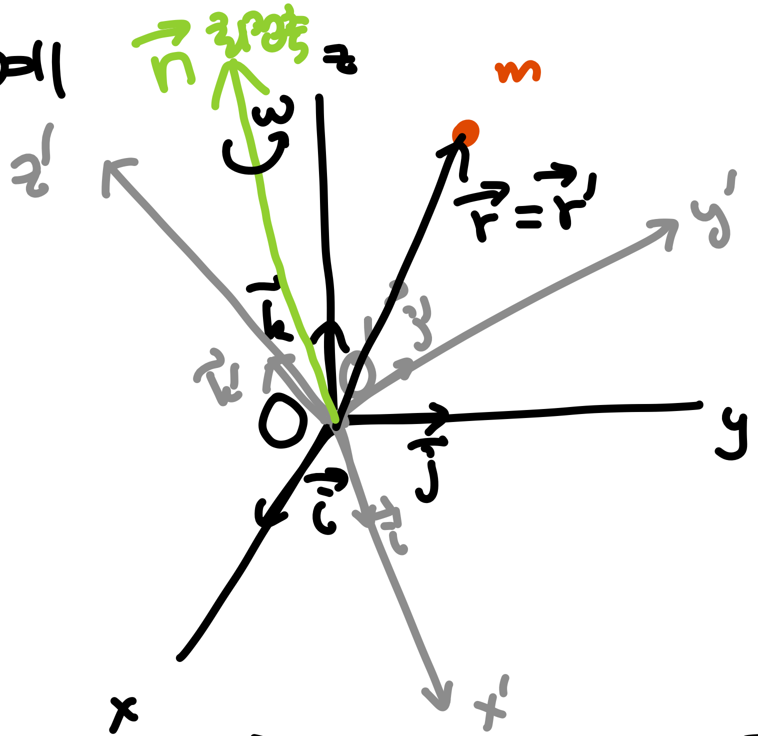
$$\rightarrow \quad x = x_0 + \dot{x}_0 t$$

$$y = y_0 + \dot{y}_0 t$$

$$y_{\text{바닥}} = \frac{1}{2}A_0 t^2$$

$$\rightarrow \quad y - y_{\text{바닥}} = y_0 + \dot{y}_0 t - \frac{1}{2}A_0 t^2$$

5.2. 회전 좌표계



$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$
$$\parallel$$
$$\vec{r}' = x'\vec{i}' + y'\vec{j}' + z'\vec{k}'$$

$$\frac{d\vec{r}}{dt} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k} = \frac{d\vec{r}'}{dt} = \dot{x}'\vec{i}' + \dot{y}'\vec{j}' + \dot{z}'\vec{k}' \qquad \rightarrow \vec{\omega}\times\vec{k}'$$
$$+ \; x'\underbrace{\dot{\vec{i}}'}_{\vec{\omega}\times\vec{i}'} + y'\underbrace{\dot{\vec{j}}'}_{\vec{\omega}\times\vec{j}'} + z'\underline{\underline{\dot{\vec{k}}'}}$$

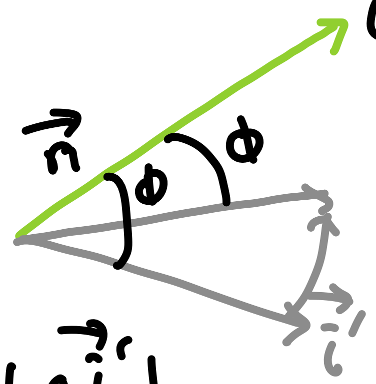
위에서

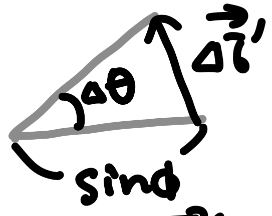
$$\boxed{\frac{d\vec{B}}{dt} = \vec{\omega}\times\vec{B}}$$

$$\frac{d\vec{i}'}{dt} \perp \vec{i}' \;\&\; \vec{\omega}$$

$$\vec{\omega} = \omega\vec{n}$$

Δt 초동안





$$\Uparrow$$

$$|\Delta\vec{i}'| = \sin\phi\,\Delta\theta$$

$$\hookrightarrow \left|\frac{\Delta\vec{i}'}{\Delta t}\right| = \sin\phi\overbrace{\frac{\Delta\theta}{\Delta t}}^{\omega} = \sin\phi\,\omega \;\Rightarrow\; \underline{\frac{d\vec{i}'}{dt} = \vec{\omega}\times\vec{i}'}$$

$$\frac{d\vec r}{dt} = \vec v = \underbrace{\dot x'\vec i' + \dot y'\vec j' + \dot z'\vec k'}_{\vec v'} + \vec\omega\times\Big(\underbrace{x'\vec i' + y'\vec j' + z'\vec k'}_{\vec r = \vec r'}\Big)$$

$$\Rightarrow \quad \vec v = \frac{d\vec r}{dt}\Big|_{fixed} = \vec v' + \vec\omega\times\vec r' = \frac{d\overset{=\vec r}{\vec r}}{dt}\Big|_{rot} + \vec\omega\times\overset{=\vec r}{\vec r'}$$

$$\boxed{\frac{d}{dt}\Big|_{fixed} = \frac{d}{dt}\Big|_{rot} + \vec\omega\times}$$

$$\frac{d\vec Q}{dt}\Big|_{fixed} = \underbrace{\frac{d\vec Q}{dt}\Big|_{rot}}_{\text{축이 회전안한다고 봄}} + \underbrace{\vec\omega\times\vec Q}_{\text{축의 회전효과}}$$

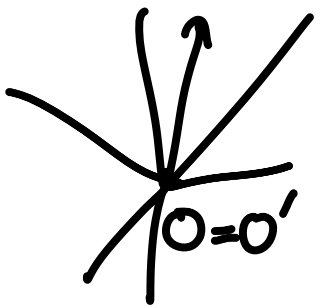
$$\vec v = \left(\frac{d\vec r}{dt}\right)_{fixed}$$

$$= \left(\frac{d\vec r}{dt}\right)_{rot} + \vec\omega\times\vec r'$$

$$\|$$

$$\vec v'$$

$\vec r = \vec r'$



$$\vec{v} = \frac{d\vec{r}}{dt}\Big|_{fixed} \rightarrow \left(\frac{d\vec{v}}{dt}\right)_{fixed} = \vec{a} = \left(\frac{d\vec{v}}{dt}\right)_{rot} + \vec{\omega}\times\vec{v}$$

$$\vec{v} = \left(\frac{d\vec{r}}{dt}\right)_{fixed} = \overset{\vec{v}'}{\overset{\equiv}{\left(\frac{d\vec{r}'}{dt}\right)_{rot}}} + \vec{\omega}\times\vec{r}'$$

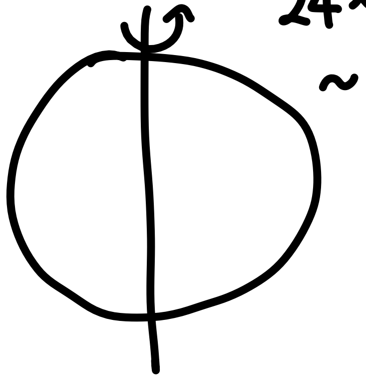
$$\overset{\vec{a}}{\overset{\equiv}{\left(\frac{d^2\vec{r}}{dt^2}\right)_{fixed}}} = \underbrace{\left(\frac{d^2\vec{r}'}{dt^2}\right)_{rot}}_{\equiv \vec{a}'} + \underbrace{\frac{d}{dt}\left(\vec{\omega}\times\vec{r}'\right)_{rot}}_{\underbrace{\left(\frac{d\vec{\omega}}{dt}\right)_{rot}}_{\dot{\vec{\omega}}}\times\vec{r}' + \vec{\omega}\times\left(\frac{d\vec{r}'}{dt}\right)_{rot}} + \vec{\omega}\times\left(\left(\frac{d\vec{r}'}{dt}\right)_{rot} + \vec{\omega}\times\vec{r}'\right)$$

$$\left(\frac{d\vec{r}'}{dt}\right)_{rot} \,\|\, \left(\frac{d\vec{r}'}{dt}\right)_{rot}$$

$$\vec{Q} = \vec{\omega}$$

$$\left(\frac{d\vec{\omega}}{dt}\right)_{fixed} = \left(\frac{d\vec{\omega}}{dt}\right)_{rot} + \underbrace{\vec{\omega}\times\vec{\omega}}_{0}$$

$$\omega = \frac{2\pi}{24\times 3600} \sim 10^{-4}$$



$$\dot{\omega} \simeq 0$$

$$\underbrace{\vec{a} = \vec{a}' + \dot{\vec{\omega}}\times\vec{r}' + \boxed{2\vec{\omega}\times\vec{v}'} + \vec{\omega}\times(\vec{\omega}\times\vec{r}')}$$

↑ transverse acc.
가로 가속도

Coriolis
가속도

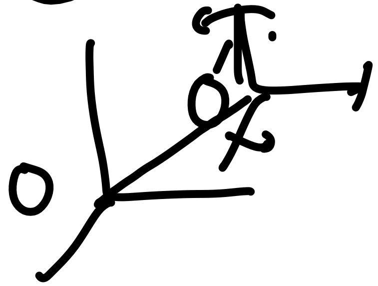
↓ 원심가속도

⇓

$$O = O'$$

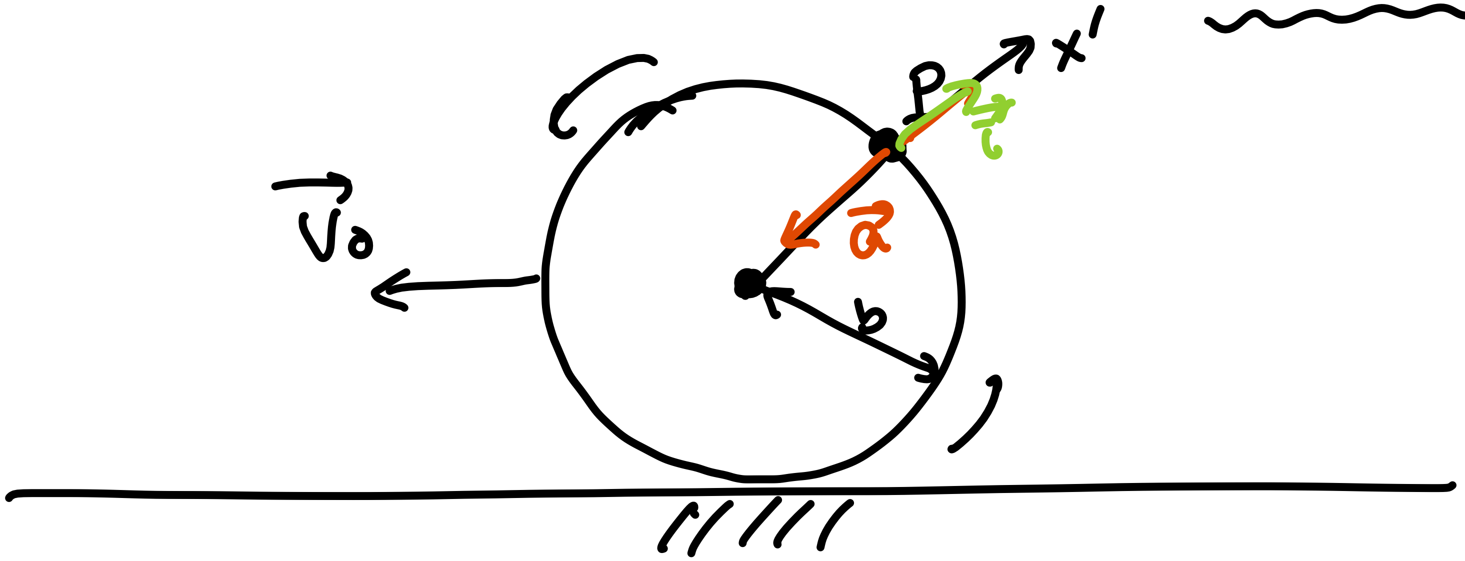
$$\vec{a} = \vec{a}' + \vec{A}_0$$

$$O \neq O'$$



$$\vec{a} = \vec{a}' + \dot{\vec{\omega}} \times \vec{r}' + 2\,\vec{\omega}\times\vec{v}' + \underline{\vec{\omega}\times(\vec{\omega}\times\vec{r}')} + \vec{A}_0$$

[Ex.]



$$\vec{r}' = b\,\vec{i}' \qquad \overset{``\vec{v}'}{\left(\frac{d\vec{r}'}{dt}\right)_{rot}} = \vec{0}$$

$$\vec{a}' = \left(\frac{d^2\vec{r}'}{dt^2}\right)_{rot} = \vec{0}$$

$$\vec{\omega} = \omega\,\vec{k}$$

$$\vec{k} = \vec{k}'$$

$$\omega b = V_0 \rightarrow \omega = \frac{V_0}{b} \qquad \therefore \vec{\omega} = \frac{V_0}{b}\vec{k}'$$

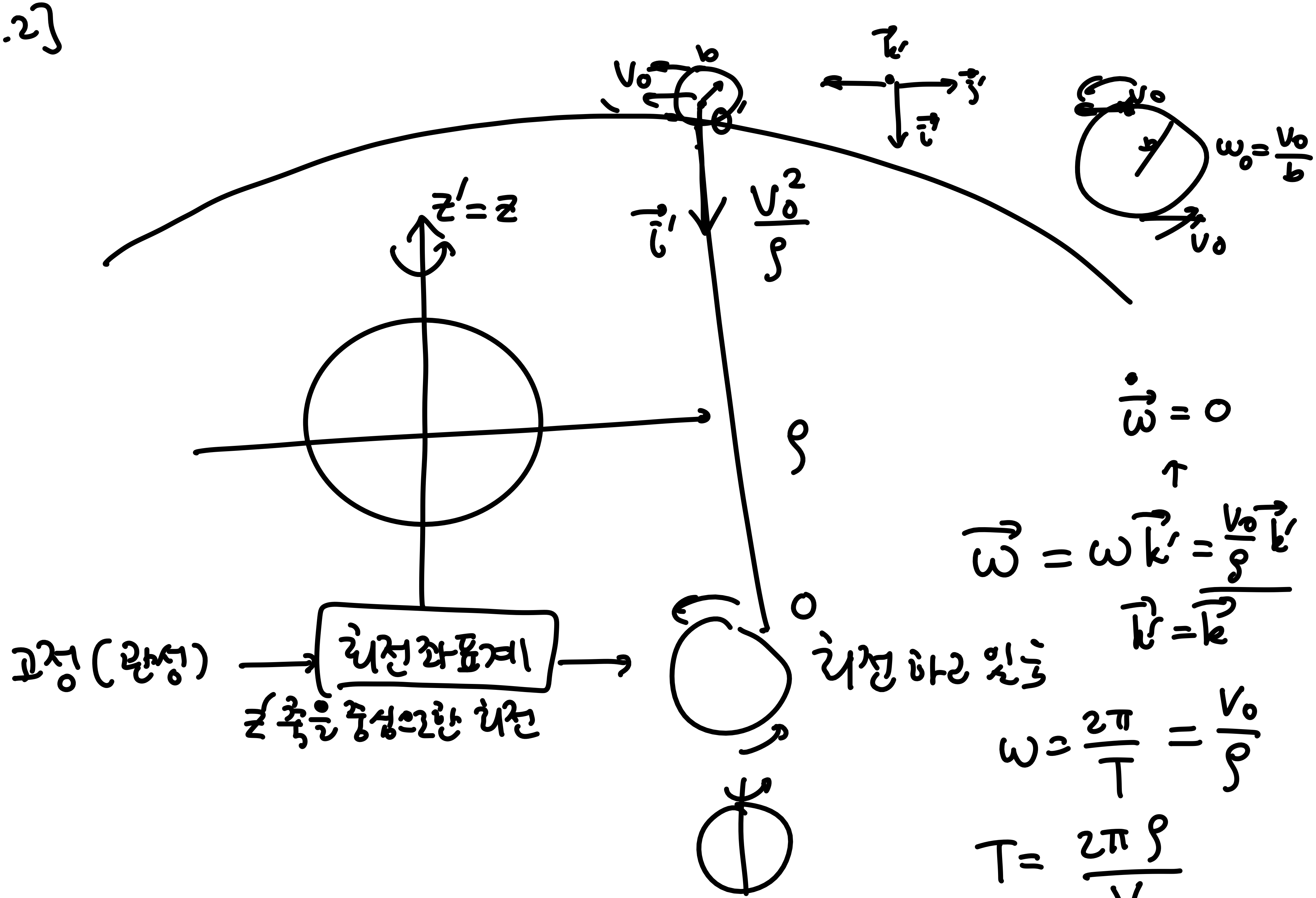
$$\dot{\vec{\omega}} = 0$$

$$\vec{A}_0 = 0$$

$$\vec{a} = \vec{0} + \vec{0} + \vec{0} + \left(\frac{V_0}{b}\right)^2 b\,\underbrace{\vec{k}'\times\underbrace{(\vec{k}'\times\vec{i}')}_{\vec{j}'}}_{-\vec{i}'}$$

$$\vec{a} = -\frac{V_0^2}{b}\vec{i}'$$

[Ex 5.2.2]



$$\dot{\vec{\omega}} = 0$$

$$\uparrow$$

$$\vec{\omega} = \omega \vec{k}' = \frac{V_0}{\rho} \vec{k}'$$

$$\vec{k}' = \vec{k}$$

$$\omega = \frac{2\pi}{T} = \frac{V_0}{\rho}$$

$$T = \frac{2\pi\rho}{V_0}$$

$$\overset{\downarrow}{\vec{a}} = \underline{\underline{\vec{a}'}} + \cancel{\dot{\vec{\omega}} \times \vec{r}'} + 2\vec{\omega} \times \underline{\underline{\vec{V}'}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}') + \vec{A}_0$$

$$\vec{A}_0 = \frac{V_0^2}{\rho} \vec{i}' \quad \vec{a}' = -b\omega_0^2 \vec{k}' = -\frac{V_0^2}{b} \vec{k}', \vec{r}' = b\vec{k}', \vec{V}' = -V_0 \vec{j}'$$

$$\vec{a} = \underline{\underline{\vec{a}'}} + \cancel{\dot{\vec{\omega}} \times \vec{r}'} + 2\,\vec{\omega}\times\underline{\underline{\vec{v}'}} + \vec{\omega}\times(\vec{\omega}\times\vec{r}') + \vec{A}_0$$

$$\vec{A}_0 = \frac{V_0^2}{\rho}\vec{i}' \qquad \vec{a}' = -b\omega_0^2\vec{k}' = -\frac{V_0^2}{b}\vec{k}', \ \vec{r}' = b\,\vec{k}', \ \vec{v}' = -V_0\vec{j}'$$

$$\vec{\omega} = \frac{V_0}{\rho}\vec{k}'$$

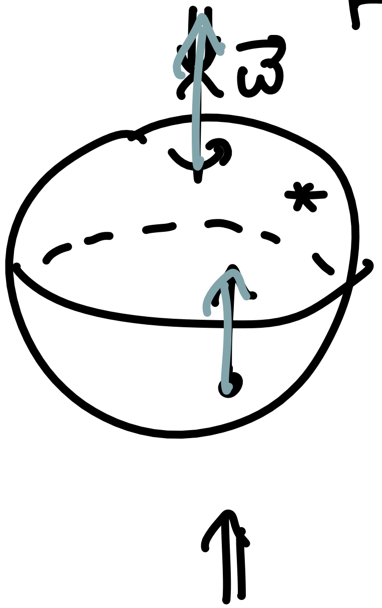
$$\vec{a} = -\frac{V_0^2}{b}\underset{\underset{\vec{k}}{\shortparallel}}{\vec{k}'} + \underbrace{2\,\frac{V_0}{\rho}\vec{k}'\times(-V_0\vec{j}')}_{2\frac{V_0^2}{\rho}\vec{i}'} + b\left(\frac{V_0}{\rho}\right)^2\vec{k}'\times\underset{\underset{0}{\shortparallel}}{(\vec{k}'\times\vec{k}')} + \frac{V_0^2}{\rho}\vec{i}'$$

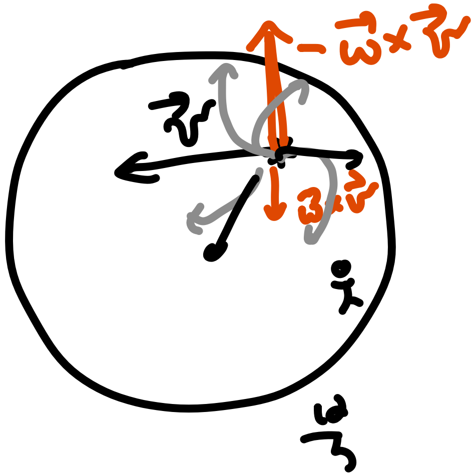
$$\underline{\vec{a} = 3\,\frac{V_0^2}{\rho}\vec{i}' - \frac{V_0^2}{b}\vec{k}'} \ 〃$$

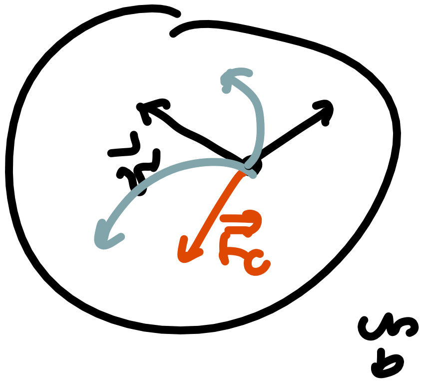
5.3. 비관성좌표계에서의 운동방정식

$$\underset{\overset{\|}{\vec F}}{m\vec a} = m\Big(\underbrace{\vec a'}_{\overset{\|}{\vec F'}} + \dot{\vec\omega}\times\vec r' + 2\vec\omega\times\vec v' + \vec\omega\times(\vec\omega\times\vec r') + \vec A_0\Big)$$

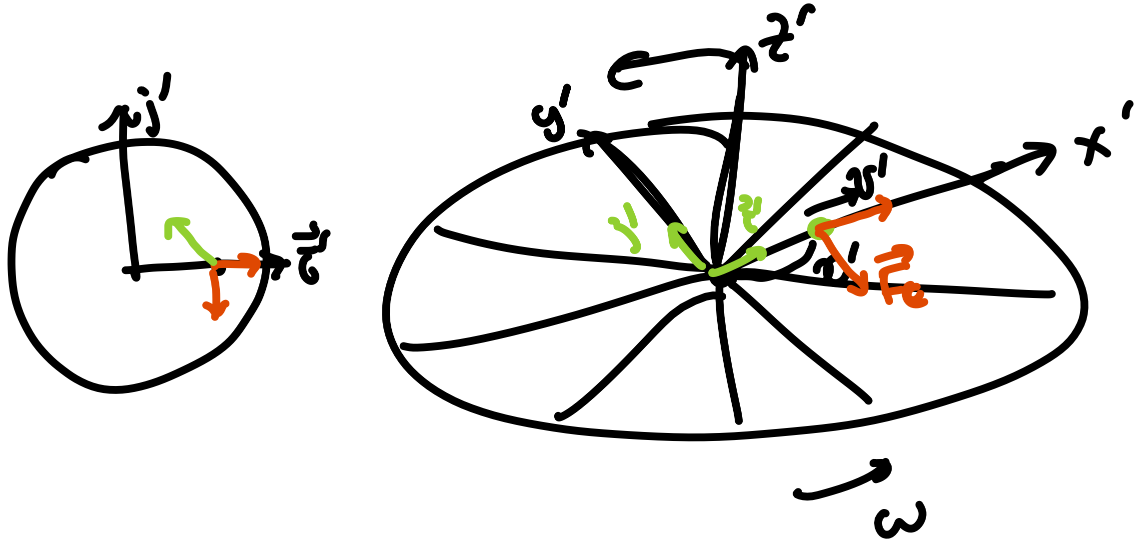
$$\vec F' = m\vec a' = \vec F \underbrace{\underbrace{\overset{\text{관성력}}{-m\vec A_0}}_{} \overset{\text{가로힘}}{-m\dot{\vec\omega}\times\vec r'} \overset{\text{코리올리힘}}{-2m\vec\omega\times\vec v'} \underset{\text{원심력}}{-m\vec\omega\times(\vec\omega\times\vec r')}}_{}$$







[Ex 5.3.1]



$$\vec{\omega} = \omega \vec{k}' \qquad \dot{\vec{\omega}} = 0$$

$$\vec{v}' = v' \vec{i}' \qquad \vec{a}' = 0$$

$$\vec{r}' = x' \vec{i}'$$

 $O = O'$

$$\vec{F}' = \underset{\underset{-mg\vec{k}'}{\parallel}}{\vec{F}} - m \underset{\underset{0}{\parallel}}{\dot{\vec{\omega}}} \times \vec{r}' - m2 \underbrace{\vec{\omega} \times \vec{V}'}_{\underset{\underset{\omega v' \vec{j}'}{\parallel}}{\omega \vec{k}' \times v' \vec{i}'}} - m \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{r}')}_{x' \omega^2 \underbrace{\vec{k}' \times (\underbrace{\vec{k}' \times \vec{i}'}_{\vec{j}'})}_{-\vec{i}'}} - m \overset{\overset{0}{\parallel}}{\vec{A}_0} + \vec{F}_{살-개미}$$

$$\vec{F}' = -mg\vec{k}' \underbrace{- 2m\omega v' \vec{j}' + m\omega^2 x' \vec{i}'}_{} + \underbrace{\vec{F}_{살-개미}}_{} = 0$$

[Ex 5.3.2]

$\vec{F}$

마찰력

$$|\vec{F}| = \mu_s mg \leq \left| -2m\omega v' \vec{j}' + m\omega^2 x' \vec{i}' \right|$$

$$\parallel$$

$$\sqrt{(m\omega^2 x')^2 + (2m\omega v')^2}$$

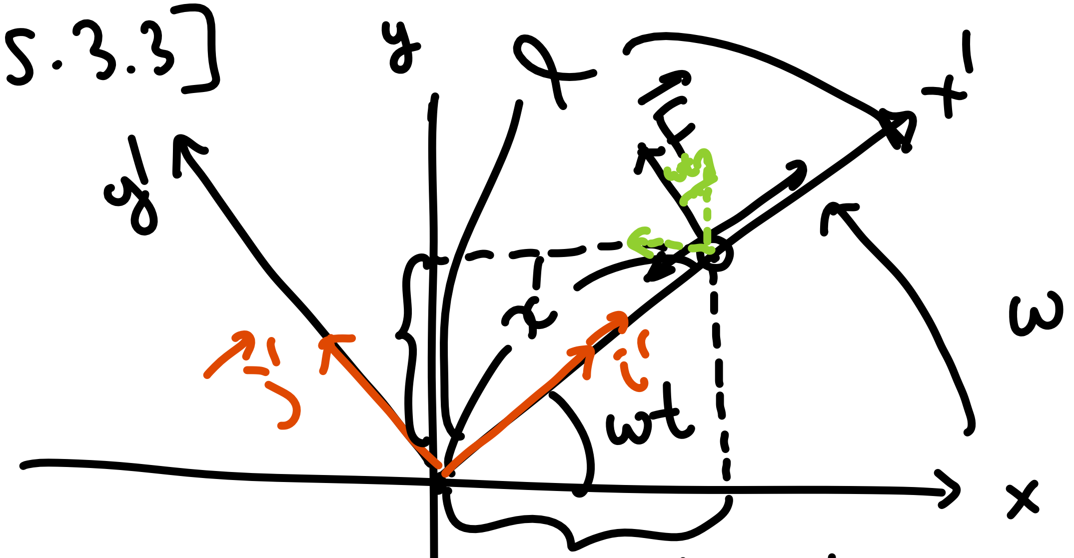
$$\mu_s mg = \sqrt{(m\omega^2 x')^2 + (2m\omega v')^2}$$

$$\mu_s^2 \cancel{m^2} g^2 = 4\cancel{m^2}\omega^2 v'^2 + \cancel{m^2}\omega^4 x'^2$$

$$\therefore x' = \sqrt{\frac{\mu_s^2 g^2 - 4\omega^2 v'^2}{\omega^4}}$$

$\vec{A}_0 = 0$

[Ex S.3.3]



$$\overset{F\vec{j}'}{\overset{''}{\vec F}} - \cancel{\dot{\omega}}\, 2m\underset{\vec{k}'}{\vec\omega}\times\vec v' - m\underset{\vec{k}'}{\vec\omega}\times\underset{\vec{i}'}{(\vec\omega\times\vec r')}$$

x'-y' 좌표계

$\vec r' = x'\vec i' \rightarrow \vec v' = \dot{x}'\vec i'$

$\vec\omega = \omega\vec k'$

$F_x = -F\sin\omega t$

$F_y = F\cos\omega t$

$x = x'\cos\omega t$

$y = x'\sin\omega t$

$$\vec F' = F\vec j' - 2m\omega\dot{x}'\vec j' + m\omega^2 x'\vec i' = m\vec a'$$

$$\vec a' = \ddot{x}'\vec i'$$

$\vec i'$: $m\ddot{x}' = m\omega^2 x' \longrightarrow \ddot{x}' = \omega^2 x' \rightarrow x' = e^{\omega t}, e^{-\omega t}$

