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2016 2학기 강의록. link.

7장. 입자계의 동역학

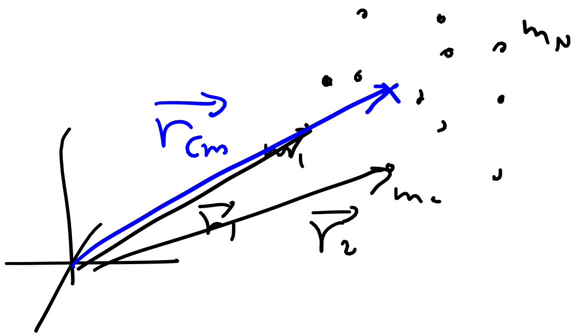
N개 $\vec{r}_a$ $a = 1, \cdots, N$ ⇒ 3N개의 좌표

$$m_a \ddot{\vec{r}}_a = \vec{F}_a(\vec{r}_1, \cdots, \vec{r}_N)$$ 3N 연립 2차 미분방정식

7.1. CM (질량 중심)

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N}{m_1 + m_2 + \cdots + m_N} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i}$$

$\vec{r}_{cm} = (x_{cm}, y_{cm}, z_{cm})$



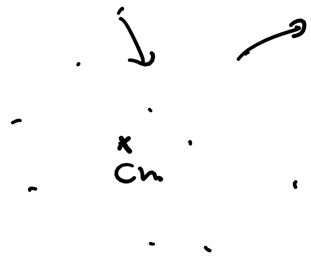
운동량 $\vec{p}_a = m_a \dot{\vec{r}}_a = m_a \vec{v}_a$

총 〃 $\sum_{a=1}^{N} \vec{p}_a = \vec{P} = \sum_a m_a \vec{v}_a = \sum_a m_a \frac{d\vec{r}_a}{dt} = \frac{d}{dt} \underbrace{\sum_a m_a \vec{r}_a}_{m \vec{r}_{cm}}$

시간에 무관

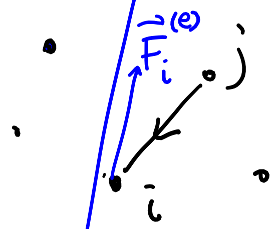
$m = \sum_{a=1}^{N} m_a$

$\therefore \vec{P} = m \frac{d\vec{r}_{cm}}{dt} = m \vec{v}_{cm}$



$\frac{d\vec{p}_i}{dt} = m_i \ddot{\vec{r}}_i = \vec{F}_i = \vec{F}_i^{(e)} + \sum_{j=1}^{N} \vec{F}_{ij}$

$(j = i \; ; \; \vec{F}_{ii} = 0)$



interaction (상호작용)

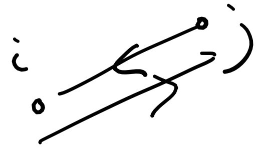
Newton 제 3 법칙 $\vec{F}_{ij} = -\vec{F}_{ji}$

$\left(j = i \quad \vec{F}_{ii} = -\vec{F}_{ii}\right)$

$\sum_{i=1}^{N} \frac{d}{dt} \vec{p}_i = \frac{d}{dt}\left(\sum_i \vec{p}_i\right) = \sum_{i=1}^{N} \left(\vec{F}_i^{(e)} + \sum_{j=1}^{N} \vec{F}_{ij}\right)$

$\sum_i \vec{p}_i = \vec{P}$

$$= \sum_{i=1}^{N} \vec{F}_i^{(e)} + \underbrace{\cancel{\sum_{i=1}^{N}\sum_{j=1}^{N}\vec{F}_{ij}}}_{=\,0}$$



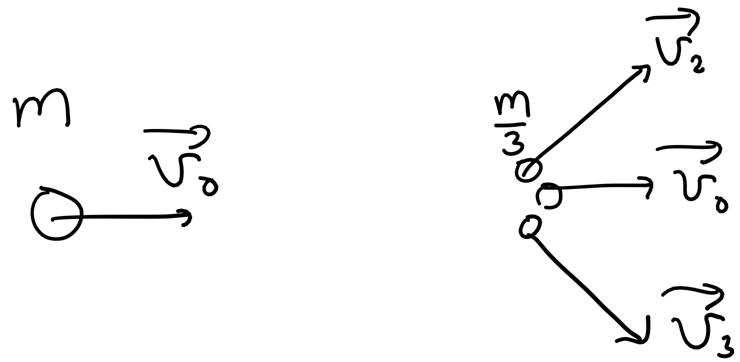
$$\sum_{i=1}^{N}\sum_{j=1}^{N}\vec{F}_{ij} = \frac{1}{2}\left(\underbrace{\sum_{i=1}^{N}\sum_{j=1}^{N}}\vec{F}_{ij} + \underbrace{\sum_{j=1}^{N}\sum_{i=1}^{N}}\vec{F}_{ji}\right)$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\Big(\underbrace{\vec{F}_{ij} + \vec{F}_{ji}}_{\vec{0}}\Big) = \vec{0}$$

$$\dot{\vec{p}} = \sum_{i=1}^{N}\vec{F}_i^{(e)} = \frac{d}{dt}\Big(m\underbrace{\vec{v}_{cm}}_{\vec{p}}\Big) = m\,\vec{a}_{cm}$$

(ex) $\vec{F}_i^{(e)} = \vec{g}\,m_i \rightarrow \sum_{i=1}^{N}(\vec{g}\,m_i) = \vec{g}\,\underbrace{\Big(\sum_{i=1}^{N} m_i\Big)} = m\vec{g} \Rightarrow \vec{a}_{cm} = \vec{g}$

[Ex 7.1.1.]