$\vec j'$: $F - 2m\omega\dot{x}' = 0$

$$x'(t) = Ae^{\omega t} + Be^{-\omega t} = A(e^{\omega t} - e^{-\omega t})$$

$$\downarrow$$

$$\dot{x}' = A\omega(e^{\omega t} + e^{-\omega t})$$

$x'(0) = 0$, $\dot{x}'(0) = \omega l = \epsilon$

$\downarrow$

$A + B = 0$

$\downarrow$

$B = -A$

$\dot{x}'(0) = 2A\omega = \omega l$

$\therefore A = \frac{l}{2}$

$$\therefore x'(t) = \frac{l}{2}(e^{\omega t} - e^{-\omega t})$$

$$= l \sinh \omega t = \frac{\epsilon}{\omega}\sinh(\omega t)$$

$$x'(T) = l = l\underbrace{\sinh \omega T}_{1} \rightarrow \omega T = \sinh^{-1} 1 \rightarrow T = \frac{1}{\omega}\underbrace{\sinh^{-1} 1}_{0.88}$$

관성좌표계:

$x = x'\cos\omega t \rightarrow \dot{x} = \dot{x}'\cos\omega t - \omega x'\sin\omega t$

① $\ddot{x} = \ddot{x}'\cos\omega t - 2\omega\dot{x}'\sin\omega t - \omega^2 x'\cos\omega t = -\frac{F}{m}\sin\omega t$

$F_x = m\ddot{x}$

$y = x'\sin\omega t$

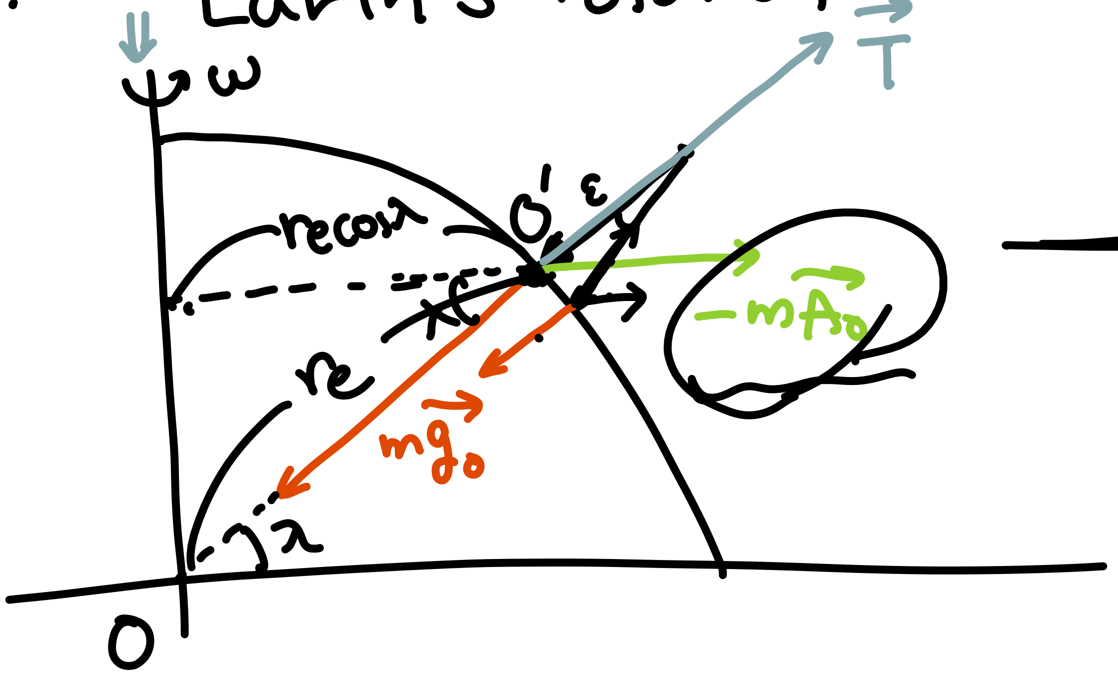
$\dot{y} = \dot{x}'\sin\omega t + \omega x'\cos\omega t$

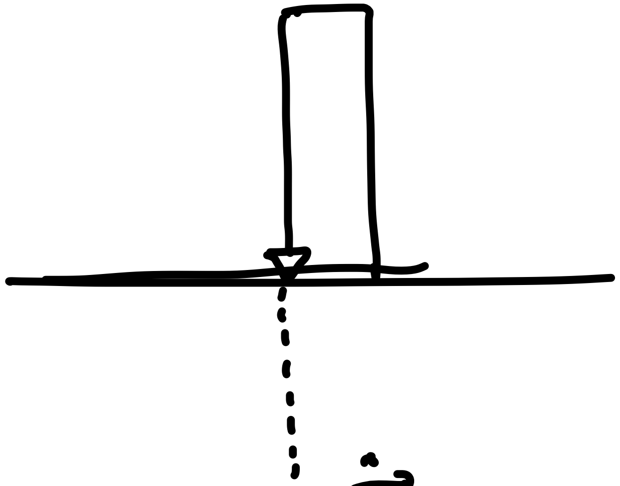
② $\ddot{y} = \ddot{x}'\sin\omega t + 2\omega x'\cos\omega t - \omega^2 x'\sin\omega t = \frac{F}{m}\cos\omega t$

① $\cos\omega t$ + ② $\sin\omega t$

$$\boxed{\ddot{x}' - \omega^2 x' = 0}$$

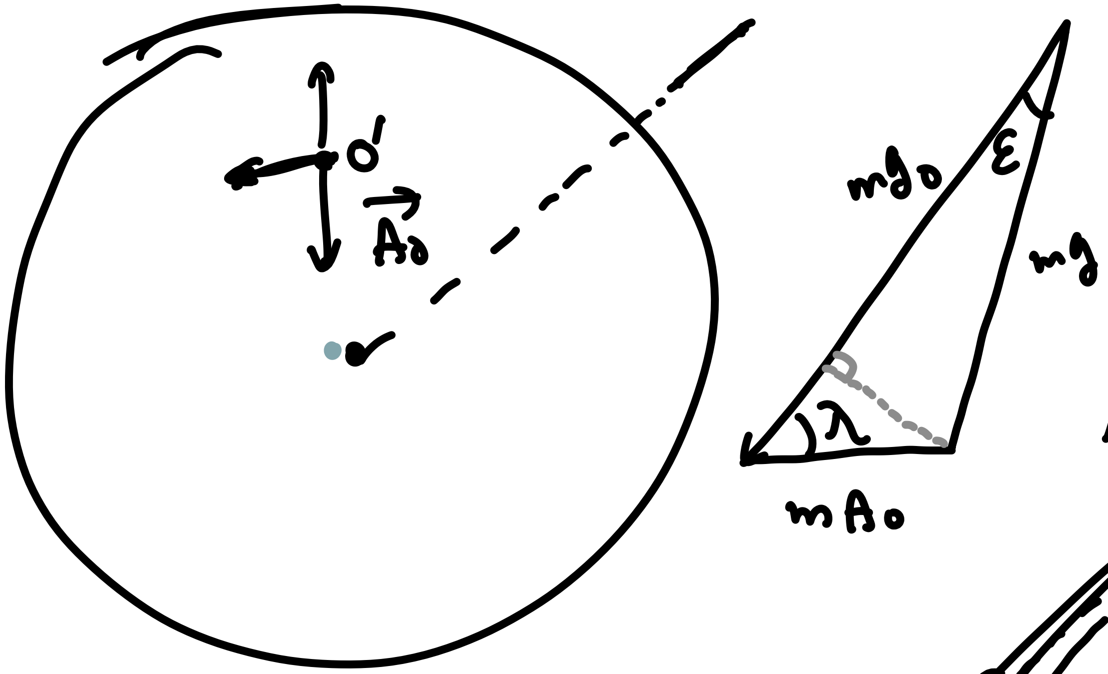
§5.4. Earth's rotation

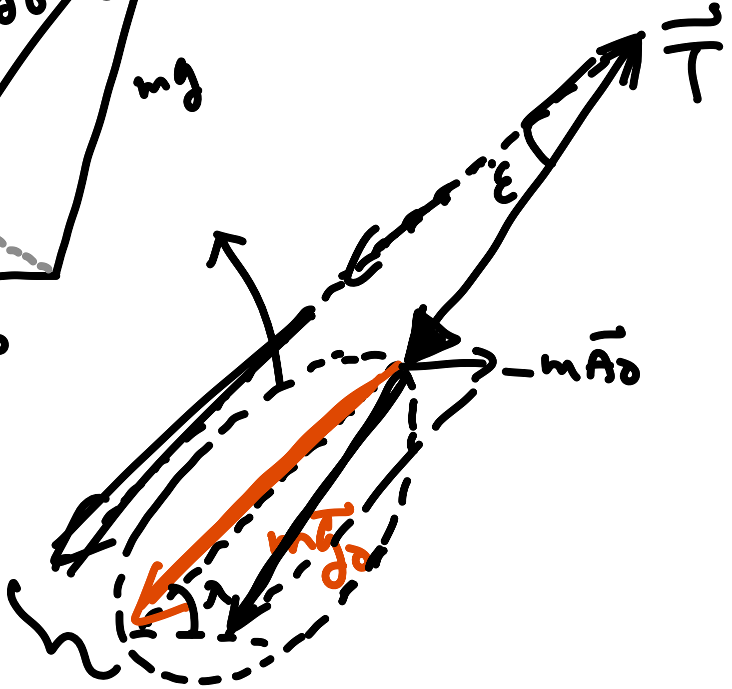




$$\left.\begin{array}{l}\vec{\omega}=0\\ \vec{v}'=0\\ \vec{r}'=0\end{array}\right\} \Rightarrow \vec{F}'=\underbrace{\vec{F}}_{\text{중력+장력}}-m\vec{A_o}$$

$$=\vec{0}$$





$$\underbrace{\vec{T}+m\vec{g_o}}_{\overset{\shortparallel}{m\vec{g}}}-m\vec{A_o}=0$$

$$\vec{g}=\vec{g_o}-\vec{A_o}$$

$$mA_o\sin\lambda = mg\sin\epsilon$$

$$\sin\epsilon \approx \epsilon \rightarrow \boxed{\epsilon \approx \frac{A_o}{g}\sin\lambda}$$

Taylor

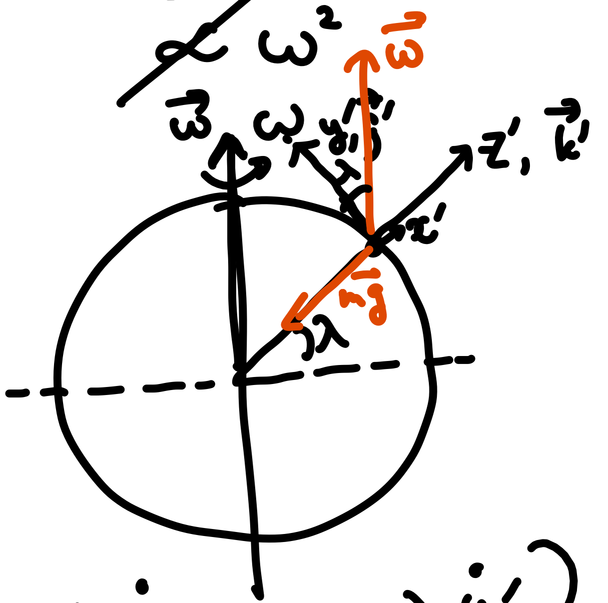
$$A_o = r_e\cos\lambda\,\omega^2$$

$$\Rightarrow \epsilon = \frac{r_e\omega^2\overbrace{\sin\lambda\cos\lambda}^{\frac{1}{2}\sin 2\lambda}}{g} \cong 0.1^\circ$$

$$\vec{F}' = m\ddot{\vec{r}}' = \vec{F} + \underbrace{m\vec{g}_0 - m\vec{A}_0}_{\to\, m\vec{g}} - 2m\vec{\omega}\times\dot{\vec{r}}' - \cancel{\underbrace{m\vec{\omega}\times(\vec{\omega}\times\vec{r}')}_{\propto\,\omega^2}}$$

$\uparrow$ 중력外 모든 실제힘 $\qquad \dot{\vec{\omega}} = 0 \qquad -mg\vec{k}'$

$$m\ddot{\vec{r}}' = \vec{F} + m\overset{\shortparallel}{\vec{g}} \underline{- 2m\vec{\omega}\times\dot{\vec{r}}'}$$



$$\vec{\omega} = \omega\cos\lambda\,\vec{j}' + \omega\sin\lambda\,\vec{k}'$$

$$\vec{\omega}\times\dot{\vec{r}}' = \begin{vmatrix} \vec{i}' & \vec{j}' & \vec{k}' \\ 0 & \omega\cos\lambda & \omega\sin\lambda \\ \dot{x}' & \dot{y}' & \dot{z}' \end{vmatrix} = \vec{i}'(\omega\cos\lambda\,\dot{z}' - \omega\sin\lambda\,\dot{y}') + \vec{j}'(\omega\sin\lambda\,\dot{x}') + \vec{k}'(-\omega\cos\lambda\,\dot{x}')$$

$\vec{F} = 0$ (air resistance is neglected)

$$\ddot{x}' = -2(\omega\cos\lambda\,\dot{z}' - \omega\sin\lambda\,\dot{y}')$$

$$\ddot{y}' = -2\omega\sin\lambda\,\dot{x}'$$

$$\ddot{z}' = -g + 2\omega\cos\lambda\,\dot{x}'$$

$$\dot{x}' = -2(\omega\cos\lambda\, z' - \omega\sin\lambda\, y') + \dot{x}_0 \quad \Big| \quad \begin{cases} \underline{\dot{y}' = -2\underset{\sim}{\omega}\sin\lambda\, x' + \dot{y}_0'} \\ \dot{z}' = -gt + \underline{2\omega\cos\lambda\, x'} + \dot{z}_0' \end{cases}$$

$$\ddot{x}' = -2(\omega\cos\lambda\, \underset{\sim}{\dot{z}'} - \omega\sin\lambda\, \underset{\sim}{\dot{y}'})$$

$+\omega^2$

$$\ddot{x}' = -2\omega\left(\cos\lambda(-gt + \dot{z}_0') - \sin\lambda\, \dot{y}_0'\right)$$

$\downarrow$

$$\dot{x}' = -2\omega\left(\cos\lambda\left(-\frac{1}{2}gt^2 + \dot{z}_0' t\right) - \sin\lambda\, \dot{y}_0' t\right) + \dot{x}_0$$

$\downarrow$

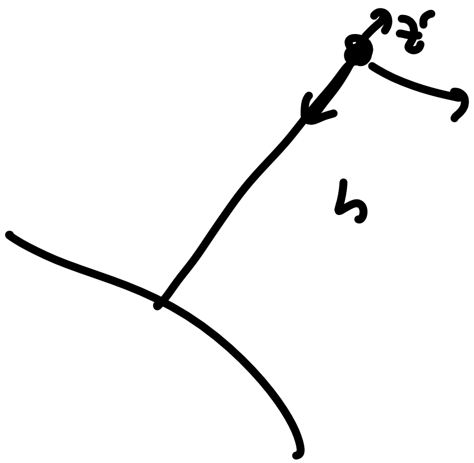
$$\rightarrow x' = -2\omega\left(\cos\lambda\left(\underset{\sim}{-\frac{1}{6}gt^3} + \frac{\dot{z}_0'}{2}t^2\right) - \sin\lambda\, \dot{y}_0' \frac{t^2}{2}\right) \underset{\sim}{+ \dot{x}_0' t + x_0'}$$

$$\dot{y}' = -2\omega\sin\lambda\left(\dot{x}_0' t + x_0'\right) + \dot{y}_0'$$

$$\rightarrow y' = -2\omega\sin\lambda\left(\frac{\dot{x}_0'}{2}t^2 + x_0' t\right) + \dot{y}_0' t + y_0'$$

$$\rightarrow z' = -\frac{1}{2}gt^2 + 2\omega\cos\lambda\left(\frac{1}{2}\dot{x}_0' t^2 + x_0' t\right) + \dot{z}_0' t + z_0'$$

[Ex 5.4.1]



$\dot{z}'_0 = \dot{x}'_0 = \dot{y}'_0 = 0$

$z'_0 = h$, $x'_0 = y'_0 = 0$

$x' = \frac{1}{3}\omega g \cos\lambda\, t^3$

$y' = 0$

$z' = h - \frac{1}{2}gt^2$

$z' = 0 \rightarrow \quad t = \sqrt{\frac{2h}{g}}$

$$x' = \frac{1}{3}\underset{\underset{\frac{2\pi}{24\times3600}}{\uparrow}}{\omega}\,\underset{\underset{9.8}{\uparrow}}{g}\,\cos\underset{\uparrow}{\lambda}\left(\sqrt{\frac{2h}{g}}\right)^3 \underset{\lambda=45^\circ}{=} 1.55\,\text{cm}$$

$h = 100\,\text{m}$ "

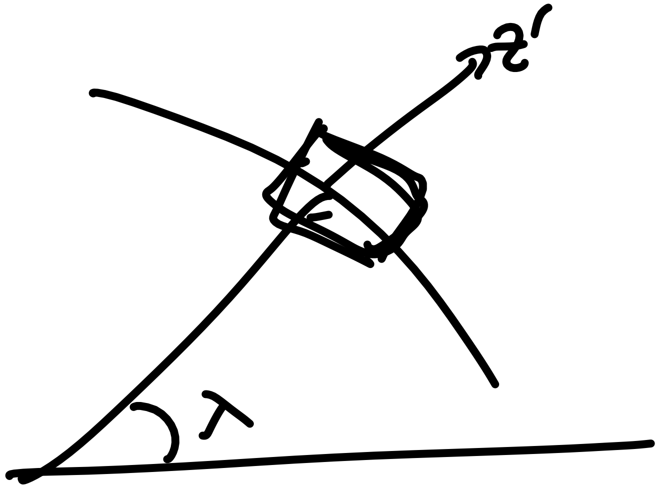
$\ominus\, \omega\, \sin\lambda \frac{H^2}{v_0}$

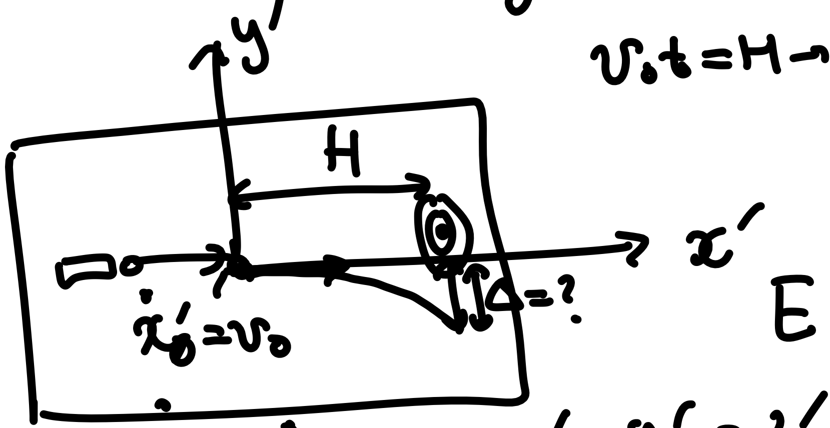
"

$y' = -\omega v_0 \sin\lambda\, t^2$

$v_0 t = H \rightarrow t = \frac{H}{v_0}$

[Ex 5.4.2]



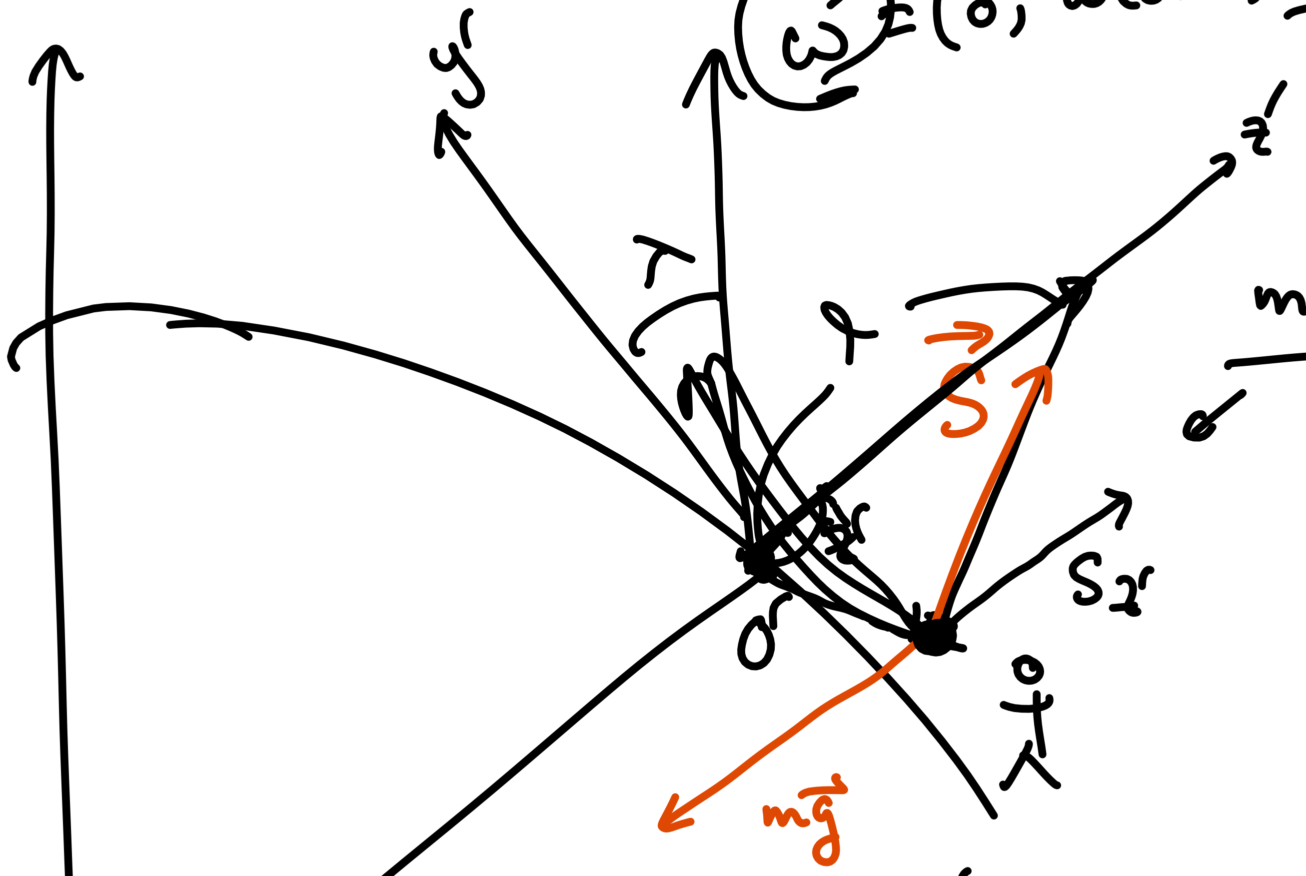


$\dot{y}'_0 = \dot{z}'_0 = 0 \quad x'_0 = y'_0 = z'_0 = 0$

§5.6 푸코의 진자.

$\boxed{\vec{\omega}} = (0, \omega\cos\lambda, \underline{\omega\sin\lambda})$

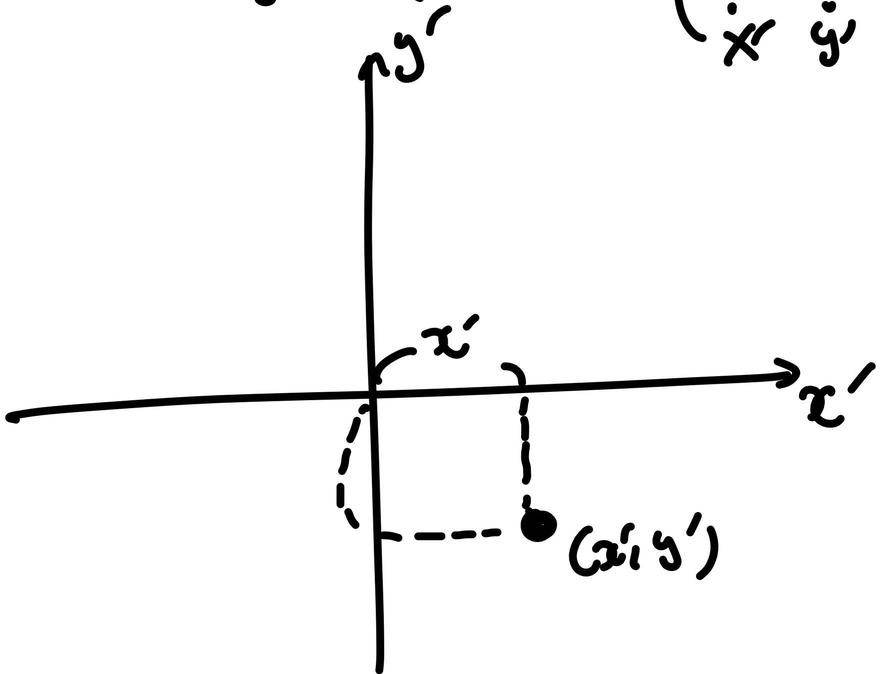
$\dot{\vec{\omega}} = 0 \qquad \omega^2 \approx 0$

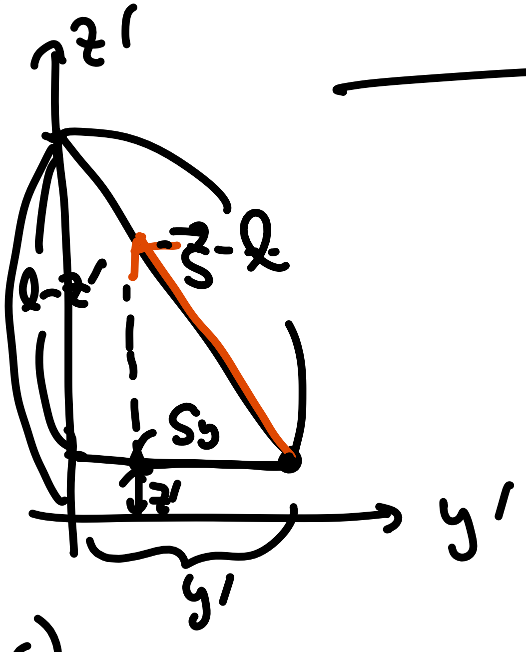


$$m\ddot{\vec{r}}' = m\vec{g} + \vec{S} - 2m\underbrace{\vec{\omega}\times\dot{\vec{r}}'}$$

$$\begin{pmatrix} i' & j' & k' \\ 0 & \omega\cos\lambda & \omega\sin\lambda \\ \dot{x}' & \dot{y}' & \dot{z}' \end{pmatrix}$$

z′에서 보면





$$m\ddot{x}' = -\frac{S}{l}x' - 2m(\omega\cos\lambda\,\dot{z}' - \omega\sin\lambda\,\dot{y}')$$

$$m\ddot{y}' = -\frac{S}{l}y' - 2m(\omega\sin\lambda\,\dot{x}')$$

$$m\ddot{z}' = -mg + \frac{l-z'}{l}S - 2m(-\omega\cos\lambda\,\dot{x}')$$

$S_y : S = -y' : l$

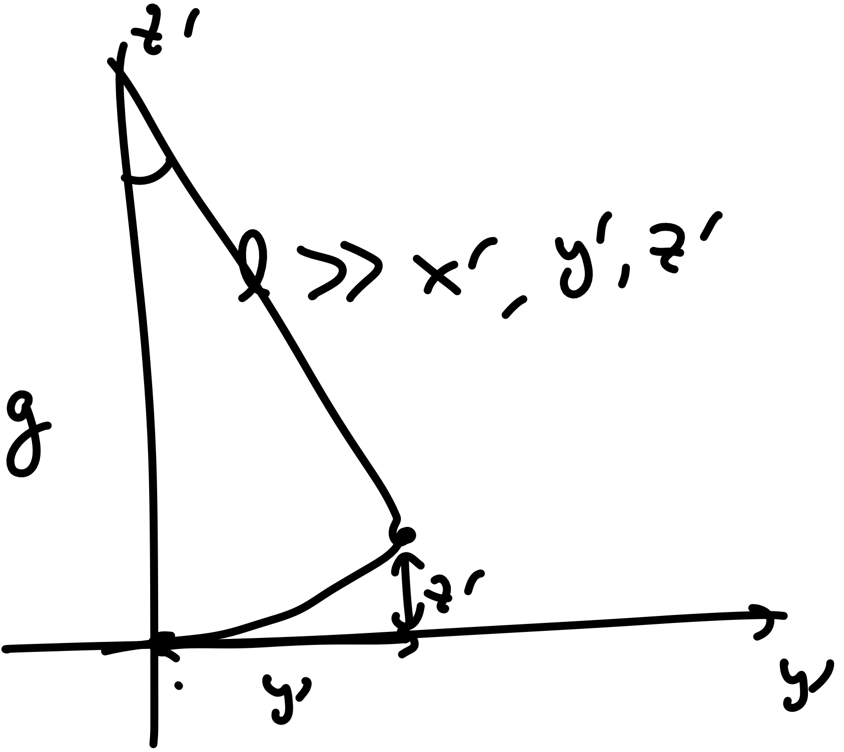
$\rightarrow S_{y'} = -\frac{y'S}{l}, \; S_{x'} = -\frac{x'S}{l}$

$S_z = \frac{l-z'}{l}S$

$$\begin{cases} \ddot{x}' = -\frac{S}{ml}x' - 2\left(\omega\cos\lambda\,\dot{z}' - \omega\sin\lambda\,\dot{y}'\right) \\ \ddot{y}' = -\frac{S}{ml}y' - 2\left(\omega\sin\lambda\,\dot{x}'\right) \end{cases}$$

$$\ddot{z}' = \underbrace{-g + \frac{l-z'}{ml}S}_{} - 2\cdot(-\omega\cos\lambda\,\dot{x}')$$

$$\approx -g + \frac{\cancel{l}S}{m\cancel{l}} = 0 \rightarrow S \approx mg$$



$$l \gg x', y', z'$$

$$\begin{cases} \ddot{x}' = -\frac{g}{l}x' + 2\underbrace{\omega\sin\lambda}_{}\dot{y}' \\ \ddot{y}' = -\frac{g}{l}y' - 2\underbrace{\omega\sin\lambda}_{\overset{\parallel}{\omega_{z'}}}\dot{x}' \end{cases}$$

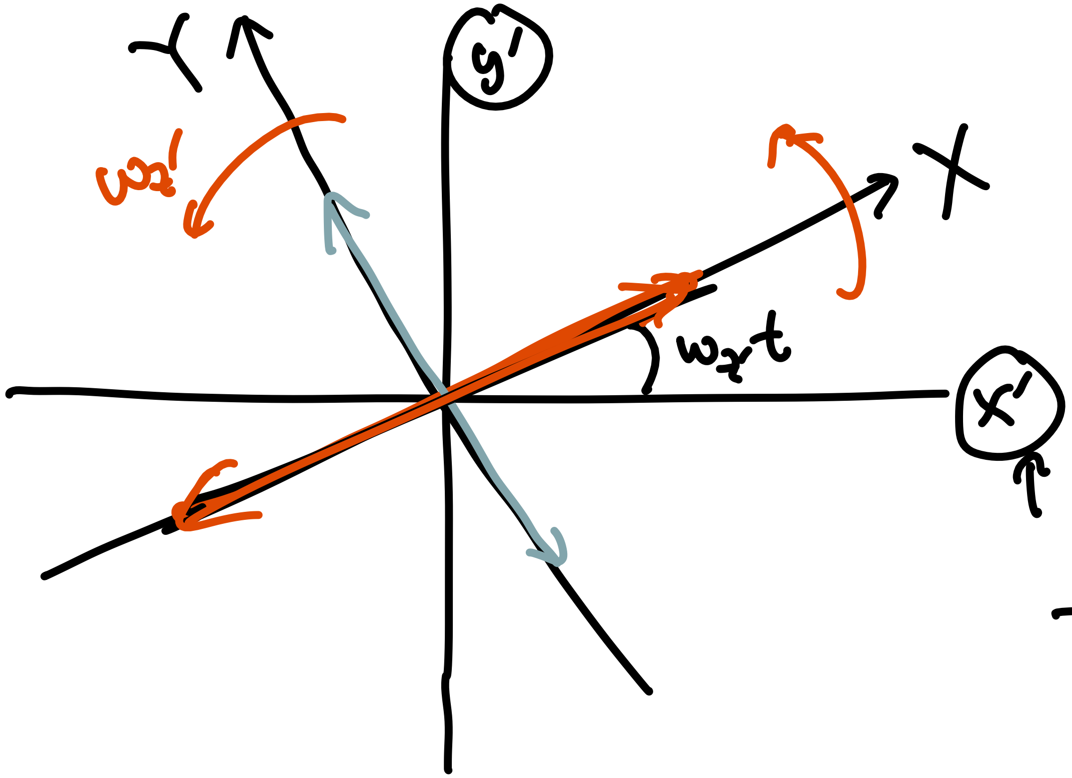
$$x' = \underset{\sim}{X}\cos(\omega_{z'}t) + \underline{Y}\sin(\omega_{z'}t)$$

$$\dot{x}' = \dot{X}\cos(\,) - \omega_{z'}X\sin(\,) + \dot{Y}\sin(\,) + \omega_{z'}Y\cos(\,)$$

$$y' = -X\sin(\omega_{z'}t) + Y\cos(\omega_{z'}t)$$

$$\left(\ddot{X} + \frac{g}{l}X\right)\cos(\omega_{z'}t) + \left(\ddot{Y} + \frac{g}{l}Y\right)\sin(\omega_{z'}t) = 0$$

$$\Rightarrow \underline{\ddot{X} + \frac{g}{l}X = 0, \quad \ddot{Y} + \frac{g}{l}Y = 0}$$



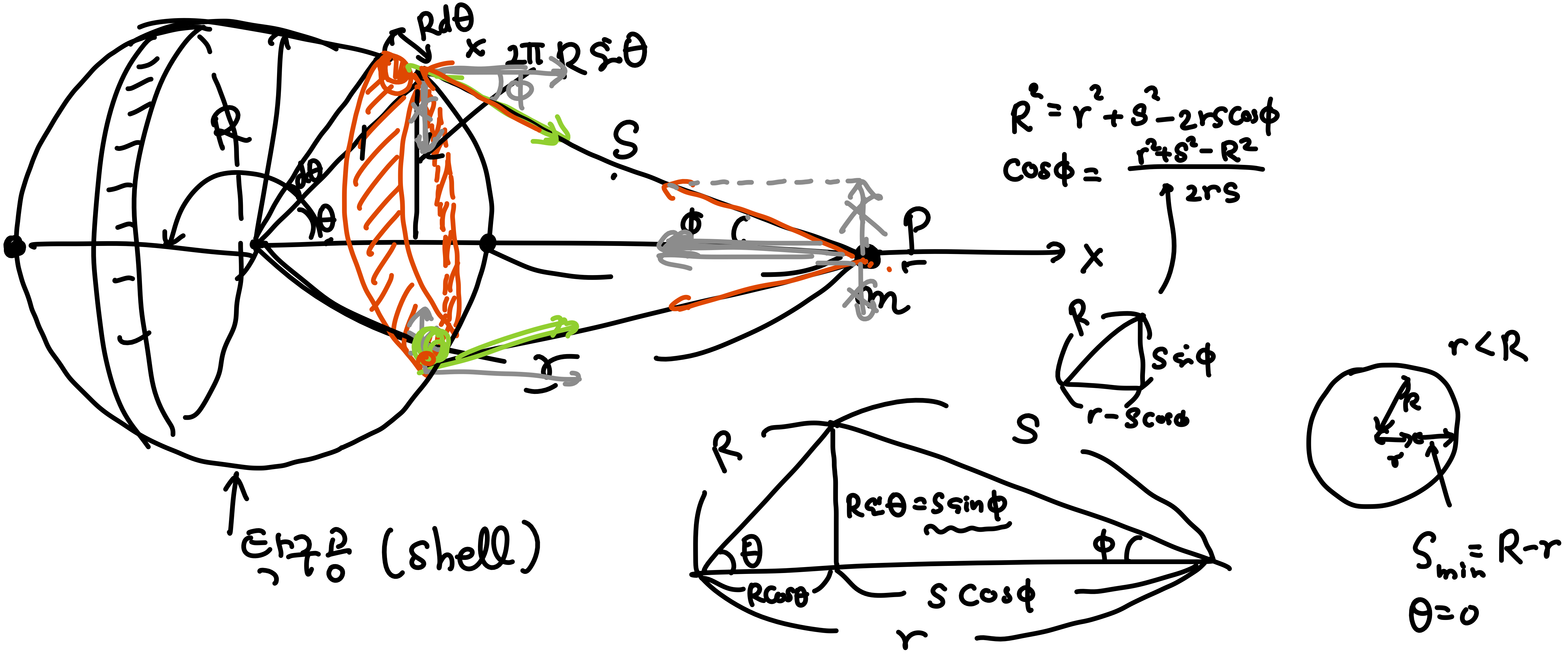
$$\frac{\ddot{x} + \frac{g}{l}x = 0}{\ddot{Y} + \frac{g}{l}Y = 0} \quad Y = 0$$

$$T = \frac{2\pi}{\omega_{z'}} \leftarrow$$ 진동면의 회전 주기

$$= \frac{2\pi}{\omega \sin\lambda} = \frac{1\ \text{day}}{\sin\lambda}$$

1day

Chap 6 . Gravitation.



$$R^2 = r^2 + s^2 - 2rs\cos\phi$$

$$\cos\phi = \frac{r^2+s^2-R^2}{2rs}$$

$$S_{min} = R - r$$

$$\theta = 0$$

$$s^2 = r^2 + R^2 - 2rR\cos\theta$$

$$dM = \rho\, 2\pi R^2 \underline{\sin\theta\, d\theta}$$

$$F_x = \int dF = \int G \frac{m\,dM}{s^2} \overset{\times}{\cos\phi} = \int_0^{\pi} G \frac{\rho\, m\, 2\pi R^2 \underbrace{\sin\theta\, d\theta}_{\frac{s\,ds}{rR}}}{s^2} \cos\phi$$

$$= \frac{G\rho m 2\pi R^2}{rR} \int_{r-R}^{r+R} \frac{s\,ds}{s^2} \underbrace{\cos\phi}_{} \quad \cos\phi = \frac{r^2+s^2-R^2}{2rs}$$

$$2s\,ds = 2rR\sin\theta\, d\theta$$

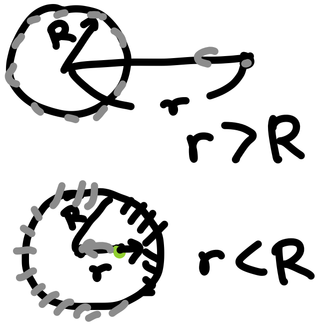
$$\sin\theta\, d\theta = \frac{s\,ds}{rR}$$

$$\underbrace{\frac{r^2-R^2}{s^2}} + 1$$

$$= \frac{G\rho m 2\pi R^2}{2r^2R}\int_{\boxed{r-R}}^{r+R} ds \underbrace{\frac{r^2+s^2-R^2}{s^2}}$$

r>R

R-r r<R



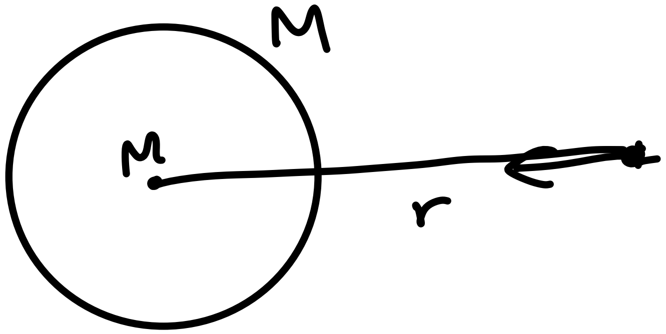
$$= \text{''}\left[\underbrace{s\Big|_{r-R}^{r+R}}_{\substack{r+R-(r-R)\\ \overset{(R-r)}{=}\\ 2R\ \cancel{2r}}} + \underbrace{(r^2-R^2)}_{(r+R)(r-R)}\left(-\frac{1}{s}\right)\Bigg|_{\substack{r-R\\ R-r}}^{r+R}\right]$$

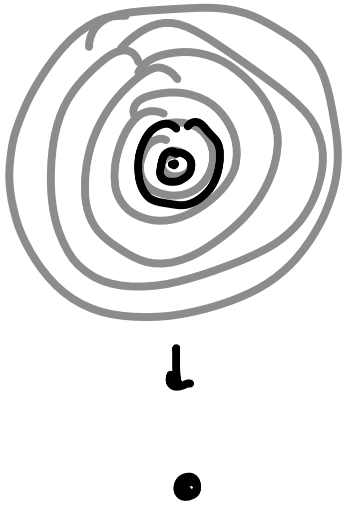
$$\underbrace{-(r-R)+(r+R) = 2R \quad -(r+R) = -\cancel{2r}}_{4R \qquad 0}$$

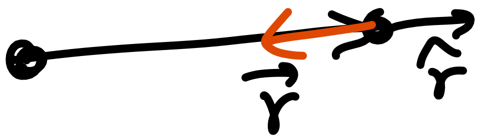
$$= \frac{G\rho m\overbrace{\boxed{2\pi R^2 \quad 2}}^{M}}{\cancel{2R}} \times \cancel{4}R \times \frac{1}{r^2}$$

$$\boxed{F_x = \frac{GmM}{r^2}}$$

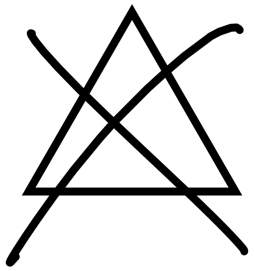
$$F_y = 0$$





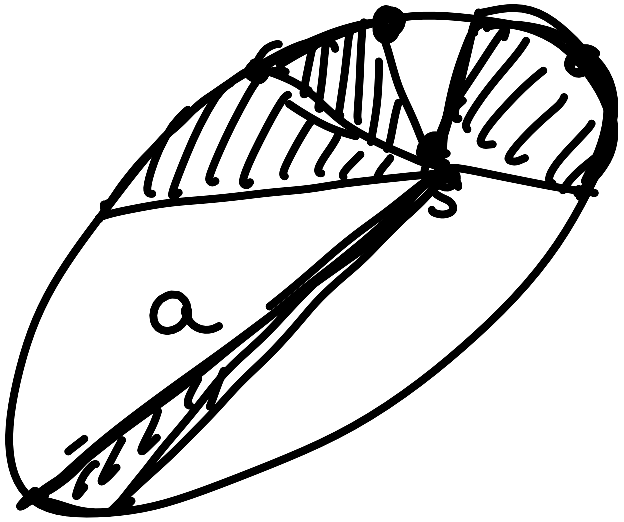


$$\boxed{\vec{F} = -G\frac{Mm}{r^2}\hat{r}}$$ 중심력



6.3. Kepler's laws.

1. ellipse
2. Equal area :

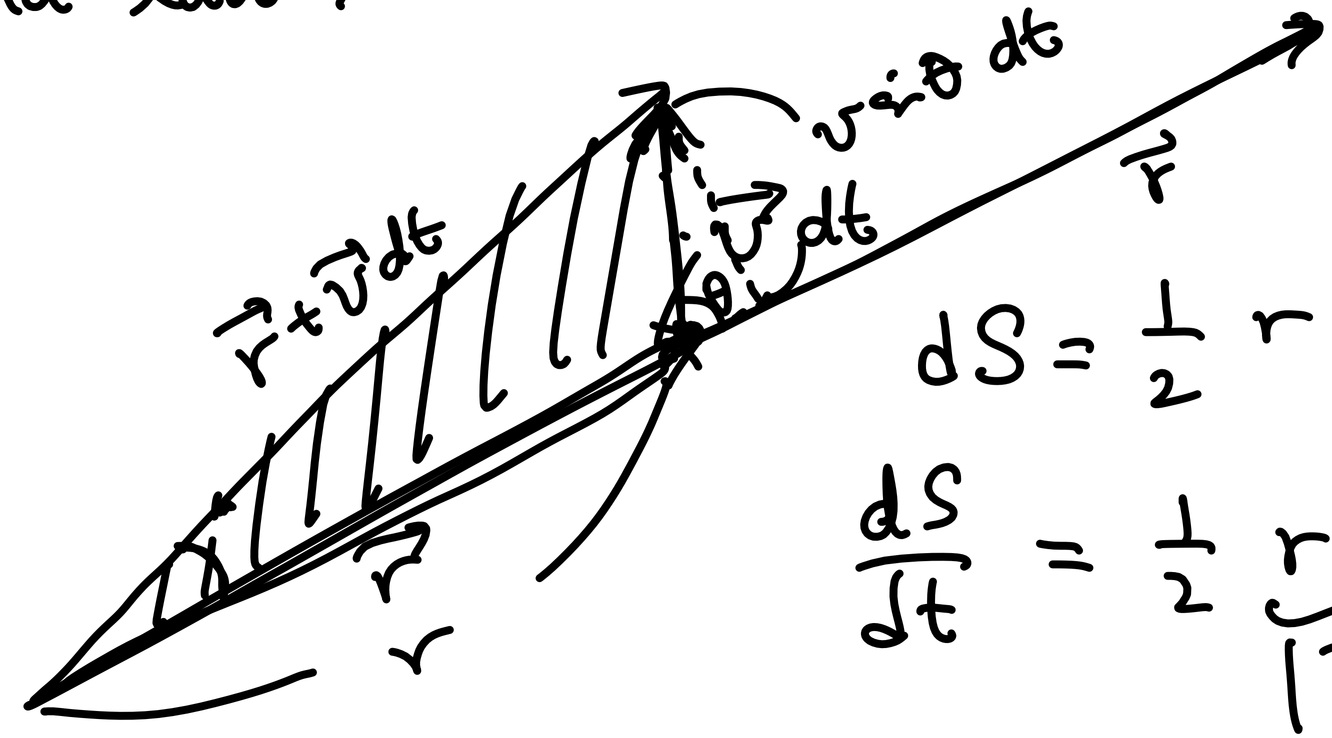


근일점 근처에서 속도 빠름

원 〃 〃 느림.

3. Harmonic law $T^2 \propto a^3$

6.4. 2nd law :



$$dS = \frac{1}{2} r v \sin\theta\,dt$$

$$\frac{dS}{dt} = \frac{1}{2}\underbrace{r v \sin\theta}_{|\vec r \times \vec v|}$$

$$= \frac{1}{2m}|\vec r \times \underbrace{m\vec v}_{\vec p}|$$

$$= \frac{1}{2m}|\vec r \times \vec p| = \frac{|\vec L|}{2m}$$

$\vec r \times \vec p = \vec L$ 각운동량.

$$\frac{d\vec L}{dt} = \cancel{\underset{\vec v = \frac{\vec p}{m}}{\underset{\parallel}{\dot{\vec r} \times \vec p}}} + \vec r \times \underset{\vec F}{\underset{\parallel}{\dot{\vec p}}} = \underset{0}{\underset{\neq}{\vec r \times \vec F}}$$

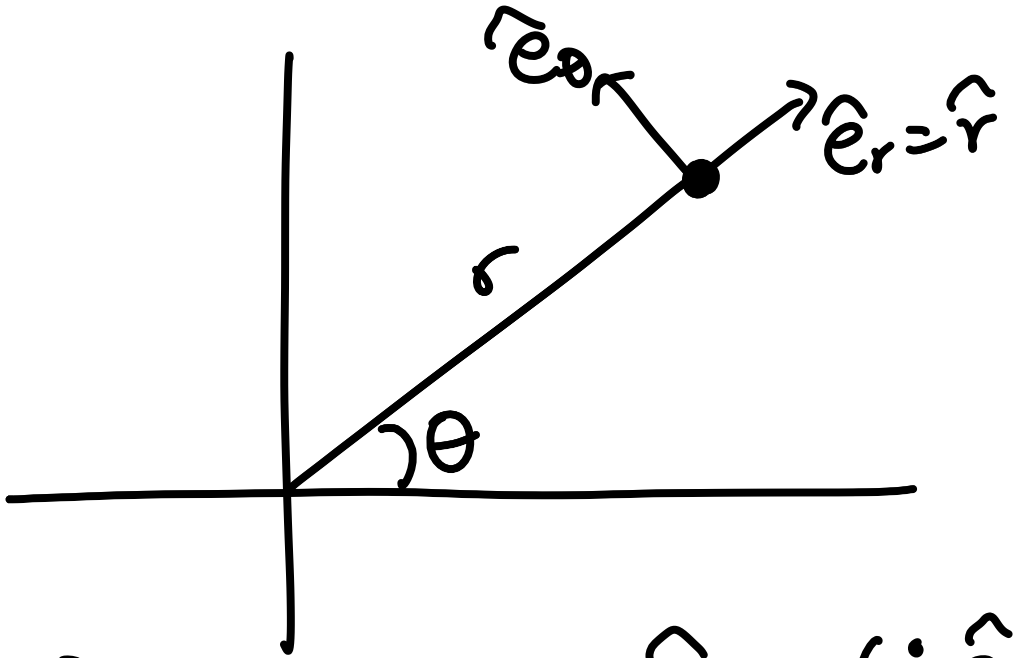
but if $\boxed{\vec F \propto \vec r}$

$\vec F = f(r)\hat r$

$\vec r \times \vec F \propto \vec r \times \vec r = 0$

$$\frac{d\vec L}{dt} = 0$$

$\vec F \propto \vec r$ (중심력) → $\underline{\vec L = \text{보존}}$



$$\vec{r} = r\hat{e}_r$$

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\downarrow$$

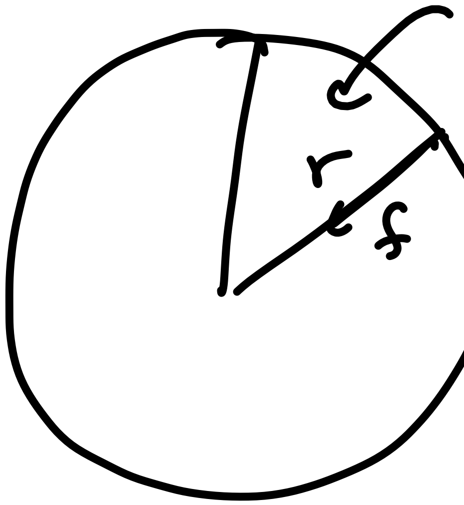
$$\vec{a} = \ddot{r}\hat{e}_r + 2\dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r$$

$$\vec{L} = \vec{r} \times m\vec{v} = mr\hat{e}_r \times (\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta)$$

$$= mr^2\dot{\theta}\ \underbrace{\hat{e}_r \times \hat{e}_\theta}_{\text{지면 나가는 방향}}$$

$$|\vec{L}| = l = mr^2\dot{\theta}$$

[Ex 6.4.1]



$$\frac{dS}{dt} = \dot{A} = \text{일정}$$

$$= \frac{l}{2m} \qquad l = 2m\dot{A}$$

$$f(r) = ?$$

$$\vec{a}_r = \frac{f(r)}{m} \quad \underset{\substack{\shortparallel\\ 0}}{\ddot{r}} - r\dot{\theta}^2 = -r\dot{\theta}^2$$

$$\vec{F} = m\vec{a}$$

$$f(r) = -mr\dot{\theta}^2$$

$$\dot{\theta} = \frac{l}{mr^2}$$

$$\therefore\ f(r) = -\frac{mrl^2}{m^2r^4} = -\frac{l^2}{mr^3} \;/\!/ = -\frac{4m\dot{A}^2}{r^3}$$

6.5. 1st law.

$$m\ddot{\vec{r}} = f(r)\hat{r} \rightarrow \boxed{m(\ddot{r}-r\dot{\theta}^2) = f(r)}$$

$$\boxed{2\dot{r}\dot{\theta}+r\ddot{\theta} = 0}$$

$$\frac{\frac{d}{dt}(r^2\dot{\theta})}{r} = \frac{2r\dot{r}\dot{\theta}+r^2\ddot{\theta}}{r}$$

$$\therefore \frac{d}{dt}(r^2\dot{\theta}) = 0$$

$\uparrow$

$$L = mr^2\dot{\theta}$$

$$\frac{L}{m} \equiv l \qquad \frac{dl}{dt} = 0$$

$$\dot{\theta} = lu^2$$

각운동량 보존

$$f(r) = -G\frac{mM}{r^2}$$

$$\ddot{r} - r\dot{\theta}^2 = \frac{f(r)}{m} = -l^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}(lu^2)^2$$

$$L = |\vec{L}|$$

$$u = \frac{1}{r} \qquad r = \frac{1}{u}$$

$$\dot{r} = -\frac{1}{u^2}\dot{u} = -l\frac{du}{d\theta}$$

$\uparrow$

$$\frac{du}{dt} = \frac{du}{d\theta}\underbrace{\frac{d\theta}{dt}}_{\dot{\theta} = \frac{L}{mr^2}} = \frac{L}{mr^2}\frac{du}{d\theta} = \underbrace{\frac{L}{m}u^2\frac{du}{d\theta}}$$

$$\ddot{r} = -l\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -l\frac{d^2u}{d\theta^2}\underbrace{\frac{d\theta}{dt}}. \rightarrow \frac{l}{r^2} = lu^2$$

$$= -l^2u^2\frac{d^2u}{d\theta^2}$$

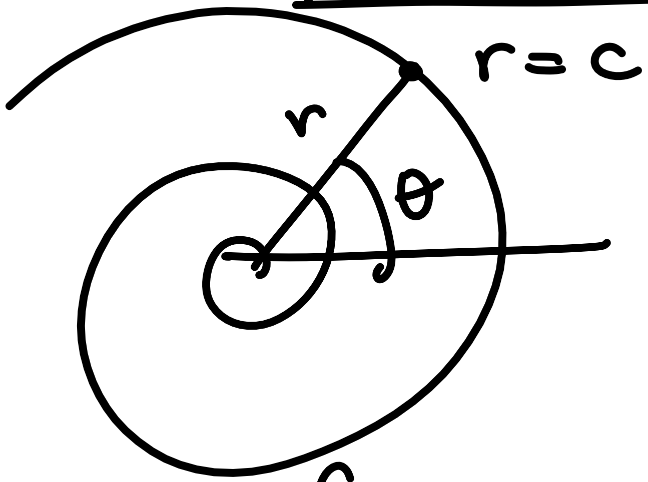
$$-l^2u^2\frac{d^2u}{d\theta^2}-\frac{1}{u}(lu^2)^2=\frac{f(\frac{1}{u})}{m}$$

$$\Rightarrow \boxed{\frac{d^2u}{d\theta^2}+u=-\frac{f(\frac{1}{u})}{ml^2u^2}} \rightarrow u=u(\theta)=\frac{1}{r}$$