$|\vec{v}_2| = |\vec{v}_3|$ $\vec{v}_2 \perp \vec{v}_3$

외력 = 0 $\vec{p}$ = 일정.

깨지기 전 $\vec{p} = m\vec{v}_0$ $=$ 깨진 후 $\vec{p} = \frac{m}{3}\left(\frac{\vec{v}_0}{2} + \vec{v}_2 + \vec{v}_3\right)$

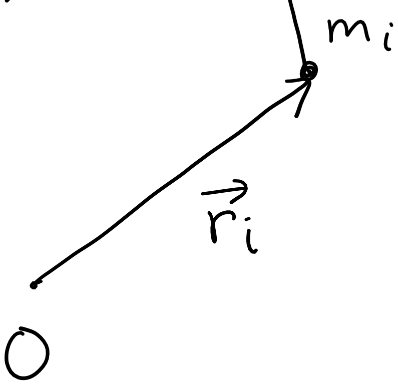
$\therefore \frac{5}{6}\cancel{m}\,\vec{v}_0 = \cancel{m}\,\frac{1}{3}(\vec{v}_2 + \vec{v}_3)$

제곱

$$\frac{25}{4}v_0^2 = v_2^2 + v_3^2 + 2\underbrace{\vec{v}_2\cdot\vec{v}_3}_{=0} = 2v_2^2$$

$$\therefore v_2 = \frac{5}{2\sqrt{2}}v_0 = v_3$$

7.2. 각운동량



$$\vec{L}_i = m_i \vec{r}_i \times \vec{v}_i$$

$$\vec{L} = \sum_{i=1}^{N} m_i \left(\vec{r}_i \times \vec{v}_i \right)$$

o

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{N} m_i \left(\underset{\vec{v}_i}{\dot{\vec{r}}_i} \times \vec{v}_i \right) + \sum_{i=1}^{N} m_i \left(\vec{r}_i \times \underset{\vec{a}_i}{\dot{\vec{v}}_i} \right)$$

$$= \sum_{i=1}^{N} \left(\vec{r}_i \times \underbrace{m_i \vec{a}_i}_{\vec{F}_i} \right) = \sum_{i=1}^{N} \vec{r}_i \times \left(\vec{F}_i^{(e)} + \sum_{j=1}^{N} \vec{F}_{ij} \right)$$

$$\vec{F}_i = \vec{F}_i^{(e)} + \sum_{j=1}^{N} \vec{F}_{ij}$$

$$= \sum_{i=1}^{N} \left[\underbrace{\vec{r}_i \times \vec{F}_i^{(e)}}_{\vec{N}_i^{(e)}} \right] + \underbrace{\sum_{i=1}^{N} \vec{r}_i \times \sum_{j=1}^{N} \vec{F}_{ij}}_{\sum_{i=1}^{N} \sum_{j=1}^{N} \vec{r}_i \times \vec{F}_{ij}} = 0 \text{ if } \vec{F}_{ij} \propto \vec{r}_{ij}$$

$$= \vec{N}^{(e)}$$

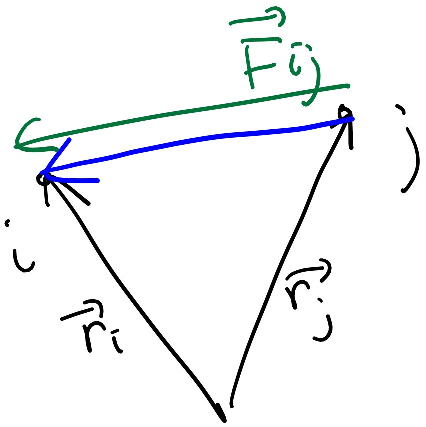
$$\therefore \boxed{\frac{d\vec{L}}{dt} = \vec{N}^{(e)}}$$

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \vec{r}_i \times \vec{F}_{ij} = \frac{1}{2}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} \vec{r}_i \times \vec{F}_{ij} + \sum_{j=1}^{N}\sum_{i=1}^{N} \vec{r}_j \times \underbrace{\vec{F}_{ji}}_{= -\vec{F}_{ij}} \right)$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij}$$

$\neq 0$ (일반적)

if $\vec{F}_{ij} \propto \vec{r}_i - \vec{r}_j \Rightarrow 0$



중심력

만약 $\vec{F}^{(e)} = 0 \Rightarrow \vec{p} =$ 일정.