중력 : $f(r)=-\frac{GmM}{r^2}=-GmMu^2$

$$\rightarrow \boxed{\frac{d^2u}{d\theta^2}+u=\frac{GM}{l^2}=\text{상수}}$$

[Ex. 6.5.1]



$r=c\theta^2 \rightarrow \frac{1}{r}=u$

$u=\frac{1}{c\theta^2} \rightarrow \theta^2=\frac{1}{cu}=\frac{r}{c}$

$\frac{du}{d\theta}=-\frac{2}{c\theta^3}$, $\frac{d^2u}{d\theta^2}=\frac{6}{c\theta^4}$

$$\frac{6}{c\theta^4}+\frac{1}{c\theta^2}=-\frac{f(r)}{ml^2u^2}$$

$$\frac{6}{c}\left(\frac{c}{r}\right)^2+\frac{1}{c}\left(\frac{c}{r}\right)=-\frac{r^2}{ml^2}f(r) \rightarrow f(r).$$

[Ex. 6.8.2] $\ell = r^2 \dot{\theta}$ $r = c\theta^2$

$$\frac{d\theta}{dt} = \dot{\theta} = \frac{d\theta}{dt} = \frac{\ell}{r^2} = \frac{\ell}{c^2\theta^4} = \frac{\ell}{c^2}\theta^{-4}$$

$$\frac{\theta^5}{5} = \int_0^\theta \theta^4 \, d\theta = \frac{\ell}{c^2}\int_0^t dt = \frac{\ell t}{c^2} \rightarrow \theta = \left(\frac{5\ell}{c^2}\right)^{\frac{1}{5}} t^{\frac{1}{5}}$$

$$\underline{f(r) = -\frac{k}{r^2}} = -ku^2$$

$u \equiv \frac{1}{r}$

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2\cancel{u^2}}\underbrace{f(u^{-1})}_{-k\cancel{u^2}} = \underbrace{\frac{k}{ml^2}}_{\equiv C} = \text{constant}$$

$$\underbrace{\frac{d^2u}{d\theta^2} + u = C} \to u(\theta) \longrightarrow u - C \equiv u' \qquad \frac{d^2u'}{d\theta^2} = \frac{d^2u}{d\theta^2}$$

$$\frac{d^2u'}{d\theta^2} + u' = 0 \longleftrightarrow$$

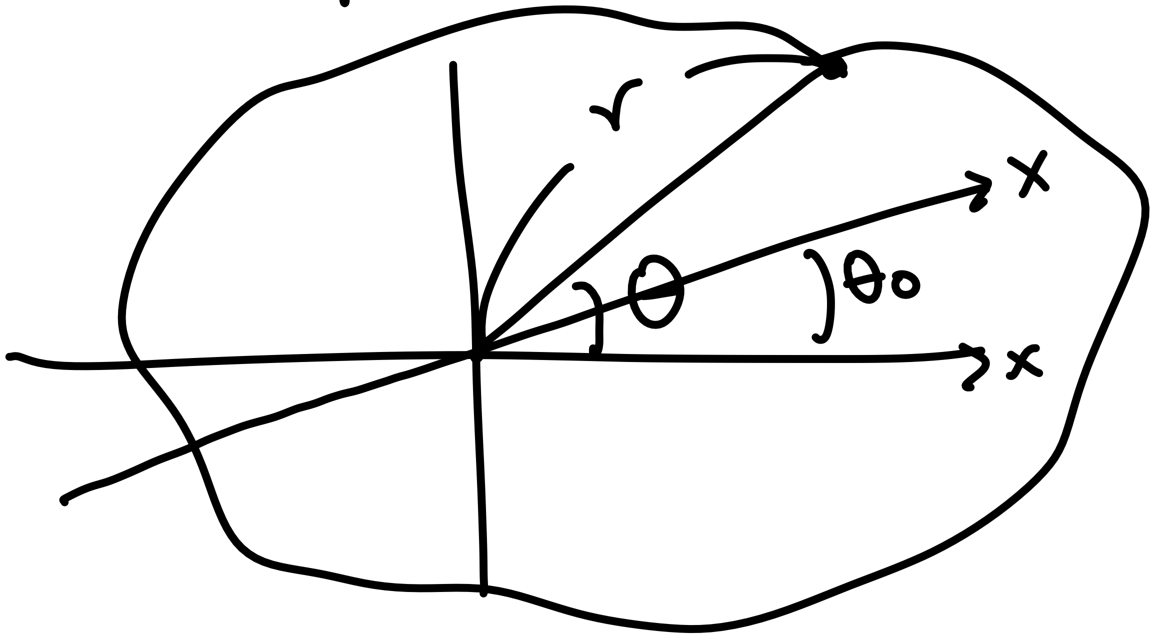
H.O. $\frac{d^2x}{dt^2} + \omega^2 x = 0$

$\to x(t) = A\cos(\omega t + \phi)$

$$\Downarrow$$

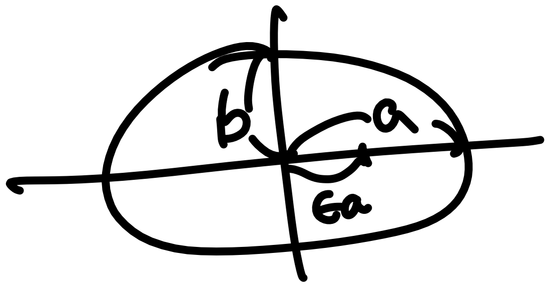
$$u - C = u' = A\cos(\theta - \theta_0)$$

$$\therefore u = \frac{1}{r} = A\cos(\theta - \theta_0) + \frac{k}{ml^2} \Rightarrow r(\theta) = \frac{1}{A\cos(\theta - \theta_0) + \frac{k}{ml^2}}$$

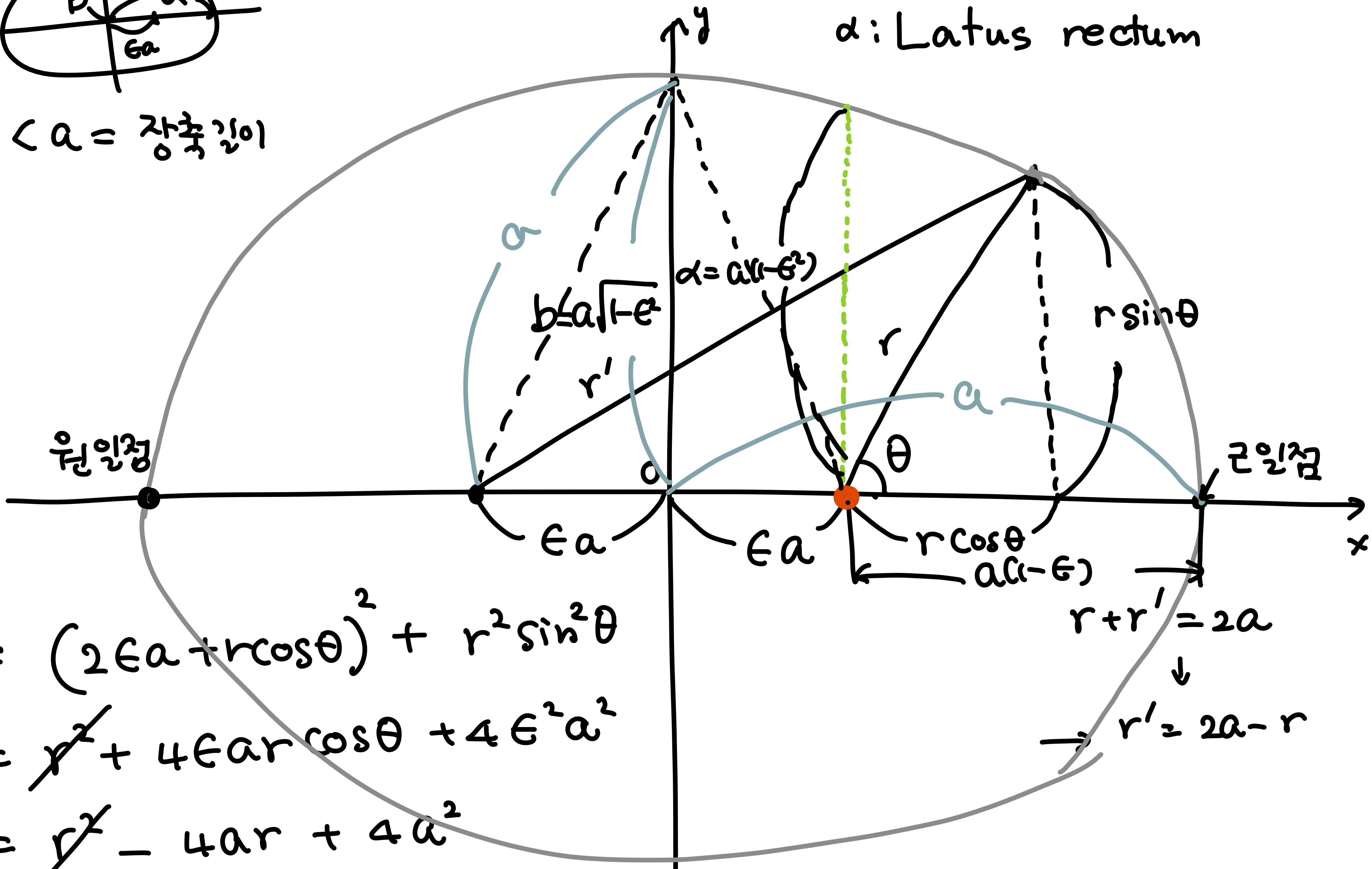


$\theta_0 = 0$

$$r = \frac{1}{\underset{\sim}{A}\cos\theta + \frac{k}{ml^2}}$$



단축의길이 = b < a = 장축길이



$r+r'=2a$

$\downarrow$

$\rightarrow r'=2a-r$

$$r'^2=(2\epsilon a+r\cos\theta)^2+r^2\sin^2\theta$$

$$=\cancel{r^2}+4\epsilon ar\cos\theta+4\epsilon^2a^2$$

$$=\cancel{r^2}-4ar+4a^2$$

$$r=\frac{4a^2(1-\epsilon^2)}{4a+4\epsilon a\cos\theta}$$

$$\boxed{r=\frac{a(1-\epsilon^2)=\alpha}{1+\epsilon\cos\theta}}\longrightarrow$$

이심율 $(\epsilon<1)$ $\theta=0\longrightarrow r=a(1-\epsilon)$

$\theta=\frac{\pi}{2}\longrightarrow r=a(1-\epsilon^2)=\alpha$

극좌표 타원: $$\boxed{r = \frac{\alpha}{1+\epsilon\cos\theta}}$$

x-y 좌표 타원: $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$r = \frac{1}{A\cos\theta+\frac{k}{ml^2}}$$ 와 비교

$$= \frac{ml^2/k}{1+\frac{Aml^2}{k}\cos\theta}$$

$$\therefore \boxed{\begin{aligned} \alpha &= a(1-\epsilon^2) = \frac{ml^2}{k} \\ \epsilon &= A\frac{ml^2}{k} \end{aligned}}$$

$\theta=\pi$

$$r_1 = \frac{a(1-\epsilon^2)}{1-\epsilon} = a(1+\epsilon)$$

원일점

$$r_0 \overset{\theta=0}{=} \frac{\alpha=a(1-\epsilon^2)}{1+\epsilon} = a(1-\epsilon)$$

근일점

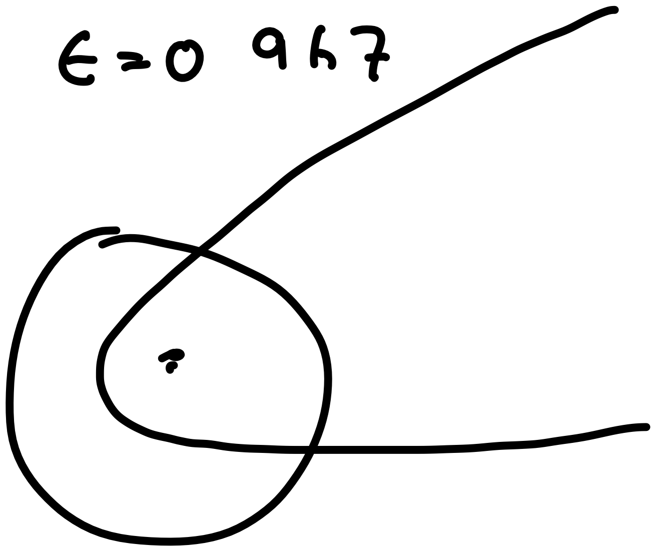
지구: $\epsilon = 0.017$

$r_0 = 91\times10^6$ mile

$r_1 = 95\times10^6$ "

Halley's comet

$\epsilon = 0\ 967$



$$r = \frac{\alpha}{1+\epsilon\cos\theta}$$

↑

항상 양수

$1+\epsilon\cos\theta_0 = 0$

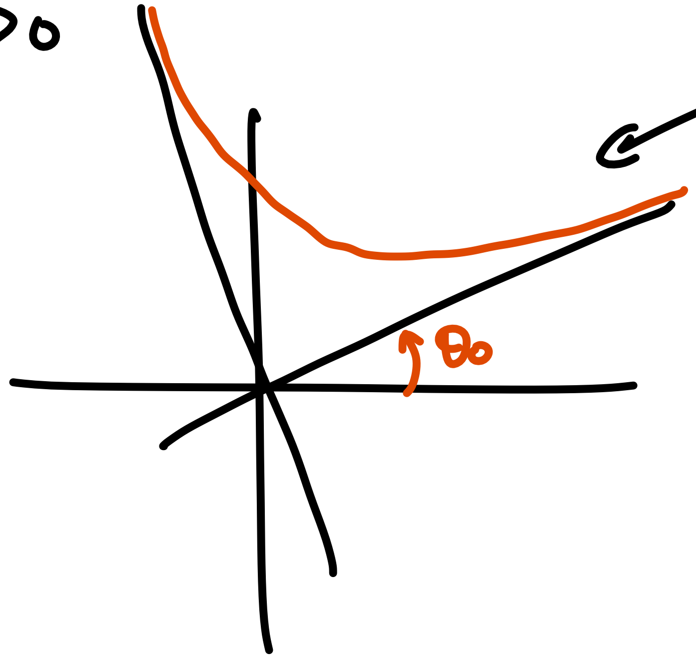
$\epsilon>0$

$0<\epsilon<1$: 타원

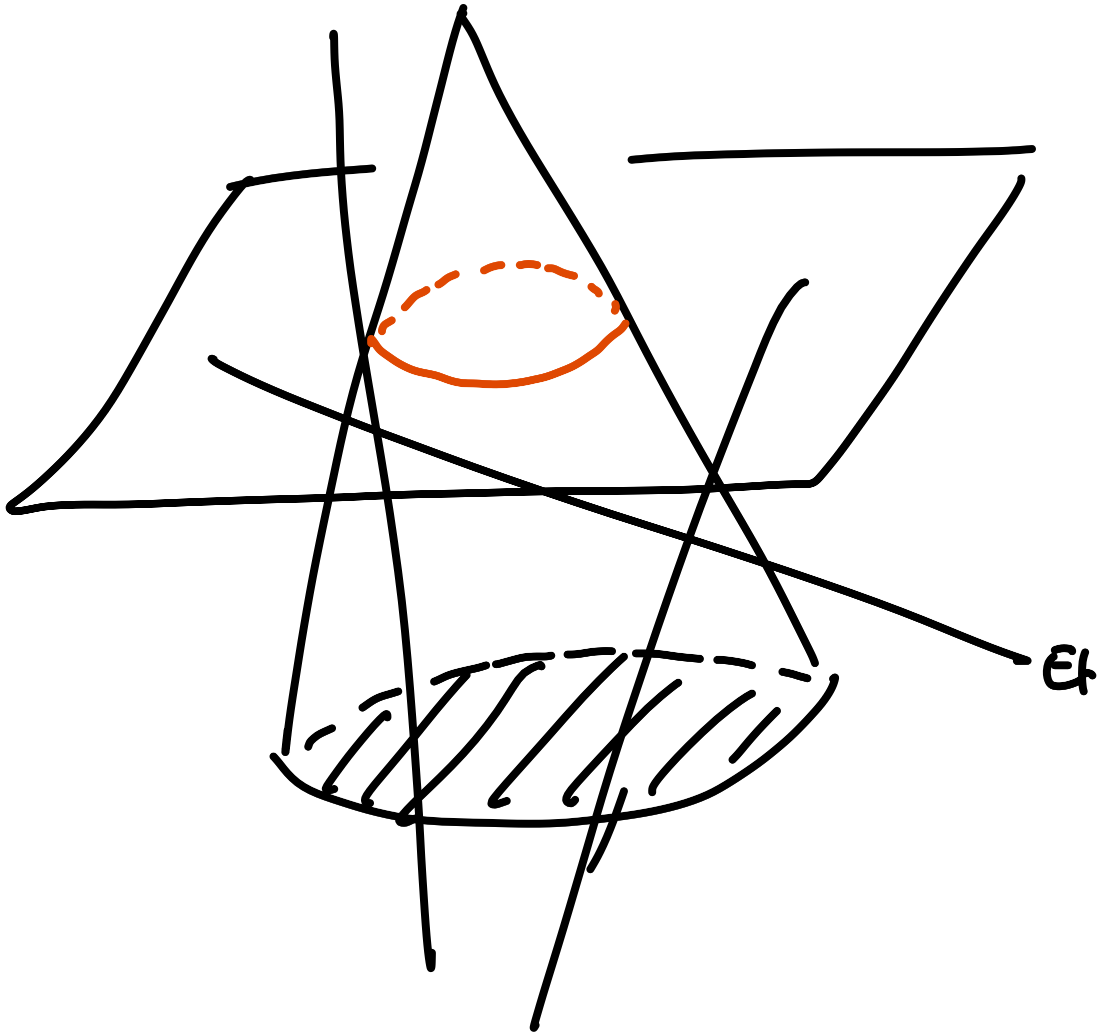
$\epsilon=0$: 원

$\epsilon=1$: parabola

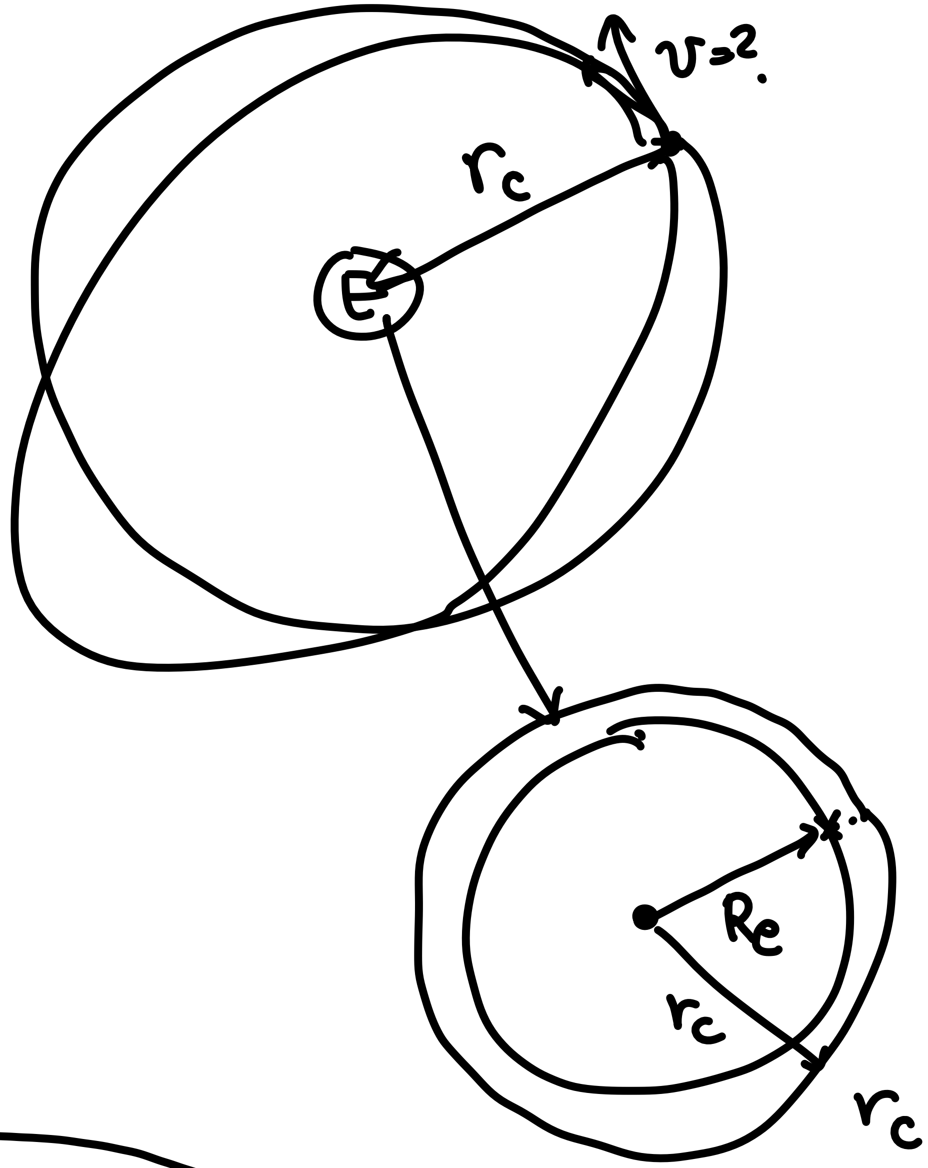
$\epsilon>1$: hyperbola



원뿔 곡선



[Ex 6.5.3]



$$r_c = \frac{ml^2}{k} \qquad k = GM_e m$$

$$r_c = \frac{(r_c v)^2}{GM_e} \rightarrow v^2 = \frac{GM_e}{r_c} = \frac{gR_e^2}{r_c}$$

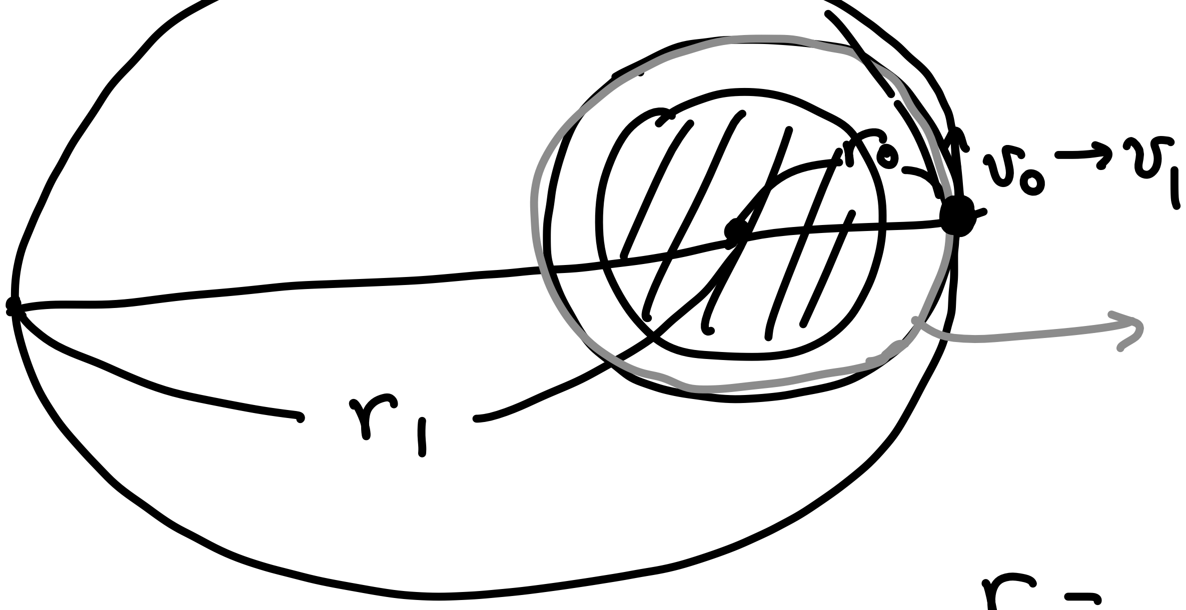
$$F = G\frac{M_e \cancel{m}}{R_e^2} = \cancel{m}g$$

$$g = \frac{GM_e}{R_e^2}$$

$r_c \approx R_e$ (저궤도위성)

$$v = \sqrt{gR_e} \approx \sqrt{9.8\times 6400\times 10^3} = 8\,\mathrm{km/s}$$

[Ex 6.5.4]



$$l_c = r_0 v_0$$

$$r = r_0 = \frac{m l_c^2}{k} = \frac{m(r_0 v_0)^2}{k}$$

$\epsilon = 0$

$$r = \frac{ml^2/k}{1+\epsilon\cos\theta} \rightarrow \begin{cases} \theta = 0 & r_0 = \dfrac{ml^2/k}{1+\epsilon} \\ \theta = \pi & r_1 = \dfrac{ml^2/k}{1-\epsilon} \end{cases}$$

$$r_0 = \frac{ml^2/k}{1+\epsilon} = \cancel{\frac{m}{k}}(r_0 v_0)^2 \Rightarrow \frac{(r_0 v_1)^2}{1+\epsilon} = (r_0 v_0)^2$$

$$r_1 = \frac{ml^2/k}{1-\epsilon}$$

$$l = r_0 v_1$$
$$\uparrow$$
$$l_0 = r_0 v_0$$

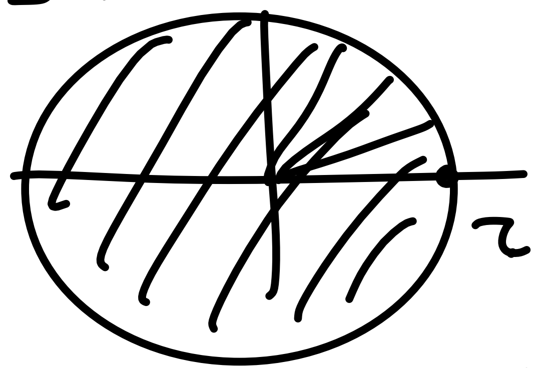
$$\downarrow$$

$$1+\epsilon = \frac{v_1^2}{v_0^2} \quad \therefore \epsilon = \frac{v_1^2}{v_0^2} - 1$$

$$\frac{r_0}{r_1} = \frac{1-\epsilon}{1+\epsilon} = \frac{2-\frac{v_1^2}{v_0^2}}{\frac{v_1^2}{v_0^2}} \Rightarrow \frac{v_1^2}{v_0^2} = \frac{2}{1+\frac{r_0}{r_1}} \Rightarrow \underbrace{v_1 = v_0\sqrt{\frac{2r_1}{r_0+r_1}}}$$

6.6. Kepler 3법칙

$$\dot{A} = \frac{L}{2m} \quad (2\text{법칙}) \longrightarrow \int_0^{\tau \leftarrow \text{주기}} \underbrace{\dot{A}}_{\frac{L}{2m}} dt = A = \pi ab$$
$$= \pi a^2\sqrt{1-\epsilon^2}$$



$$= \frac{L}{2m}\tau = \frac{l}{2}\tau$$

law

Harmonic

$$\frac{l^2}{4}\tau^2 = \pi^2 a^4(1-\epsilon^2) = \pi^2 a^3 \underbrace{a(1-\epsilon^2)}_{ml^2/k}$$

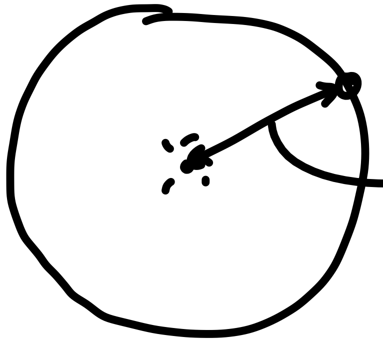
$$\alpha = a(1-\epsilon^2) = \frac{ml^2}{k}$$

$$\therefore \frac{\cancel{l^2}}{4}\tau^2 = \pi^2 a^3 \frac{m\cancel{l^2}}{k = GM_\odot m}$$

$$\Rightarrow \tau^2 = \boxed{\frac{4\pi^2}{GM_\odot}} a^3 \Rightarrow \tau^2 \propto a^3$$

행성의 질량과 무관

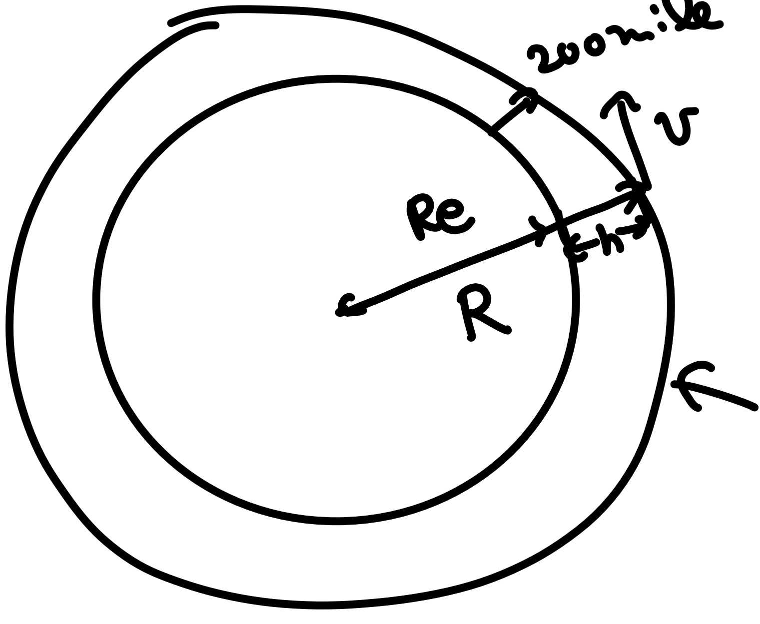
$$\underset{\underline{1\ \text{yr}}}{\underset{\uparrow}{\tau^2}} = \underbrace{1}_{\frac{4\pi^2}{GM_\odot}} \cdot \underset{1}{\underset{\uparrow}{a^3}}$$



$\rightarrow$ 1 AU = 1.5×10^8 km

[Ex. 6.6.1] $a = 4$ AU $\longrightarrow \tau = ?$ $\quad \tau^2 = a^3 \rightarrow \tau = a^{\frac{3}{2}} = 4^{\frac{3}{2}} = 8\text{yr}$

[Ex 6.6.2]



$Re \gg h$ $\quad \tau^2 = \frac{4\pi^2}{GMe} \underset{\overset{\shortparallel}{R}}{a^3} = \frac{4\pi^2}{GMe} R^3$

원 : $\quad R = \frac{\cancel{m} l^2}{GMe\cancel{m}} = \frac{(Rv)^2}{GMe}$

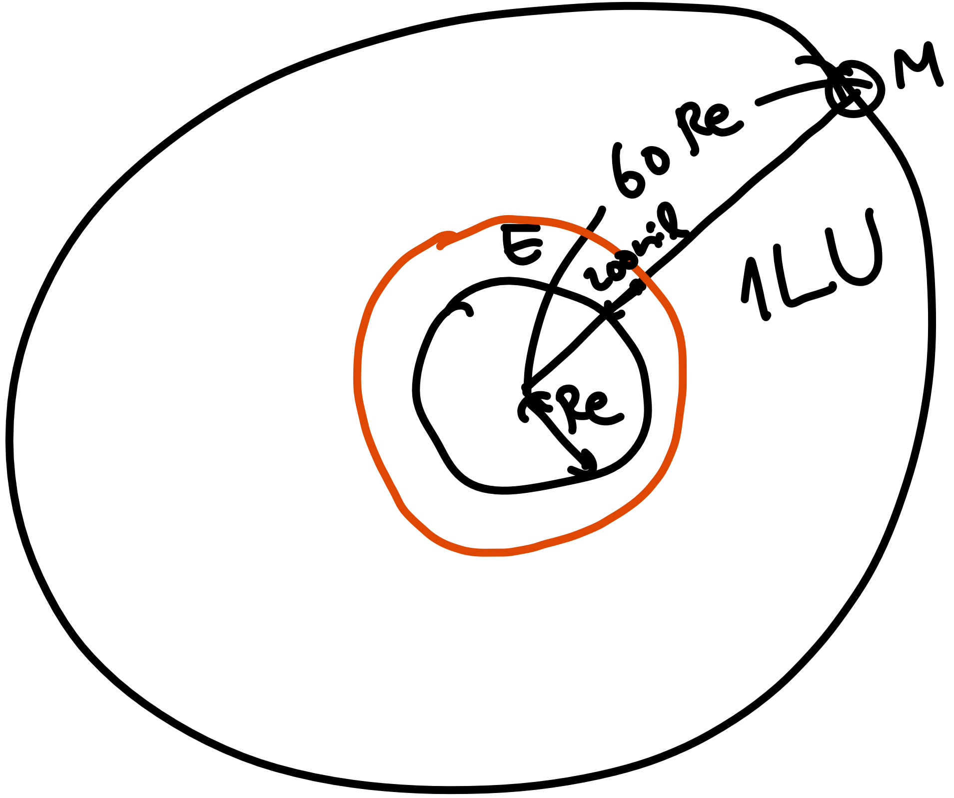
$R = Re + h$

$$\therefore \tau^2 = \frac{4\pi^2}{GMe}(Re+h)^3$$

$$GMe = g\,Re^2$$

$$\tau^2 = \frac{4\pi^2}{g\,Re^2}(Re+h)^3 = \frac{4\pi^2 Re}{g}\left(1+\frac{h}{Re}\right)^3$$

$$(1+\alpha)^{\frac{3}{2}} \approx 1 + \frac{3}{2}\alpha \longrightarrow \rightarrow \tau \approx 2\pi\sqrt{\frac{Re}{g}}\left(1+\frac{3}{2}\frac{h}{Re}\right)$$



$$\tau^2 = \frac{4\pi^2}{\underbrace{GMe}_{1}} \underset{\uparrow\ 1}{a^3}$$

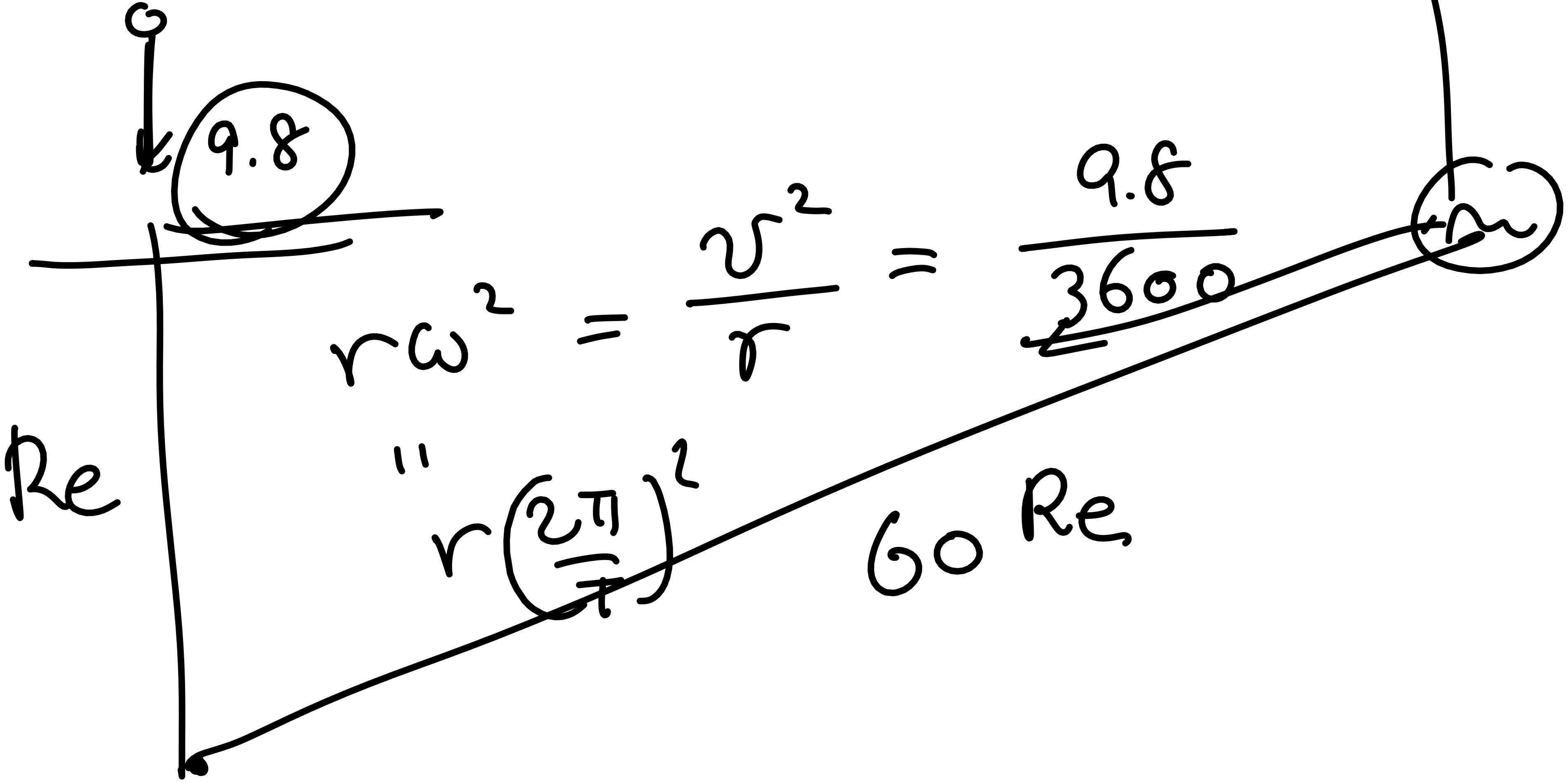
Month

$1 =$

$$\frac{Re + h}{60\,Re} = \frac{1 + \frac{h}{Re}}{60}$$

$= 0.017$ (LU)

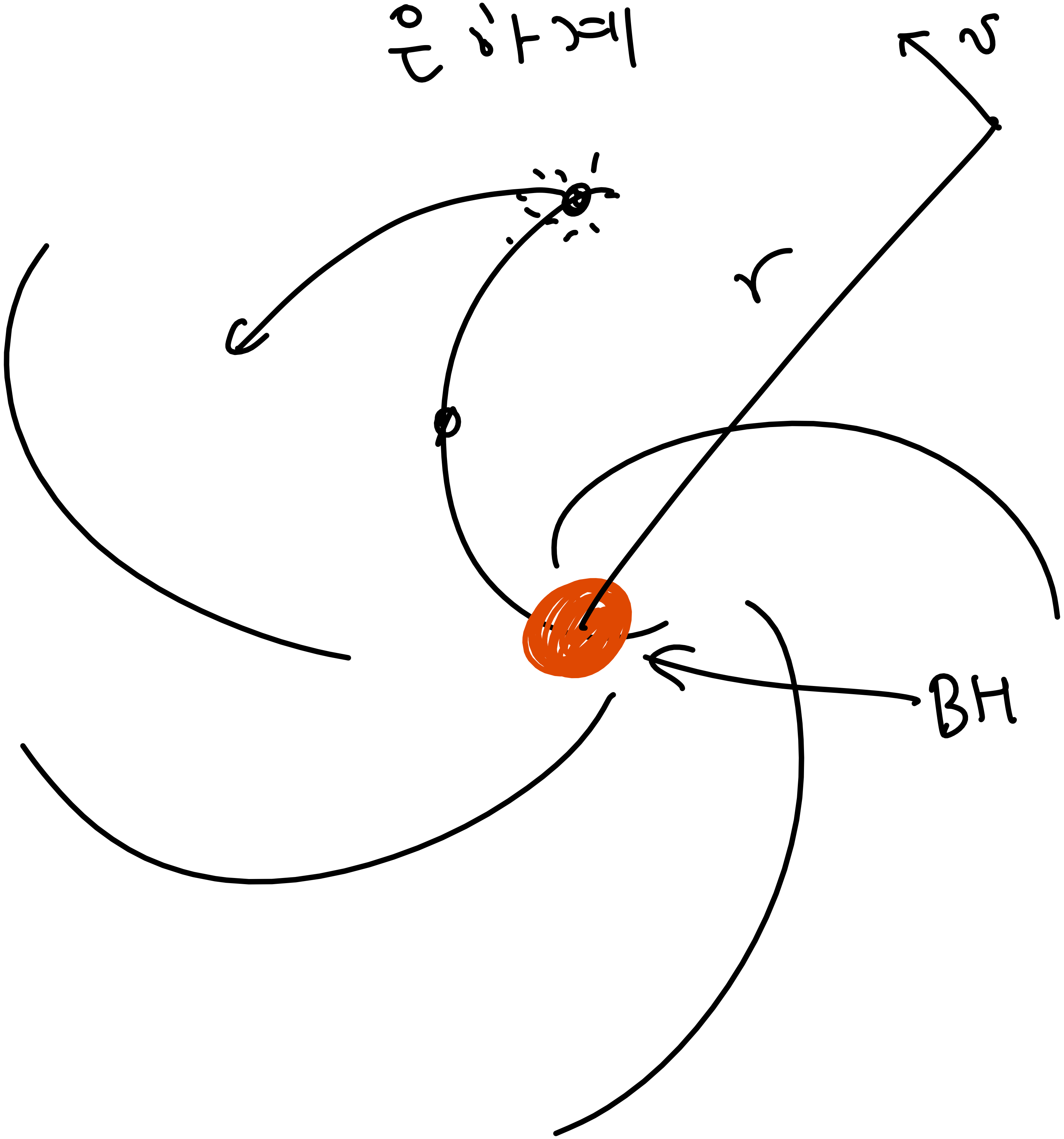
$\tau = (0.017)^{\frac{3}{2}}$ (M)



$$r\omega^2 = \frac{v^2}{r} = \frac{9.8}{3600}$$

$$= r\left(\frac{2\pi}{\tau}\right)^2$$

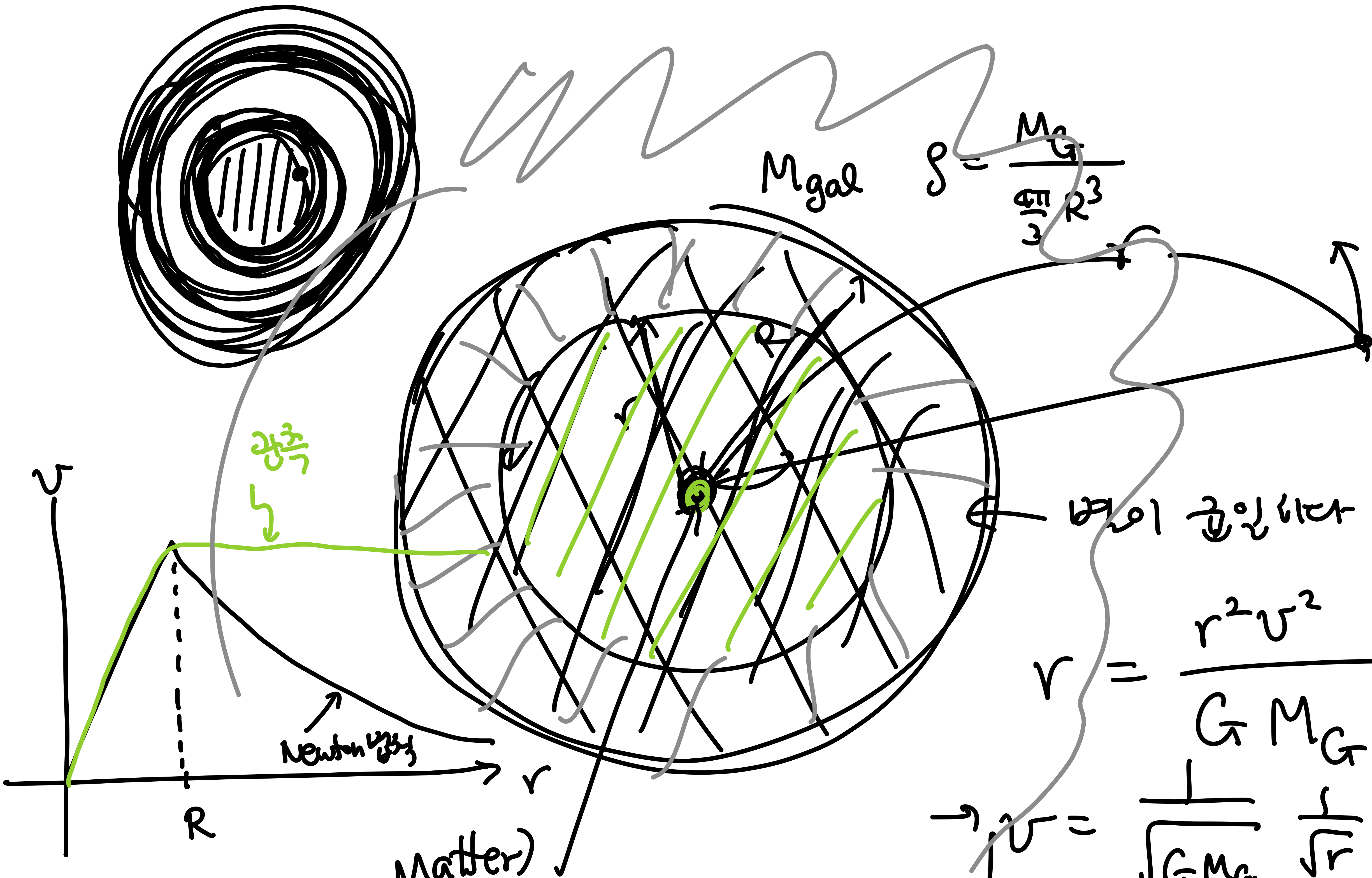
은하계



$$r = \frac{m l^2}{k}$$

$$= \frac{(rv)^2}{G M_{BH}}$$

$$\rightarrow v \propto \frac{1}{\sqrt{r}}$$



$$\rho = \frac{M_G}{\frac{4\pi}{3}R^3}$$

$$r = \frac{r^2v^2}{GM_G}$$

암흑물질 (Dark Matter)

의 직접적 증거

정량.

$$M = \rho\frac{4\pi}{3}r^3$$

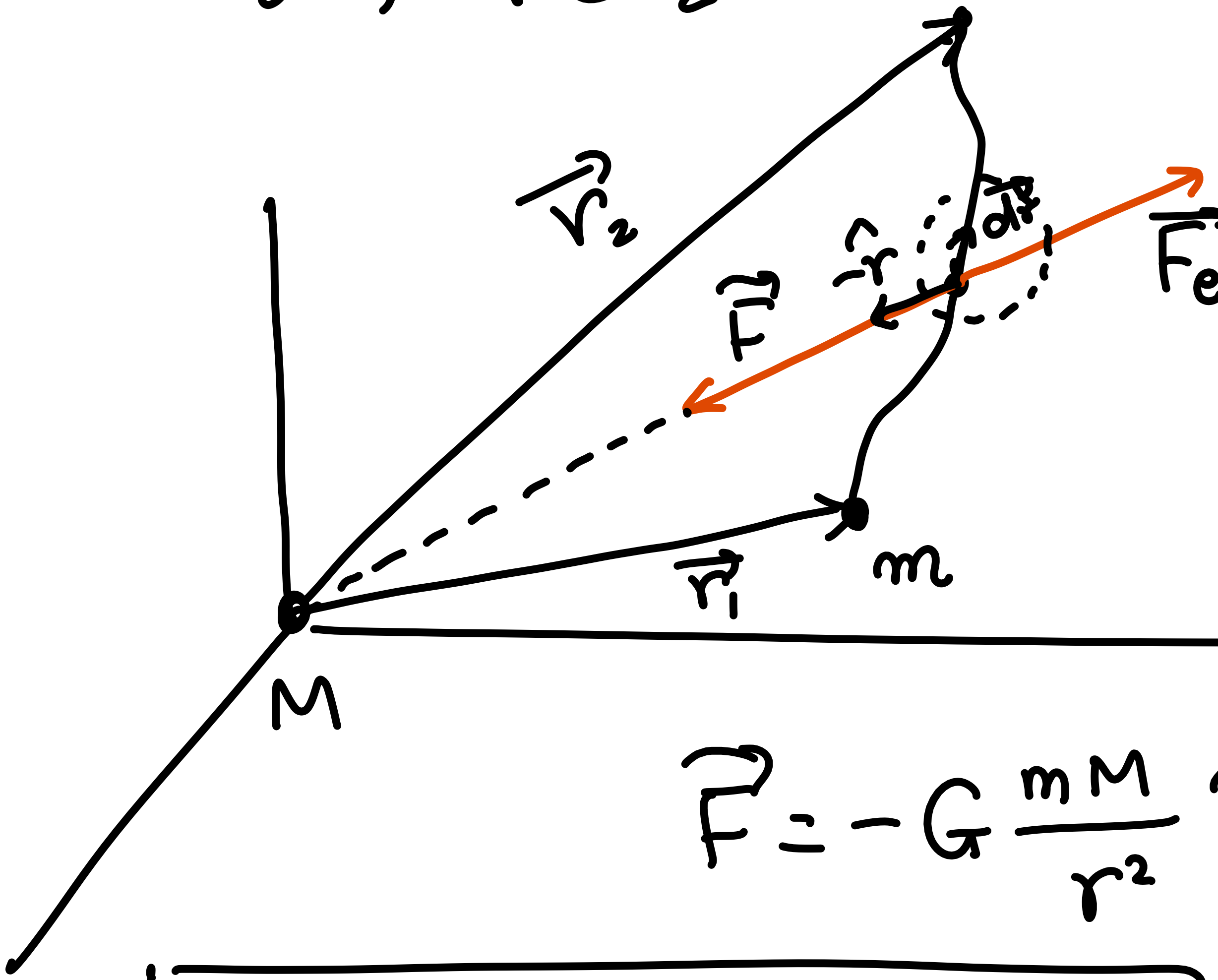
$$r = \frac{m(rv)^2}{G\underbrace{\rho\frac{4\pi}{3}r^3}_{M}m}$$

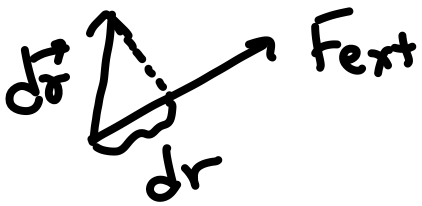
$$\rightarrow v = \frac{1}{\sqrt{GM_G}}\frac{1}{\sqrt{r}} \quad (r>R)$$

$$v = \sqrt{\frac{4\pi\rho}{3}G}\; r \quad (r<R)$$

$$= \sqrt{\frac{GM_G}{R^3}}\; r$$

6.7. 중력 퍼텐셜





$$\vec{F}_{ext} = G\frac{mM}{r^2}\hat{r}$$

$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_{ext}\cdot d\vec{r}$$: 외부에서 가해준 E.

$$= \int_{r_1}^{r_2} G\frac{mM}{r^2}\,dr$$

$$= -GmM\left(\frac{1}{r_2}-\frac{1}{r_1}\right) > 0$$

$r_2 > r_1$

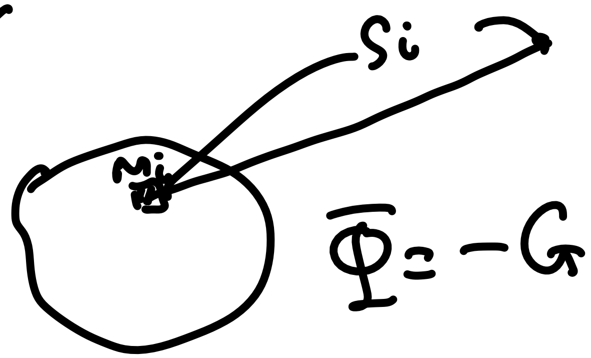
$$= V(r_2) - V(r_1)$$

$$\vec{F} = -G\frac{mM}{r^2}\hat{r}$$

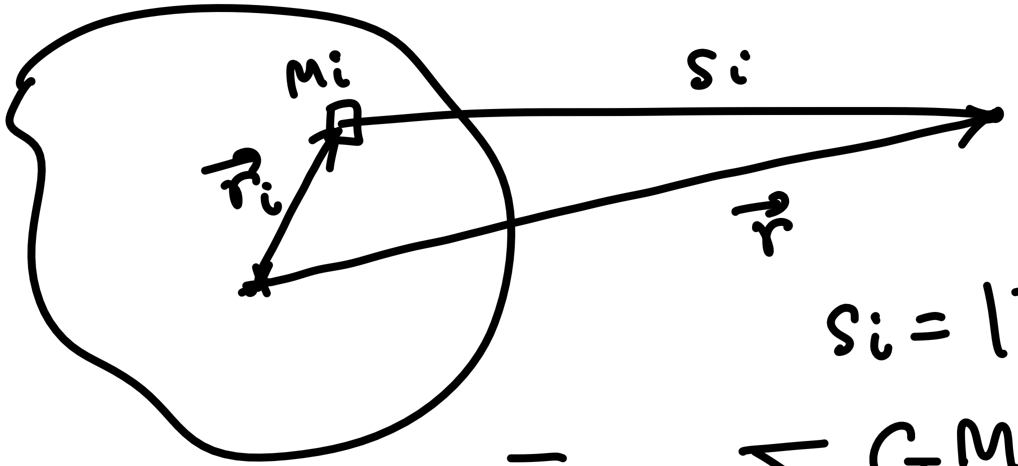
$$\therefore \boxed{V(r) = -\frac{GmM}{r}}$$

중력 위치에너지.

중력 퍼텐셜 : $\frac{V}{m} = \Phi = -\frac{GM}{r}$



$$\Phi = -G\sum_i \frac{M_i}{s_i}$$



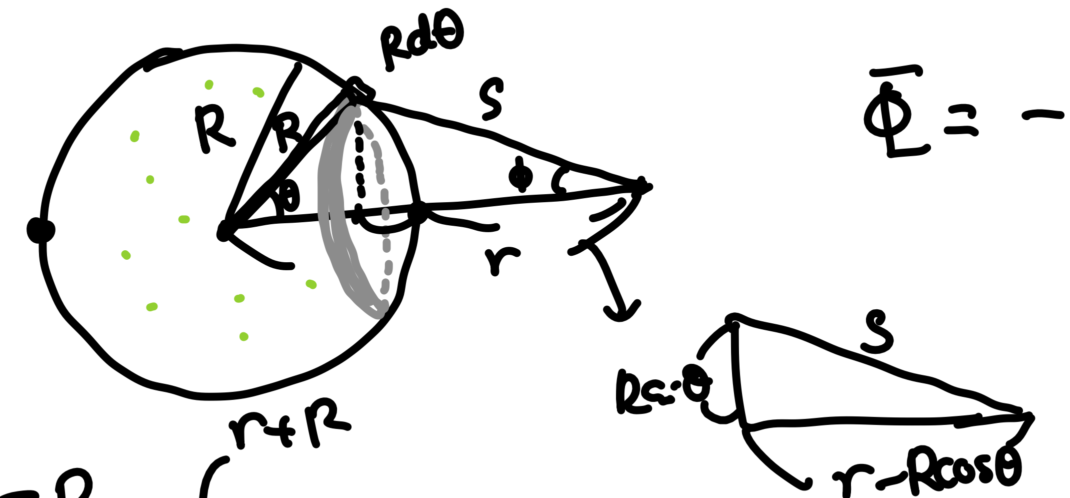
$$s_i = |\vec{r} - \vec{r_i}|$$

$$\Phi = -\sum_i \frac{GM_i}{|\vec{r} - \vec{r_i}|} = -\int \frac{G\rho(\vec{r_i})\, d^3 r_i}{|\vec{r} - \vec{r_i}|}$$

$$= -\int \frac{G\rho(\vec{r}')\, d^3 r'}{|\vec{r} - \vec{r}'|}$$

$$\vec{F} = -\vec{\nabla} V = -m\vec{\nabla}\Phi = m\vec{g} \rightarrow \vec{g} = -\vec{\nabla}\Phi$$

[Ex 6.7.1]



$$\Phi = -G\rho \int_0^{2\pi} \frac{2\pi R^2 \boxed{\sin\theta\, d\theta}}{s} \quad \frac{sds}{rR}$$

$$s^2 = r^2 + R^2 - 2rR\cos\theta$$

$$\cos\theta = \frac{r^2 + R^2 - s^2}{2rR}$$

$$-\sin\theta\, d\theta = \frac{-2sds}{2rR}$$

r>R

$$\therefore \Phi = -G\rho \frac{2\pi R}{r} \int_{r-R\ (r>R)}^{r+R} ds = -G\rho \frac{2\pi R}{r} \underbrace{(r+R-(r-R))}_{2R}$$

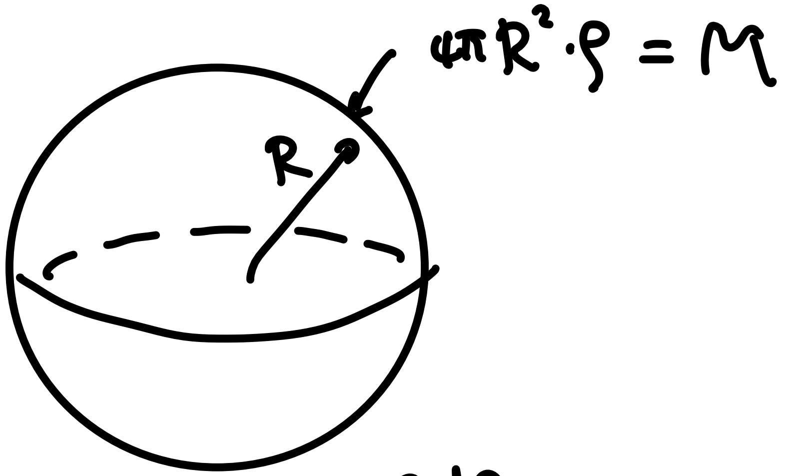
R−r (R>r)   −(R−r)   2r

$$= -G \frac{\boxed{4\pi R^2 \rho}\ M}{r}$$

$$\Phi = -\frac{GM}{r}\,_{\prime\prime}$$

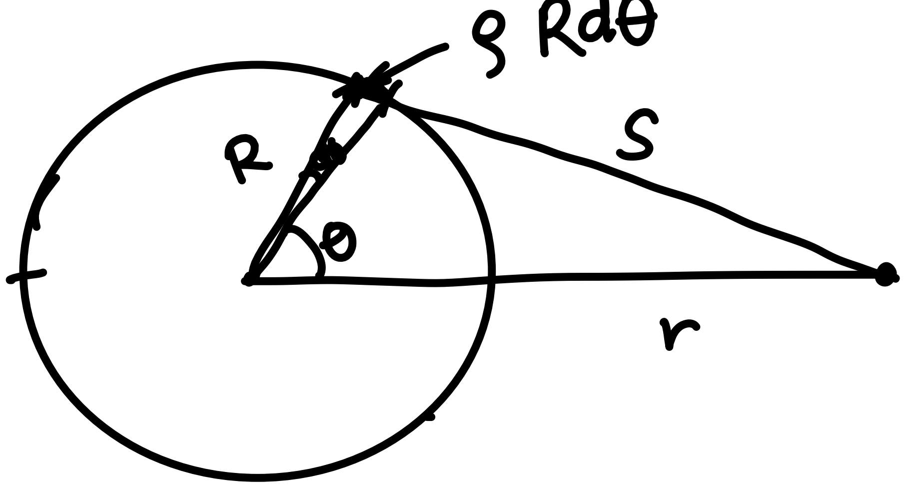
$$-G4\pi\rho R$$

$$= -\frac{GM}{R}$$ (상수)



$4\pi R^2 \cdot \rho = M$

[Ex 6.7.2]



같은 평면

$$\Phi = -G2\int_0^{\pi} \frac{\rho R d\theta}{S}$$

$$s^2 = r^2 + R^2 - 2rR\cos\theta$$

$$\Phi = -2G\rho R\int_0^{\pi} \frac{d\theta}{\sqrt{r^2+R^2-2rR\cos\theta}}$$

① $\underline{r \gg R}$ $\quad \frac{R}{r} \equiv x \ll 1$

$$\Phi = -\frac{2G\rho R}{r}\int_0^{\pi} \frac{d\theta}{\sqrt{1+\underbrace{\frac{R^2}{r^2}}_{x^2}-2\underbrace{\frac{R}{r}}_{x}\cos\theta}} \rightarrow \frac{1}{\sqrt{1-\underbrace{\left(2x\cos\theta-x^2\right)}_{\epsilon}}} = \frac{1}{\sqrt{1-\epsilon}} \approx 1+\frac{\epsilon}{2}+\frac{3\epsilon^2}{8}+\cdots$$

$$= 1 + \frac{1}{2}(1-\epsilon)^{-\frac{3}{2}}\Big|_{\epsilon=0}\cdot\epsilon + \frac{1}{2}\frac{3}{4}\cdot 1\cdot\epsilon^2$$

$$= 1 + \frac{1}{2}\left(2x\cos\theta - x^2\right) + \frac{3}{8}\cdot 4\cos^2\theta\, x^2 + O(x^3)$$

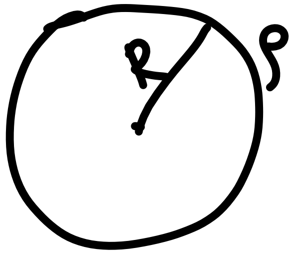
$$\Phi = -\frac{2G\rho R}{r}\int_0^{\pi}\left(1+\frac{1}{2}\left(2x\cos\theta - x^2\right)+\frac{3}{8}\cdot 4\cos^2\theta\, x^2 + O(x^3)\right)d\theta$$

$$= \quad '' \quad \left(\pi + 0 \underbrace{-\frac{x^2}{2}\pi + \frac{3}{2}x^2\cdot\frac{\pi}{2}}_{\left(\underbrace{\frac{3}{4}-\frac{2}{4}}_{\frac{1}{4}}\right)\pi x^2}\right)$$

$$= -\frac{G\overset{M}{\boxed{2\rho R\pi}}}{r}\left(1+\frac{1}{4}\left(\frac{R}{r}\right)^2+\cdots\right)$$

$$\int_0^{\pi}\cos\theta\, d\theta = \sin\theta\Big|_0^{\pi} = 0$$

$$\underbrace{\int_0^{\pi}\cos^2\theta\, d\theta = \int_0^{\pi}\sin^2\theta\, d\theta}_{\frac{\pi}{2}}$$



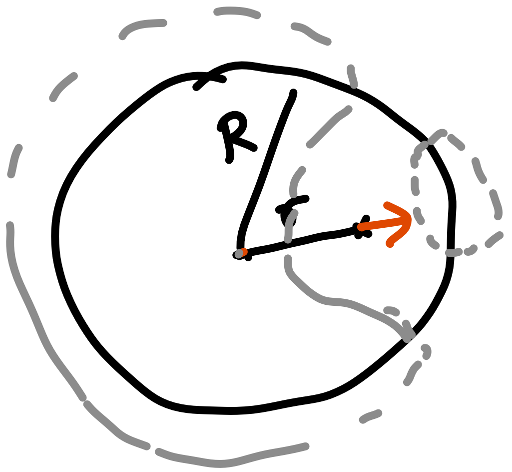
$$M = 2\pi R\rho$$

$$\vec{g} = -\vec{\nabla}\Phi \simeq -\vec{\nabla}\left(-\frac{GM}{r}-\frac{GMR^2}{4r^3}+\cdots\right)$$

$$\vec{\nabla} \leftarrow \vec{e}_r\frac{\partial}{\partial r} \qquad = \underset{GM}{\vec{e}_r}\left(-\frac{1}{r^2}-\frac{3}{4}\frac{R^2}{r^4}\right)$$

② $r < R$

$$x \equiv \frac{r}{R}$$



$$\Phi = -\frac{2G\rho \cancel{R}}{\cancel{R}}\int_0^{\pi}\frac{d\theta}{\sqrt{\underbrace{\frac{r^2}{R^2}}_{x}+1-2\underbrace{\frac{r}{R}}_{x}\cos\theta}}$$

$$\underbrace{\qquad\qquad}_{\pi\left(1+\frac{1}{4}\left(\frac{r}{R}\right)^2+\cdots\right)}$$

$$\Phi = -\frac{GM}{R}\left(1+\frac{r^2}{4R^2}+\cdots\right)$$

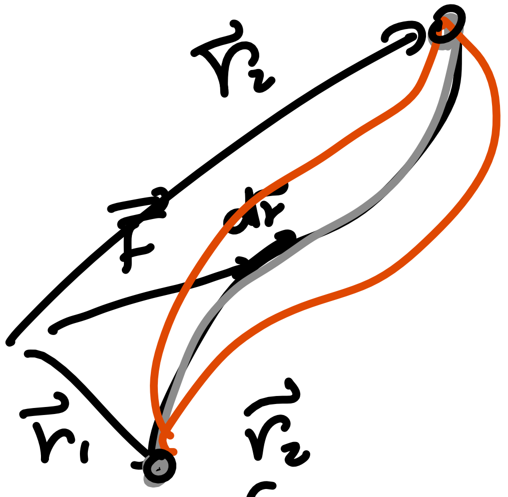
$$\vec{g} = -\vec{\nabla}\Phi = +\frac{GM}{R}\cdot\frac{\overset{\vec{r}}{\boxed{r\,\vec{e}_r}}}{2R^2}$$

6.8. Potential of ~~Central field~~ (중심력)

$\vec F \propto \vec r$

$$\vec\nabla\times\vec F = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec e_r & r\vec e_\theta & r\sin\theta\,\vec e_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix} \qquad \vec F = f(r)\,\underset{\hat r}{\overset{}{\vec e_r}} \qquad (r,\theta,\phi)$$

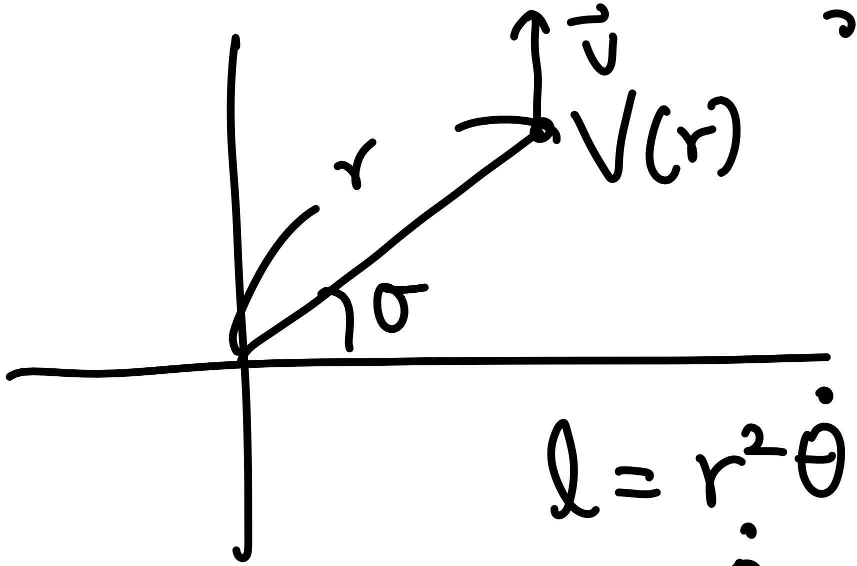
$$= \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec e_r & r\vec e_\theta & r\sin\theta\,\vec e_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ f(r) & 0 & 0 \end{vmatrix} = 0$$



$$-\int_{\vec r_1}\vec F\cdot d\vec r = V(\vec r_2)-V(\vec r_1)$$

, path, indep

6.9. Energy.



극좌표

$$\vec v = \frac{d\vec r}{dt} = \dot r\vec e_r + r\dot\theta\vec e_\theta$$

$$v^2 = \dot r^2 + r^2\dot\theta^2 \qquad T = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2)$$

$$T+V = \frac{1}{2}m(\dot r^2+r^2\dot\theta^2) + V(r) = E$$

$$\frac{1}{2}m\dot r^2 + \underbrace{\frac{m\cancel{r^2}}{2}\frac{l^2}{\cancel{r^4}\,r^2} + V(r)}_{V_{eff}(r) = V(r)+\frac{ml^2}{2r^2}} = E$$

"유효 위치 에너지"

$$l = r^2\dot\theta = \frac{L}{m} = \text{상수}$$

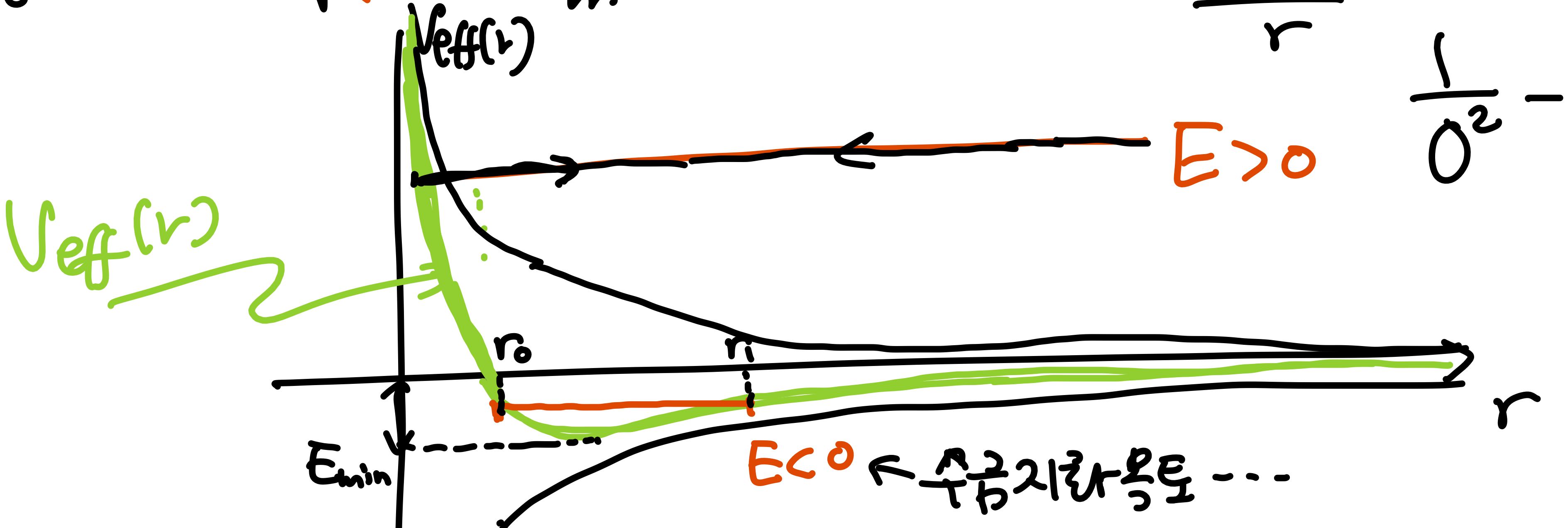
$$\dot\theta = \frac{l}{r^2}$$

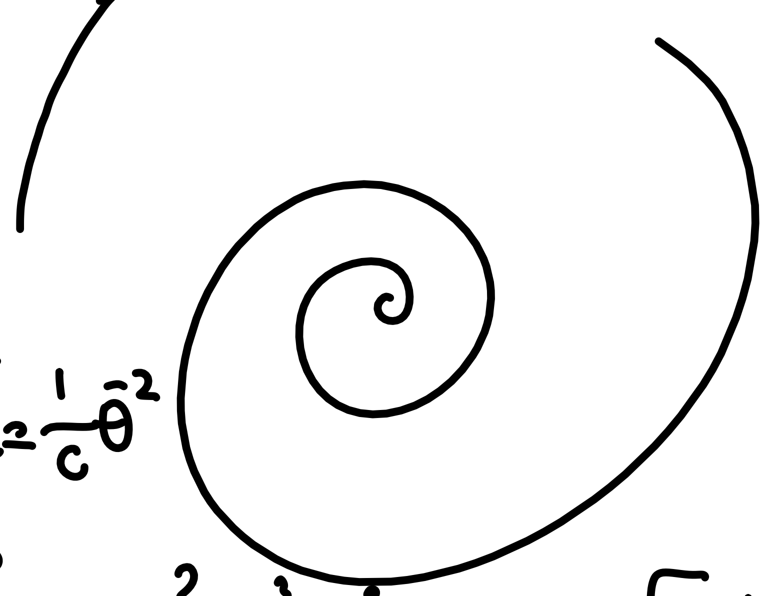
$$E = \frac{1}{2} m \dot{r}^2 + V_{eff}(r)$$

$$V_{eff}(r) = \underbrace{V(r)}_{= -\frac{GMm}{r}} + \frac{ml^2}{2 \cdot r^2}$$

$$\frac{dr}{dt} = \dot{r} = \pm\sqrt{\frac{2(E - V_{eff}(r))}{m}}$$

$$\frac{1}{0^2} - \frac{1}{0}$$





[Ex 6.9.1]

$$r = c\theta^2 = \frac{1}{u}$$

$$l = r^2\dot{\theta} \rightarrow \dot{\theta} = lu^2 \longrightarrow u = \frac{1}{c}\theta^{-2}$$

$$\dot{r} = \underbrace{\frac{dr}{du}}_{-\frac{1}{\cancel{u^2}}} \underbrace{\frac{du}{d\theta}}_{-2\sqrt{c}\,u^{\frac{3}{2}}} \frac{d\theta}{dt}\ l\cancel{u^2}$$

$$= 2l\sqrt{c}\,u^{\frac{3}{2}}$$

$$\frac{du}{d\theta} = -\frac{2}{c}\theta^{-3} = -\frac{2}{c}\left(\frac{1}{\sqrt{cu}}\right)^{-3} = -\frac{2}{c}c^{\frac{3}{2}}u^{\frac{3}{2}} = -2\sqrt{c}\,u^{\frac{3}{2}}$$

$$E = \frac{1}{2}m(4l^2c\,u^3) + \underset{+\frac{ml^2}{2r^2}}{V(r)} \longrightarrow V(r) = E - \frac{ml^2}{2r^2} - \frac{2ml^2c}{r^3}$$ //

6.10. Orbital Energy

$$V(r) = -\frac{k}{r} = -ku \qquad u = \frac{1}{r}$$

$$l = r^2\dot{\theta} \rightarrow \dot{\theta} = \frac{l}{r^2} = lu^2$$

$$E = \frac{1}{2}m\dot{r}^2 + V_{ef}(r) \qquad V_{ef}(r) = V(r) + \frac{ml^2}{2\cdot r^2}$$

$$\dot{r} = \frac{dr}{dt} = \frac{d\theta}{dt}\frac{dr}{d\theta} = lu^2\frac{d}{d\theta}\left(\frac{1}{u}\right) = l\cancel{u^2}\left(-\frac{1}{\cancel{u^2}}\right)\frac{du}{d\theta} = -l\frac{du}{d\theta}$$

$$E = \frac{1}{2}ml^2\left(\frac{du}{d\theta}\right)^2 - ku + \frac{ml^2}{2}u^2 = \frac{1}{2}ml^2\left(\underbrace{\left(\frac{du}{d\theta}\right)^2 + u^2}\right) - ku$$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2(E+ku)}{ml^2} \qquad \left(\frac{du}{d\theta}\right) = \sqrt{-u^2 + \frac{2(E+ku)}{ml^2}}$$

$$\int \frac{du}{\sqrt{\underbrace{-u^2 + \frac{2(E+ku)}{ml^2}}_{au^2+bu+c}}} = \int_{\theta_0}^{\theta} d\theta = \theta - \theta_0$$

$$au^2 + bu + c \quad \left(a=-1,\ b=\frac{2k}{ml^2},\ c=\frac{2E}{ml^2}\right)$$

$$au^2+bu+c = a\left(\underbrace{u^2 + \frac{b}{a}u + \frac{b^2}{4a^2}} - \frac{b^2}{4a^2}\right) + c$$

$$= a\left(u + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

$$= \underbrace{\frac{b^2}{4} + c}_{A^2} - (\underbrace{u - \frac{b}{2}}_{u'})^2 \qquad u' = u - \frac{b}{2} \qquad du' = du$$

$$= A^2 - u'^2$$

$$\int \frac{du'}{\sqrt{A^2 - u'^2}}$$

$$\int \frac{du'}{\sqrt{A^2-u'^2}} = \int \frac{-A\sin\alpha\, d\alpha}{\sqrt{A^2-A^2\cos^2\alpha}} = -\int d\alpha = -\alpha$$

$$\underbrace{u' = A\cos\alpha} \rightarrow du' = -A\sin\alpha\, d\alpha$$

$$\therefore \ -\alpha = \theta-\theta_0 \qquad \alpha = \theta_0-\theta \qquad \therefore \ u' = \underset{\uparrow \frac{1}{r}}{u} - \frac{b}{2} = A\cos(\theta_0-\theta)$$

$$\therefore \ \frac{1}{r} = \frac{b}{2} + A\cos(\theta_0-\theta) \qquad A = \sqrt{\frac{b^2}{4}+c} \qquad b = \frac{2k}{ml^2},\ c = \frac{2E}{ml^2}$$

$$\boxed{r = \left[\frac{k}{ml^2} + \sqrt{\frac{k^2}{(ml^2)^2} + \frac{2E}{ml^2}}\ \cos(\theta-\theta_0)\right]^{-1}}$$

$$r = \frac{ml^2/k}{1 + \underbrace{\sqrt{1+\frac{2E\,ml^2}{k^2}}}_{\epsilon}\ \cos(\theta-\theta_0)}$$

$$\epsilon = \sqrt{1+\frac{2E\,ml^2}{k^2}}$$

$\epsilon < 1 \rightarrow E < 0$

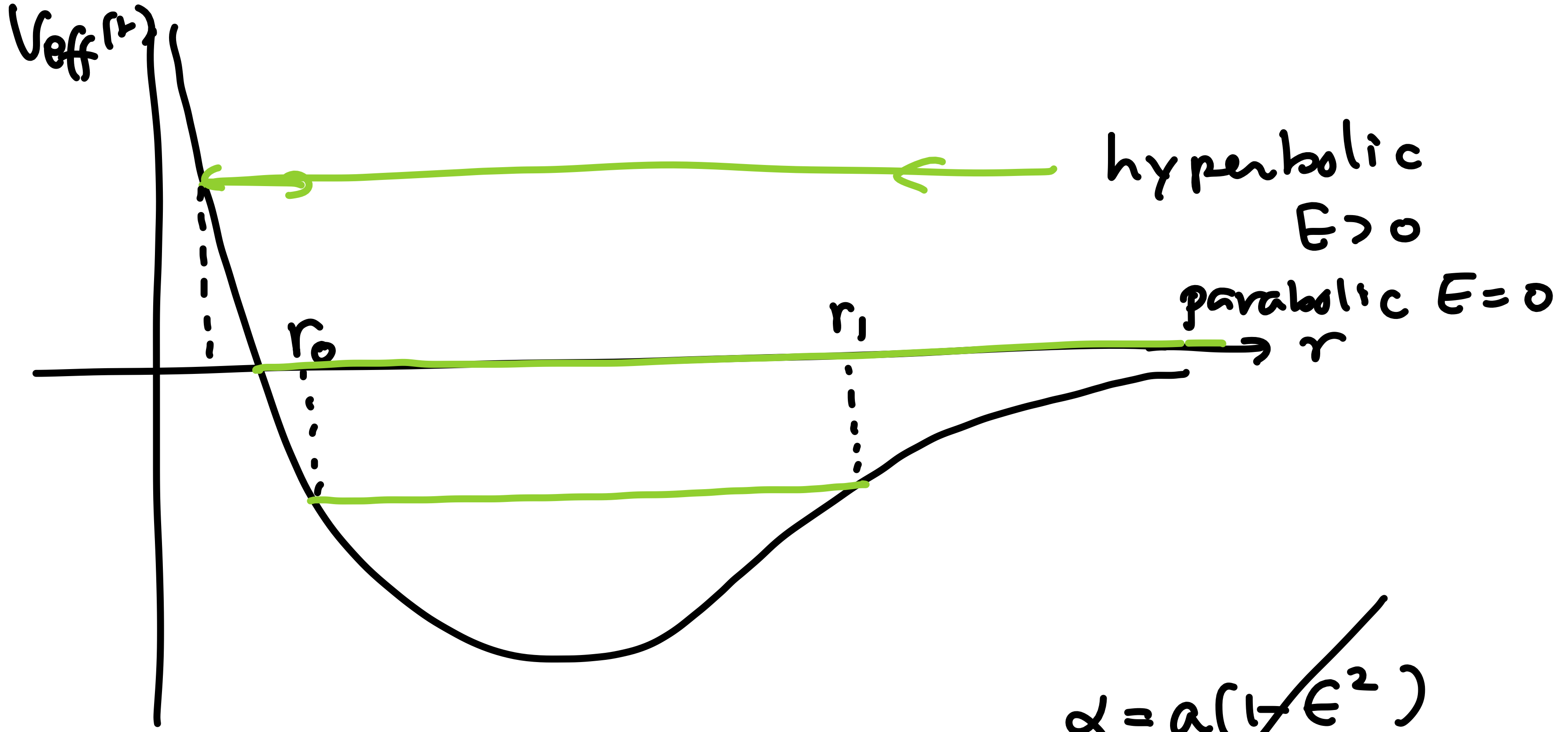
$\epsilon > 1 \rightarrow E > 0$

$\epsilon = 1 \rightarrow E = 0$

$$(\text{cf})\quad r(\theta) = \frac{1}{A\cos(\theta-\theta_0) + \frac{k}{ml^2}} = \frac{\overset{\alpha}{ml^2/k}}{1 + \underbrace{A\frac{ml^2}{k}}_{\epsilon}\cos(\theta-\theta_0)}$$

$$\alpha = a(1-\epsilon^2) = \frac{ml^2}{k}$$

$$\epsilon = A\,\frac{ml^2}{k}$$



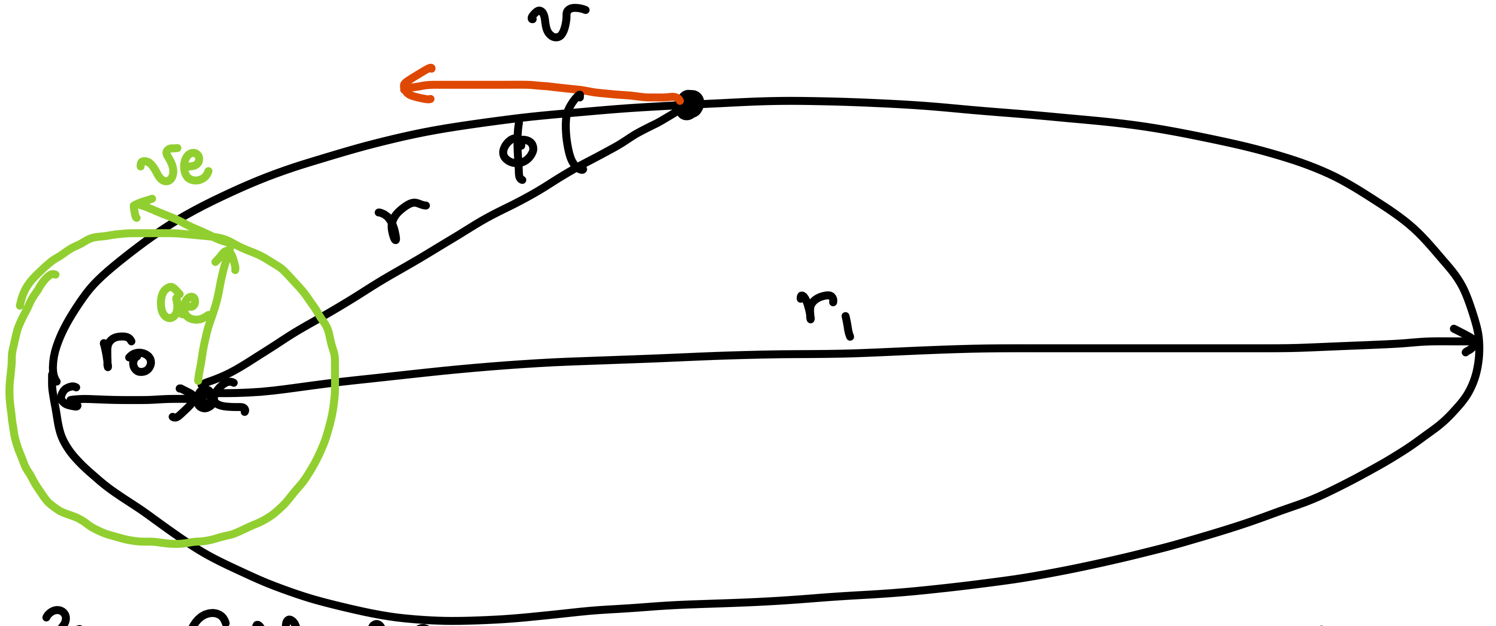
$$\alpha = a(\cancel{1-\epsilon^2})$$

$$\epsilon = \sqrt{1+\frac{2E\,ml^2}{k^2}} \rightarrow \cancel{1-\epsilon^2} = -\frac{2E}{k}\boxed{\frac{ml^2}{k}}$$

$$1 = -\frac{2Ea}{k}$$

$$\therefore E = -\frac{k}{2a} \,/\!/$$

[Ex 6.10.1]



$$V(r) = -\frac{k}{r} = -\frac{GM_\odot m}{r}$$

$$k = GM_\odot m$$

$$E = \frac{1}{2}mv^2 - \frac{GM_\odot m}{r} \qquad l = vr\sin\phi \qquad GM_\odot = a_e v_e^2$$

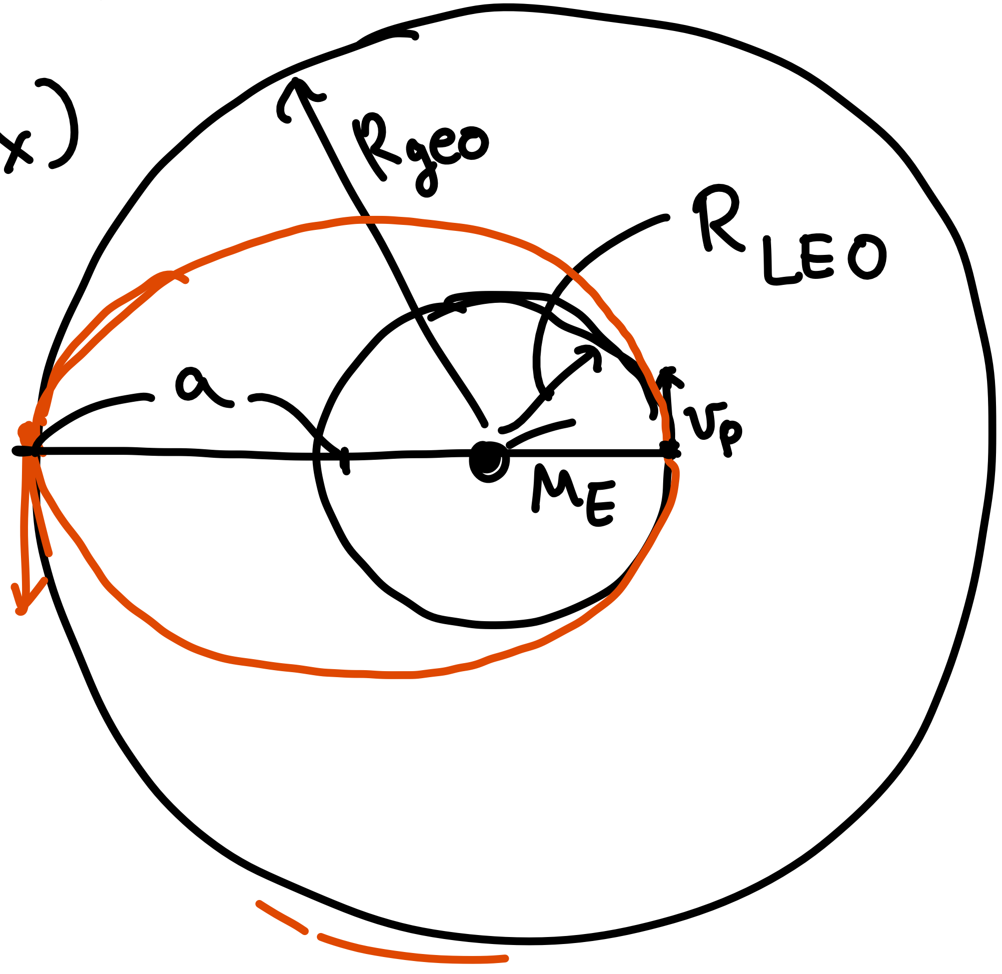
$$\epsilon = \sqrt{1+\frac{2}{(GM_\odot m)^2}\left(\frac{1}{2}mv^2 - \frac{GM_\odot m}{r}\right)m(vr\sin\phi)^2}$$

$$\epsilon = \sqrt{1 + \cdot\left(\frac{v^2}{v_e^2} - \frac{a_e \cdot 2}{r}\right)\left(\frac{v}{v_e}\frac{r}{a_e}\sin\phi\right)^2} = \sqrt{1 + \left(V^2 - \frac{2}{R}\right)\left(VR\sin\phi\right)^2}$$

$$V = \frac{v}{v_e} \qquad R = \frac{r}{a_e}$$

$\phi = 30°$ $\quad V = 0.5 \quad R = 4 \quad \longrightarrow \epsilon = \ ---$

(Ex)



$R_{LEO} =$ 근일점$\overset{\text{거리}}{\vee}$ $= a(1-\epsilon)$

$R_{geo} =$ 원일점$\overset{''}{\vee}$ $= a(1+\epsilon)$

$\Big\}\ a = \frac{R_{LEO} + R_{geo}}{2}$

○ : $E = \frac{1}{2} m v_p^2 - \frac{G M_E m}{R_{LEO} = a(1-\epsilon)} = -\frac{G M_E m}{2a}$

$$v_p^2 = \frac{2}{\cancel{m}} G M_E \cancel{m}\left(\frac{1}{a(1-\epsilon)} - \frac{1}{2a}\right)$$

$$= \frac{G M_E}{a}\left(\underbrace{\frac{2}{1-\epsilon} - 1}_{\frac{1+\epsilon}{1-\epsilon}}\right)$$

$v_{LEO} \longrightarrow v_p$

$$\boxed{\Delta v = v_p - v_{LEO} = \sqrt{\frac{G M_E}{a}}\left(\sqrt{\frac{1+\epsilon}{1-\epsilon}} - \frac{1}{\sqrt{1-\epsilon}}\right)}$$

○ : $\frac{G M_E m}{R_{LEO}^2} = m \frac{v_{LEO}^2}{R_{LEO}} \rightarrow v_{LEO}^2 = \frac{G M_E}{R_{LEO}}$

$a(1-\epsilon) \rightarrow R_{LEO}$

$$\Delta v = v_p - v_{LEO} = \sqrt{\frac{GM_E}{\frac{R_L+R_g}{2}}}\left(\sqrt{\frac{R_g}{R_L}} - \frac{\sqrt{\frac{R_L+R_g}{2}}}{\sqrt{R_L}}\right) \leftarrow$$

$$1-\epsilon = \frac{\overset{R_L}{R_{LEO}}}{a} \qquad 1+\epsilon = \frac{R_{geo}}{a} = R_g$$

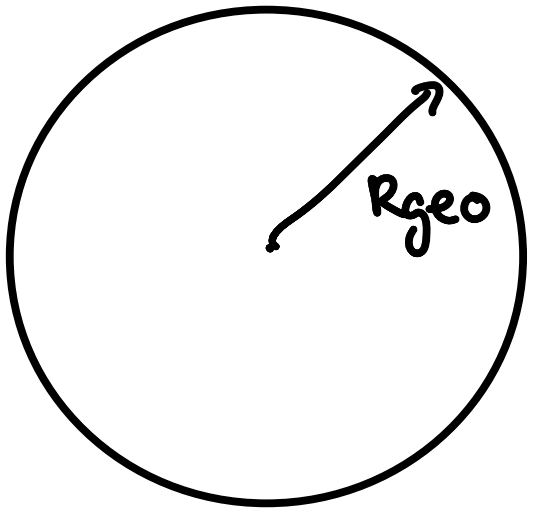
$$= \frac{\sqrt{GM_E}}{\sqrt{R_{LEO}}}\left(\sqrt{\frac{2R_{geo}}{R_{LEO}+R_{geo}}} - 1\right)$$

R_{LEO} = 근일점 거리 $= a(1-\epsilon)$

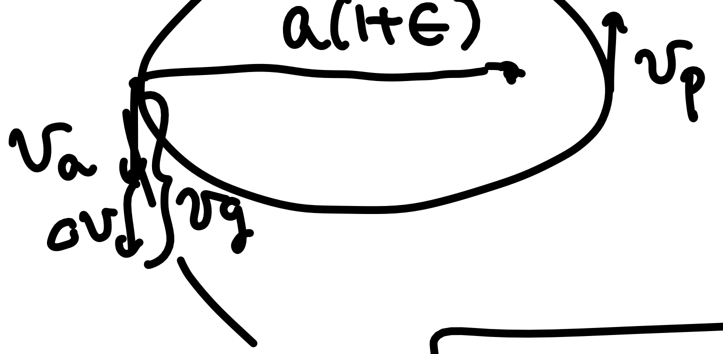
R_{geo} = 원일점 ″ $= a(1+\epsilon)$

$$\left.\begin{matrix} \\ \end{matrix}\right\} \underset{\sim\sim\sim}{a = \frac{R_{LEO}+R_{geo}}{2}}$$

(b)



$$\frac{GM_E m}{R_{geo}^2} = m\frac{v_g^2}{R_g} \rightarrow \underset{\sim}{v_g} = \underline{\underline{\sqrt{\frac{GM_E}{R_{geo}}}}}$$



$$E = \frac{1}{2}\cancel{m}v_a^2 - \frac{GM_E\cancel{m}}{a(1+\epsilon)} = -\frac{GM_E\cancel{m}}{2a}$$

$$v_a^2 = \frac{2GM_E}{R_{geo}} - \frac{GM_E}{\frac{R_{LEO}+R_{geo}}{2}}$$

$$\downarrow$$

$$v_a^2 = \frac{2GM_E}{R_{LEO}+R_{geo}}\left(\frac{R_{LEO}+R_{geo}}{R_{geo}} - 1\right)$$

$$″ \; \frac{R_{LEO}}{R_{geo}}$$

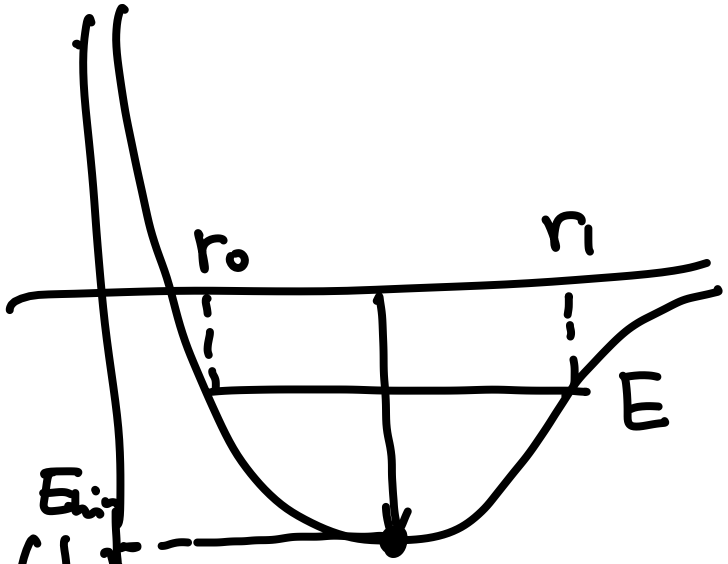
$$v_a^{-} = \sqrt{\frac{2GM_E}{R_{LEO}+R_{geo}}\frac{R_{LEO}}{R_{geo}}} \Leftarrow$$

$$\Delta v_2 = v_g - v_a$$

6.11.

$$E = \frac{1}{2}m\dot{r}^2 + V_{eff}(r)$$

$$V_{eff}(r) = \frac{ml^2}{2r^2} + V(r)$$



$$\dot{r} = 0 \longrightarrow E = V_{eff}(r = r_0, r_1)$$

$$V = -\frac{k}{r} \qquad 0 = \underbrace{\frac{ml^2}{2}}_{a}\left(\frac{1}{r}\right)^2 \underbrace{-k}_{b}\left(\frac{1}{r}\right) \underbrace{-E}_{c}$$

$$\frac{1}{r} = \frac{k \pm \sqrt{k^2 + 2Eml^2}}{ml^2}$$

$$r = \frac{ml^2}{k \pm \sqrt{}} = \frac{ml^2(k \mp \sqrt{})}{(k \pm \sqrt{\cdots})(k \mp \sqrt{})} = -\frac{\cancel{ml^2}(k \mp \sqrt{})}{2E\cancel{ml^2}}$$

$$\rightarrow k^2 - (k^2 + 2Eml^2)$$

$$\left(\frac{1}{r}\right) = \frac{k}{ml^2} \qquad r_0(-) = \frac{k \mp \sqrt{k^2 + 2Eml^2}}{-2E}$$

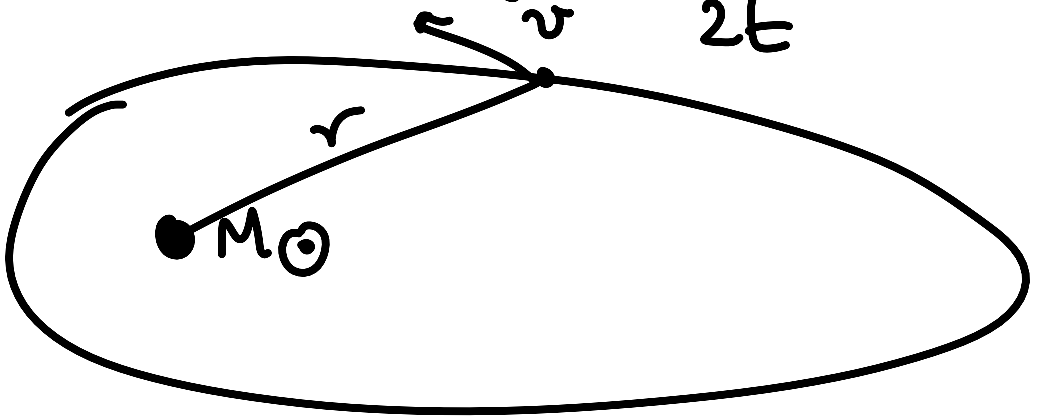
$$r_1(+)$$

$$\rightarrow V_{eff}\Big|_{min} = \frac{k^2}{2ml^2} - \frac{k^2}{ml^2} = -\frac{k^2}{2ml^2}$$

$$= E_{min} \qquad \rightarrow E_{min} = -\frac{k^2}{2ml^2},$$

[Ex6.11]

$a = -\frac{k}{2E}$ $\quad E = -\frac{k}{2a}$ $\quad k = GM_{\odot} m$



$$a = -\frac{GM_{\odot} m}{2\left(\frac{mv^2}{2} - \frac{GM_{\odot} m}{r}\right)} = -\frac{a_e v_e^2}{v^2 - \frac{2a_e v_e^2}{r}}$$

$$= \frac{a_e}{-V^2 + \frac{2}{R}}$$

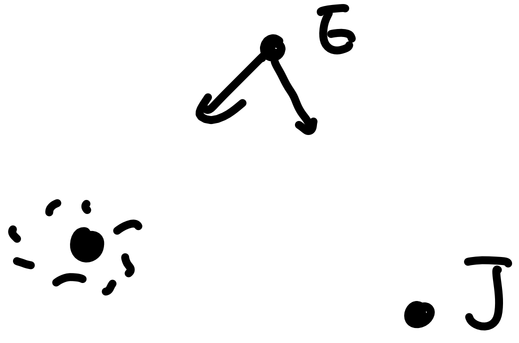
$$GM_{\odot} = a_e v_e^2$$

$$V \equiv \frac{v}{v_e}$$

$$R = \frac{r}{a_e}$$

6.12. $V(r) = -\frac{k}{r}$ $\quad \underline{f(r) = \frac{k}{r^2}}$

태양만



$$\underline{m\ddot{r} = \frac{ml^2}{r^3} + f(r)}$$

$$\uparrow$$

$$V_{eff} = \frac{ml^2}{2r^2} + V(r)$$

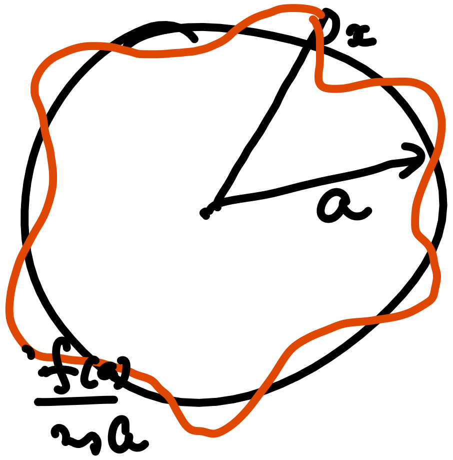
$$f_{eff}(r) = -\frac{\partial}{\partial r} V_{eff}$$

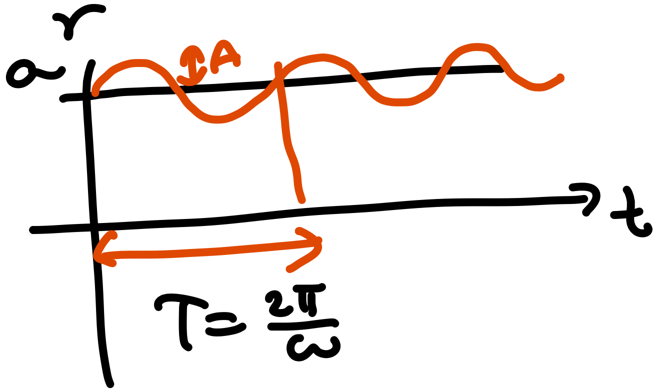
원궤도.

$r = a$

$\dot r = 0 \quad \ddot r = 0$

$$\underline{\frac{ml^2}{a^3} + f(a) = 0} \rightarrow \frac{l^2}{a^4} = -\frac{f(a)}{ma}$$





$r = a + \underset{\sim}{x}$

$a \gg x$

$\frac{x}{a} \ll 1$

$$m\ddot r = \frac{ml^2}{r^3} + f(r)$$

$$\|$$

$$m\ddot x = \frac{ml^2}{(a+x)^3} + f(a+x)$$

$$= \underset{\sim}{\frac{ml^2}{a^3}}\underbrace{\frac{1}{(1+\frac{x}{a})^3}}_{1-3\frac{x}{a}+\cdots} + \underset{\sim}{f(a)} + xf'(a) + \cdots$$

$$\frac{ml^2}{a^3} = -f(a)$$

$$\frac{l^2}{a^4} = -\frac{f(a)}{ma}$$

$$m\ddot x = -\frac{ml^2}{a^3}\frac{3}{a}\,. + f'(a)\,x$$

$$= -3\frac{f(a)}{ma}$$

$$\Rightarrow \ddot x + \underbrace{\left(\frac{3l^2}{a^4} - \frac{f'(a)}{m}\right)}_{V''_0} x = 0$$

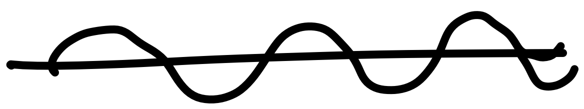
$$\rightarrow = \omega^2$$

$$\ddot x + \omega^2 x = 0$$

$$x(t) = A\cos(\omega t + \phi)$$

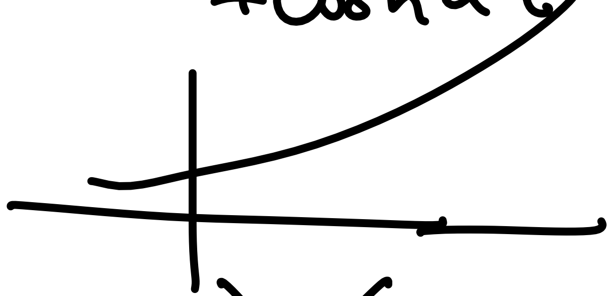
$$\ddot{x} + \underbrace{\left(-3\frac{f(a)}{ma} - \frac{f'(a)}{m}\right)}_{\substack{\wedge \qquad\qquad \vee\!\shortparallel \\ 0 \qquad\qquad 0}} x = 0$$

$\longrightarrow$ H.O.



$$\ddot{x} - \alpha^2 x = 0 \rightarrow \ddot{x} = \alpha^2 x \rightarrow x = \sinh\alpha t + \cosh\alpha t$$

$t \rightarrow$ 증가



$$\ddot{x} = 0 \rightarrow x = \cancel{x_0} + \alpha t$$

$$\omega^2 = \frac{1}{m}\left(-3\frac{f(a)}{a} - f'(a)\right) > 0$$

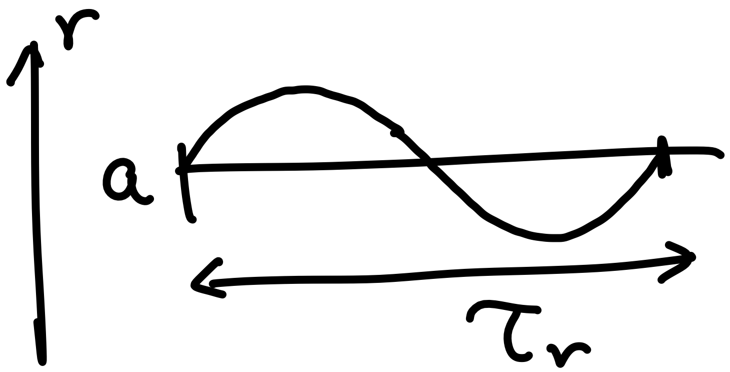
$$\boxed{3\frac{f(a)}{a} + f'(a) < 0}$$

$$f(r) = -c r^n \rightarrow f'(a) = -n c a^{n-1}$$

$$-3c\frac{a^n}{a} - n c a^{n-1} = -a^{n-1} c\,(3+n) < 0$$

$$\boxed{3+n > 0} \qquad \underline{\underline{n = -2, -1, \cdots}}$$

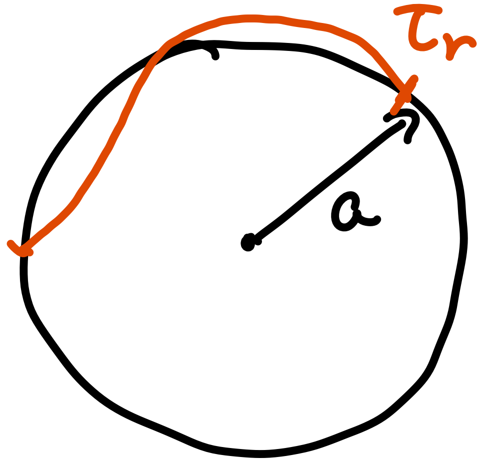
$$\omega^2 = \frac{1}{m}\left(-3\,\frac{f(a)}{a} - f'(a)\right)$$



$$\tau_r = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{-3\frac{f(a)}{a} - f'(a)}}$$

$$\frac{\tau_r}{2}\underbrace{\dot\theta} = \frac{\tau_r}{2}\frac{l}{a^2} = \psi = \underline{\text{apsidal angle}}$$

(극지각)



$$l = r^2\dot\theta \approx a^2\dot\theta$$

$$r = a + \underbrace{x}_{x \ll a} \approx a$$

$$\frac{l^2}{a^4} = -\frac{f(a)}{ma} \rightarrow \frac{l}{a^2} = \sqrt{-\frac{f(a)}{ma}}$$

$$\psi = \pi\sqrt{-\frac{f(a)}{\cancel{m}a}}\sqrt{\frac{\cancel{m}}{-3\frac{f(a)}{a} - f'(a)}}$$

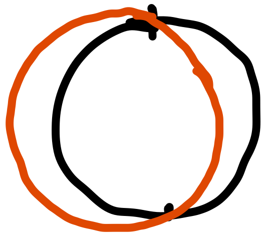
$$= \pi\sqrt{\frac{1}{3 + \underbrace{\frac{af'(a)}{f(a)}}_{n}}} \;,,$$

$$f(r) = -cr^n \qquad \frac{af'(a)}{f(a)} = \frac{-n\cancel{c}\cancel{a^n}}{-\cancel{c}\cancel{a^n}} = n$$

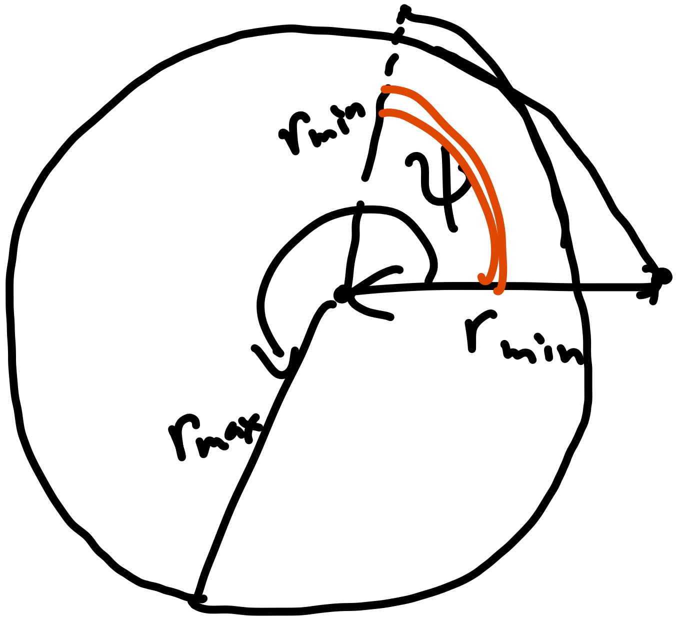
$$\underline{\psi = \pi(3+n)^{-\frac{1}{2}}}$$

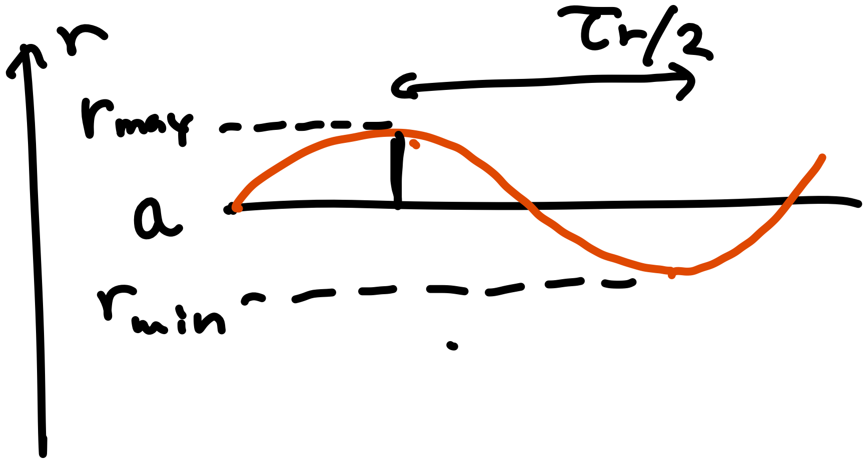
$$(3+n)^{-1} > 1 \qquad 1 > 3+n$$

$n = -2$; $\psi = \pi$ $\longrightarrow$



$n < -2$: $\psi > \pi$, $n > -2$: $\psi < \pi$



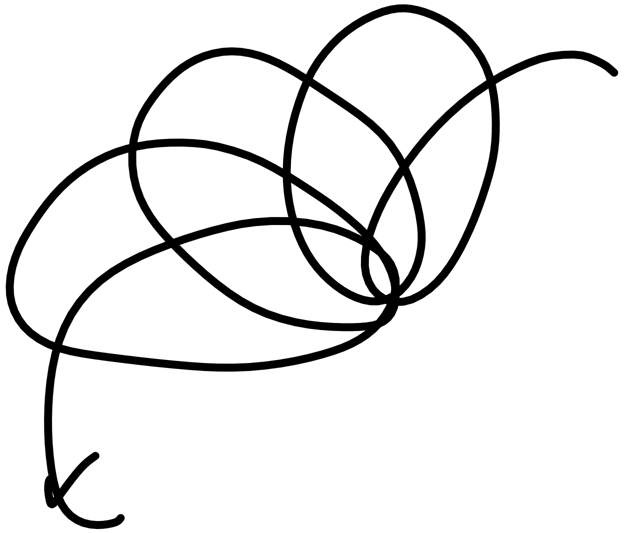


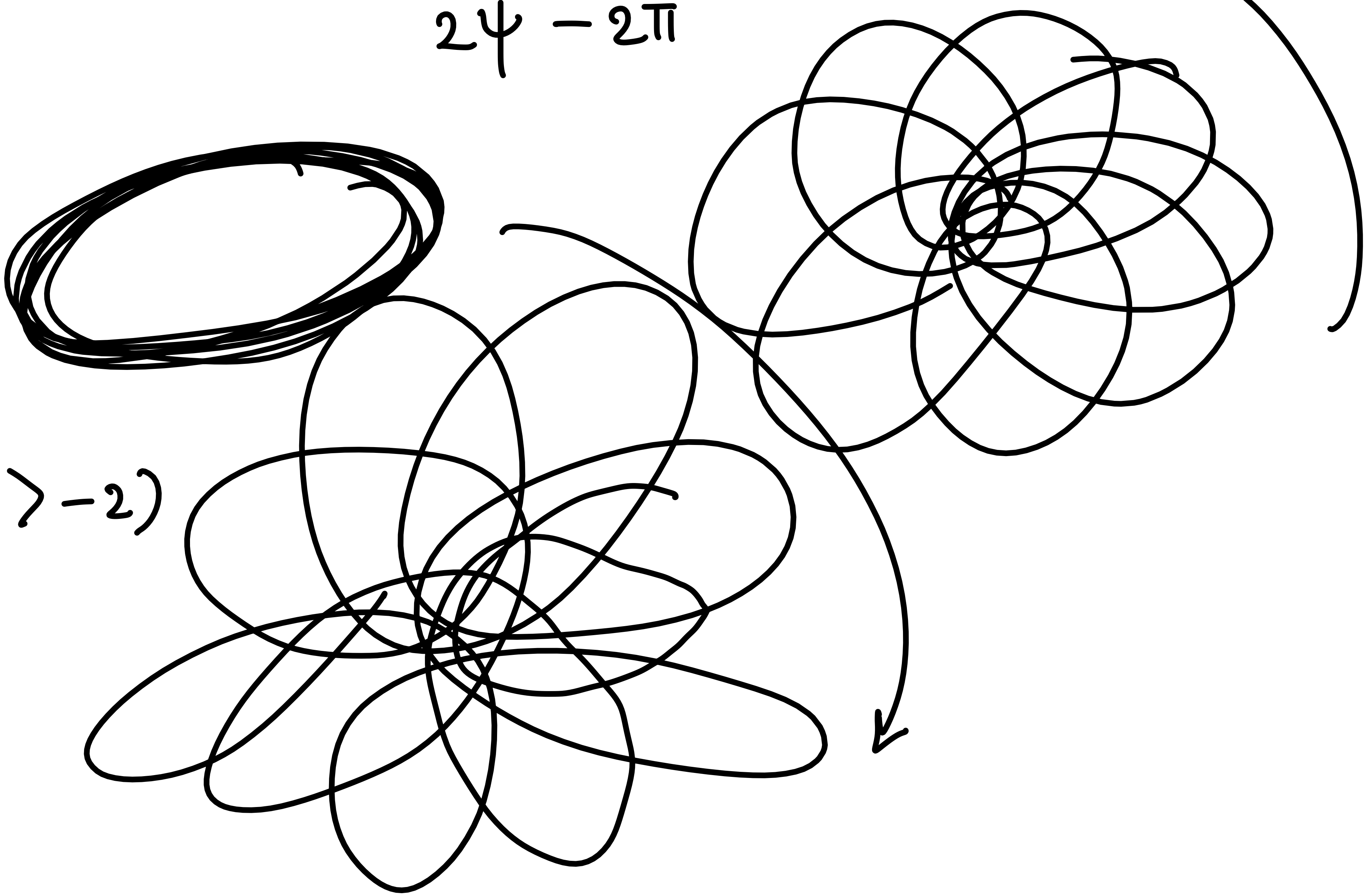
$\psi > \pi$

세차운동(precession)

$2\psi - 2\pi$

$\psi > \pi$





$\psi < \pi \quad (n > -2)$

[Ex 6.13.1] $f(r) = -\frac{k}{r^2} + \epsilon r \rightsquigarrow \psi = ?$

$$f'(r) = \frac{2k}{r^3} + \epsilon$$

$$\psi = \pi \sqrt{\frac{1}{3 + \frac{a f'(a)}{f(a)}}} \;,, = \pi \left(3 + \underbrace{\frac{a\left(\frac{2k}{a^3} + \epsilon\right)}{-\frac{k}{a^2} + \epsilon a}} \right)^{-\frac{1}{2}}$$

$$\frac{\cancel{2k}\,\frac{}{\cancel{a^2}}\left(1 + \frac{a^3}{2k}\epsilon\right)}{-\frac{\cancel{k}}{\cancel{a^2}}\left(1 - \frac{a^3}{k}\epsilon\right)}$$

$$= -2\left(1 + \underbrace{\left(\frac{a^3}{2k} + \frac{a^3}{k}\right)}\epsilon\right)$$

$$\left(\frac{3 + = -2\left(1 + \frac{3}{2k}a^3\epsilon\right)}{\left(1 - \frac{3}{k}a^3\epsilon\right)}\right)^{-\frac{1}{2}}$$

$$= 1 + \frac{3a^3}{2k}\epsilon$$

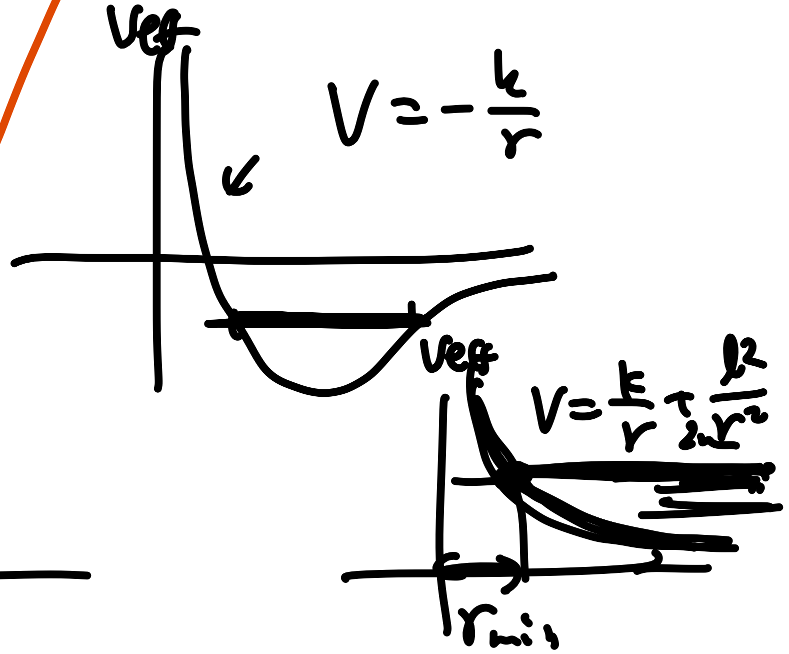
$$\therefore \psi = \pi\left(1 + \frac{3a^3}{2k}\epsilon\right) \;,,$$

6.14. Repulsive force (척력)

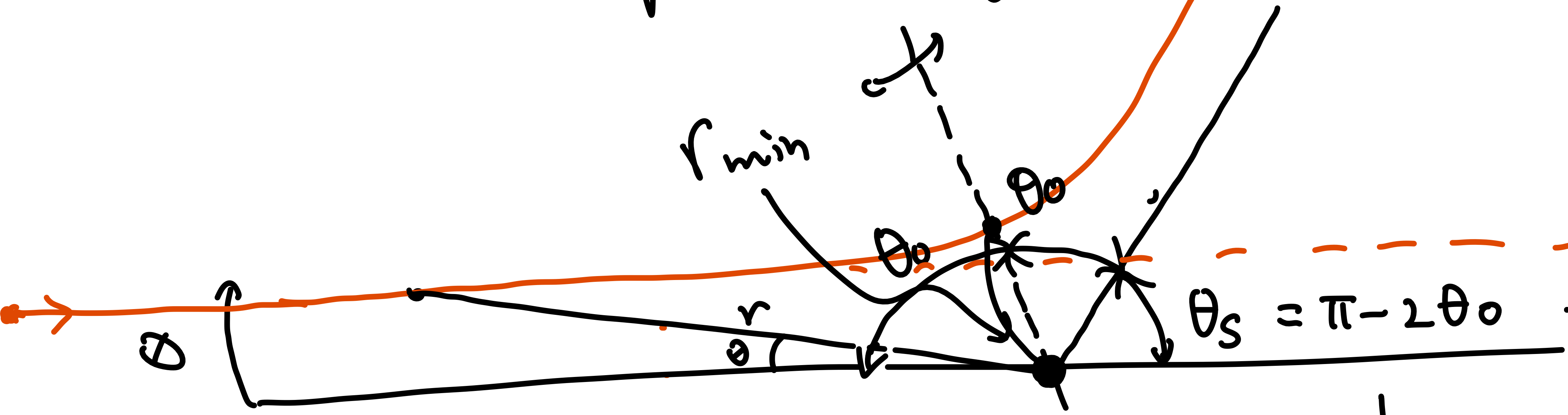
$$f(r) = -\frac{k}{r^2} \longrightarrow \frac{Qq}{r^2}$$

$k \rightarrow -Qq$

$$r = \frac{1}{\left[\frac{-Qq}{ml^2} + \sqrt{\frac{Q^2q^2}{(ml^2)^2} + \frac{2E}{ml^2}}\cos(\theta-\theta_0)\right]}$$



$$= \frac{ml^2/Qq}{-1 + \sqrt{1 + \frac{2Eml^2}{Q^2q^2}}\cos(\theta-\theta_0)}$$



$$\theta_s = \pi - 2\theta_0 \rightarrow \theta_0 = \frac{\pi - \theta_s}{2}$$

$$\cos\theta_0 = \frac{1}{\sqrt{1 + \frac{2Eml^2}{Q^2q^2}}} = \sin\frac{\theta_s}{2}$$

$\theta = 0 \quad r = \infty$

$\theta = 2\theta_0 \quad r = \infty$

$$\frac{1}{\sqrt{1+\frac{2Eml^2}{Q^2g^2}}} = \sin\frac{\theta_s}{2}$$

$$\cot\frac{\theta_s}{2} = \sqrt{\frac{2Eml^2}{Q^2g^2}}$$

$E = \frac{1}{2}mv_0^2$

$l = v_0 b$

QM

$k = \frac{mv_0}{\hbar}$

$\psi \sim e^{ikz} + \frac{f(\theta)e^{ikr}}{r}$

$\frac{d\sigma}{d\Omega} \sim |f(\theta)|^2$

$r\sin\theta = b$

↑ impact parameter (충돌 변수)

b가 유일한 변수

$\theta_s(b)$

$b(\theta_s)$

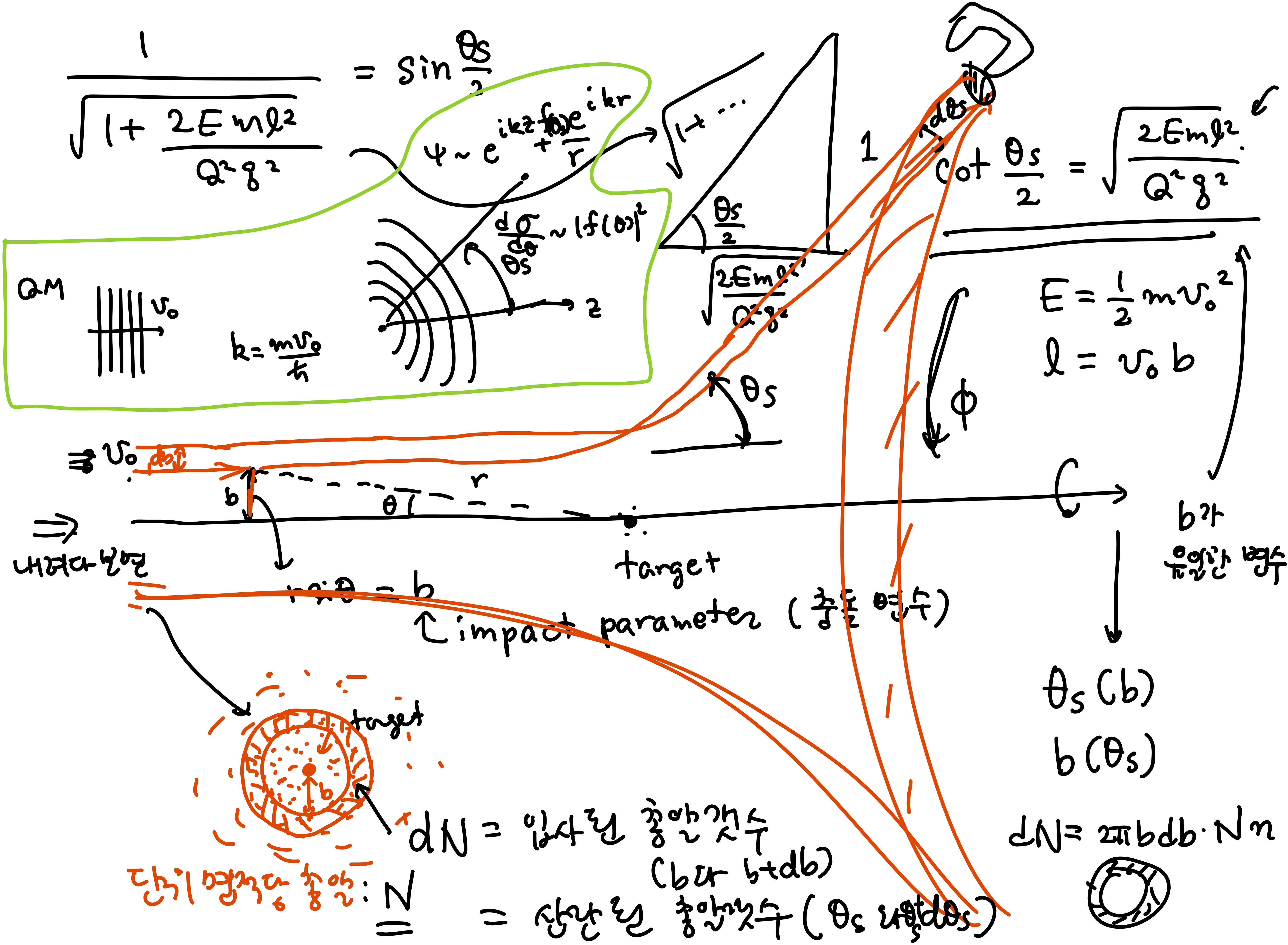
$dN = 2\pi b\,db \cdot Nn$

내려다보면

단위 면적당 총알 : N

dN = 입사된 총알 갯수 (b와 b+db)

= 산란된 총알 갯수 (θ_s 와 $\theta_s + d\theta_s$)



$$dN = 2\pi b\,db \cdot N n \longrightarrow \frac{dN}{N} = 2\pi b\,db\, n$$

개수

단위면적당 갯수

$$= n\,\sigma(\theta_s)\,\underline{d\Omega}$$

입체각

θ_s & $\theta_s + d\theta_s$

$$\underline{d\Omega = 2\pi \sin\theta_s\, d\theta_s}$$

$$\Rightarrow \quad \sigma(\theta_s) = 2\pi b \left|\frac{db}{d\Omega}\right|$$

$$\sigma(\theta_s) = \frac{b}{\sin\theta_s}\left|\frac{db}{d\theta_s}\right|$$

: 단위 $\underline{\text{면적}}$ (m^2, bahn, ...)

← cross section

$$\boxed{\left(\frac{Qq}{4E}\right)^2 \frac{1}{\sin^4\frac{\theta_s}{2}}} \;=\; \sigma(\theta_s)$$

$$= \frac{b(\theta_s)}{\sin\theta_s}\left|\frac{db(\theta_s)}{d\theta_s}\right|$$

$b = b(\theta_s)$

$$= \frac{(Qq)^2}{2E\cdot 4E}\,\frac{\cot\frac{\theta_s}{2}}{\underset{2\sin\frac{\theta_s}{2}\cancel{\cos\frac{\theta_s}{2}}}{\sin\theta_s}}\,\frac{1}{\sin^2\frac{\theta_s}{2}}$$

$$\cot\frac{\theta_s}{2} = \frac{\cancel{\cos}}{\sin}$$

4

$$\cot\frac{\theta_s}{2} = \sqrt{\frac{2Em l^2}{Q^2 q^2}}$$

$$E = \frac{1}{2} m v_0^2, \quad l = v_0 b$$

$$= \sqrt{\frac{m^2 v_0^2 v_0^2 b^2}{Q^2 q^2}} = \boxed{\frac{m v_0^2}{Qq}}\, b$$

상수

$$b = \frac{Qq}{2E}\cot\frac{\theta_s}{2}$$

$$\frac{db}{d\theta_s} = \frac{Qq}{4E}\,\frac{1}{\sin^2\frac{\theta_s}{2}}$$

[Ex 6.14.1] $\theta_s = \frac{\pi}{2}$

$$\cot\frac{\theta_s}{2} = \sqrt{\frac{2Em\ell^2}{Q^2q^2}} = \frac{2E^{\ell}}{\underset{79e}{\underset{\uparrow}{Q}}\,\underset{2e}{\underset{\uparrow}{q}}}\,b$$

[Ex 6.14.2]

$$r_{min} = \frac{m\ell^2/Qq}{-1 + \sqrt{1 + \frac{2Em\ell^2}{Q^2q^2}}}$$

기말시험

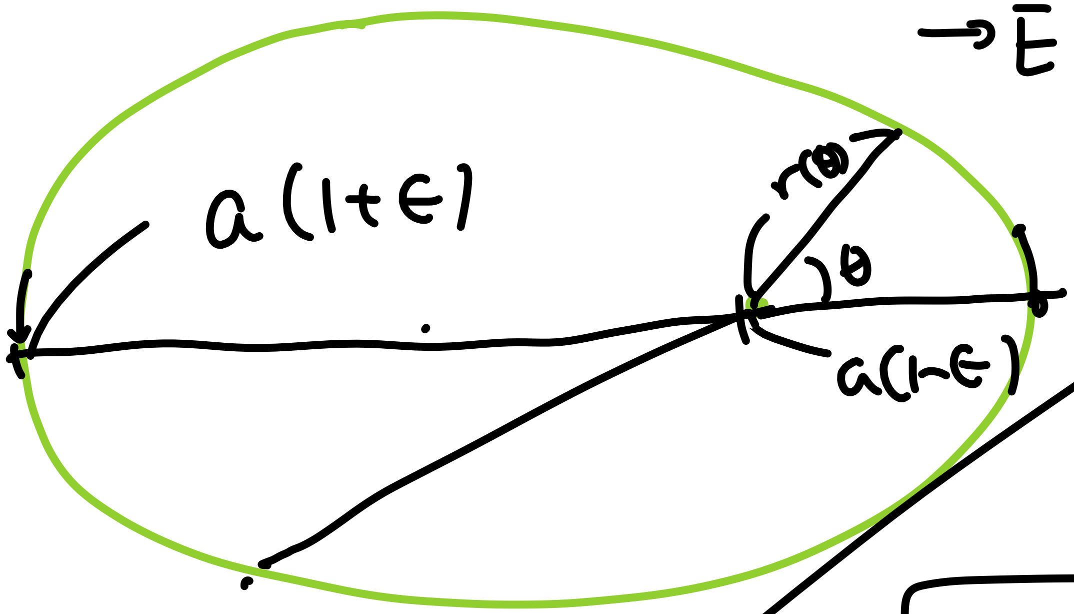
5 문제.

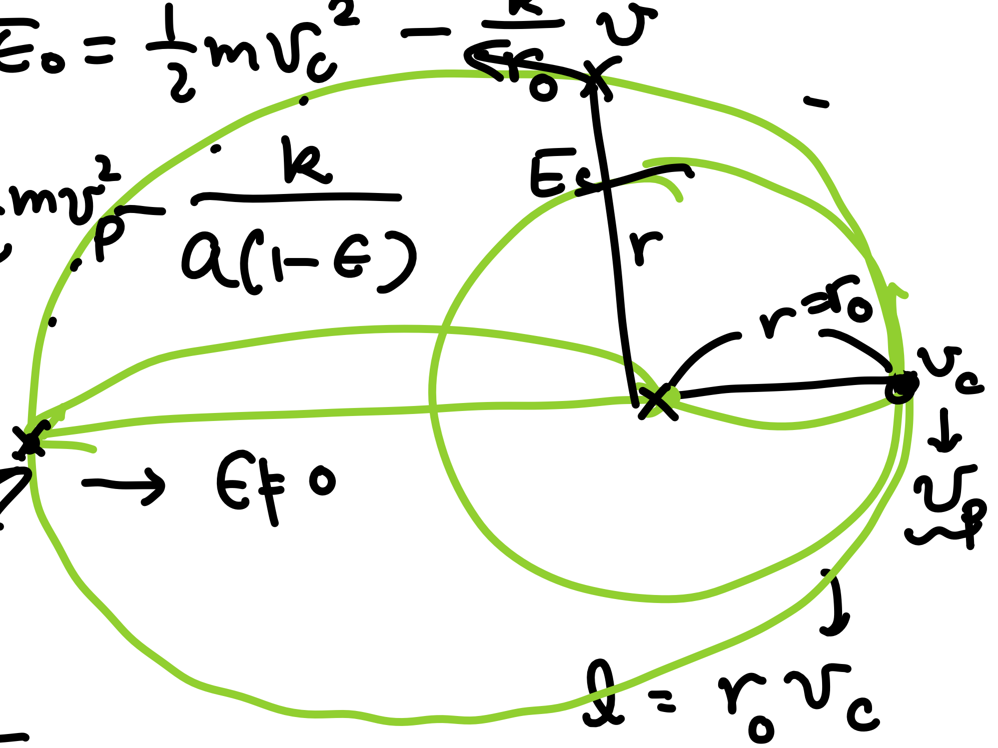
4장 1 문제
5장 2 〃
6장 2 〃

(예제) 1 문제
(숙제) 2 〃
(창작) 2 〃

$$E_0 = \frac{1}{2} m v_c^2 - \frac{k}{r_0}$$

$$\rightarrow E = \frac{1}{2} m v_p^2 - \frac{k}{a(1-\epsilon)}$$





$$\ell = r_0 v_c$$

$$\epsilon = \sqrt{1 + \frac{2Em\ell^2}{k^2}}$$

$$\boxed{\ell = r^2 \dot{\theta}}$$

$$\dot{\theta} = \frac{\ell}{r^2(\theta)}$$

$$\ell = r_0^2 \dot{\theta}$$

$$E = \frac{1}{2} m v_a^2 - \frac{k}{a(1+\epsilon)}$$

$$= \frac{1}{2} m v_p^2 - \frac{k}{a(1-\epsilon)}$$

$$= \frac{1}{2} m v^2 - \frac{k}{r}$$

$$\ddot{x}' = -\frac{g}{l} x' + 2\ \omega_z\ \dot{y}'$$

$$+)\ i\ddot{y}' = -\frac{g}{l} i y' - 2\ \omega_z i \dot{x}'$$

$$\ddot{x}' + i\ddot{y}' = -\frac{g}{l}\left[x' + iy'\right] - 2\omega_z i\left(\dot{x}' + i\dot{y}'\right)$$

$$x' + iy' \equiv Z$$

$$\boxed{\ddot{Z} = -\underbrace{\frac{g}{l}}_{\Omega^2} z - \underbrace{2\omega_z i \dot{Z}}}$$

$$\ddot{z} + \Omega^2 z + \underbrace{2\omega_z i}_{2\gamma}\ \dot{z} = 0$$

$$\downarrow$$

$$Z = Ae^{\alpha t} = A\, e^{(\cdots)t}$$

$$\alpha^2 + 2\omega_z i \alpha + \Omega^2 = 0$$

$$\alpha = \cdots$$