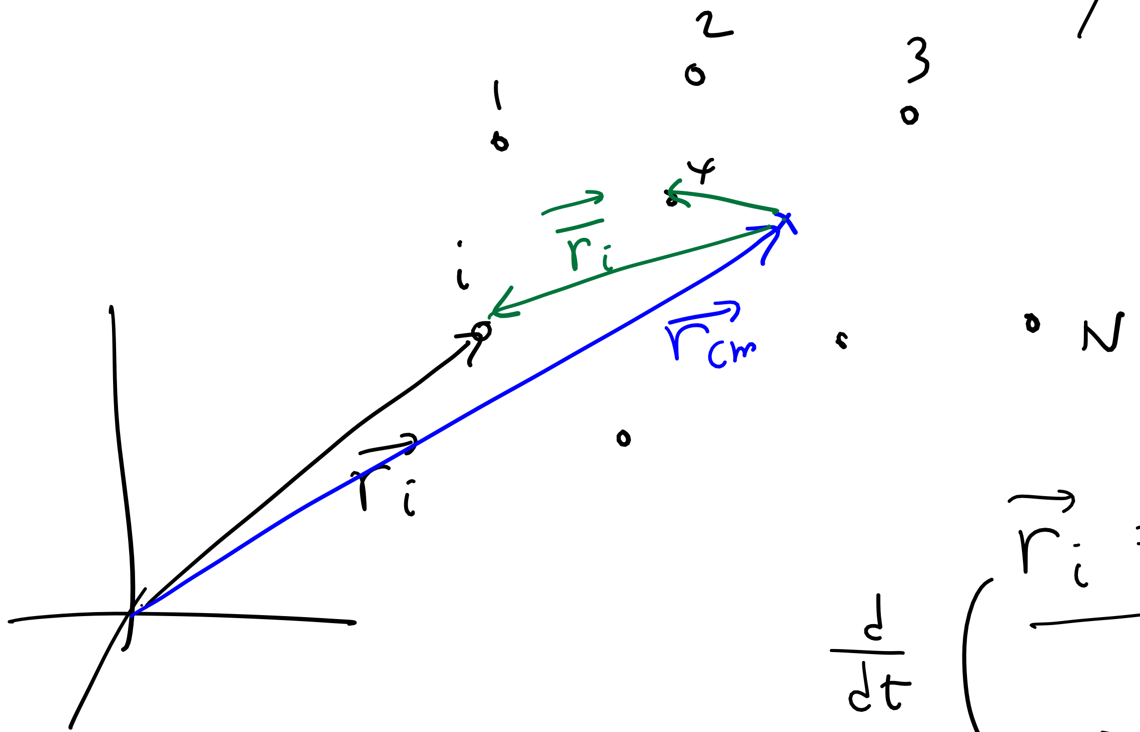
$\vec{F}^{(e)} = 0 \to \vec{N}^{(e)} = 0 \Rightarrow \vec{L} =$ 일정 // $\vec{F}_{ij} \propto \vec{r}_{ij}$

relative coord. (상대좌표)

$$\cancel{\vec{r}_{cm}} + \frac{\overbrace{\sum_i m_i \bar{\vec{r}}_i}^{=0}}{m} = \frac{1}{m}\sum_i m_i(\vec{r}_{cm} + \bar{\vec{r}}_i)$$

$$\parallel \qquad\qquad \parallel$$

$$\cancel{\vec{r}_{cm}} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$$



$$\vec{r}_i = \vec{r}_{cm} + \bar{\vec{r}}_i$$

$$\frac{d}{dt} \Big\downarrow$$

$$\vec{v}_i = \vec{v}_{cm} + \bar{\vec{v}}_i$$

$$\therefore \sum_i m_i \bar{\vec{r}}_i = 0 \xrightarrow{\frac{d}{dt}} \sum_i m_i \bar{\vec{v}}_i = 0$$

$$\vec{p}_i = m_i \vec{v}_i = m_i\left(\vec{v}_{cm} + \vec{v}\,'_i\right)$$

$$\vec{p} = \sum_i \vec{p}_i = \underbrace{\left(\sum_i m_i\right)}_{= m} \vec{v}_{cm} + \underbrace{\sum_i m_i \vec{v}\,'_i}_{= 0} = m\,\vec{v}_{cm}$$

$$\vec{L}_i = m_i\, \vec{r}_i \times \vec{v}_i = m_i\left(\vec{r}_{cm} + \vec{r}\,'_i\right) \times \left(\vec{v}_{cm} + \vec{v}\,'_i\right)$$

$$= m_i\left(\vec{r}_{cm} \times \vec{v}_{cm} + \vec{r}\,'_i \times \vec{v}_{cm} + \vec{r}_{cm} \times \vec{v}\,'_i + \vec{r}\,'_i \times \vec{v}\,'_i\right)$$

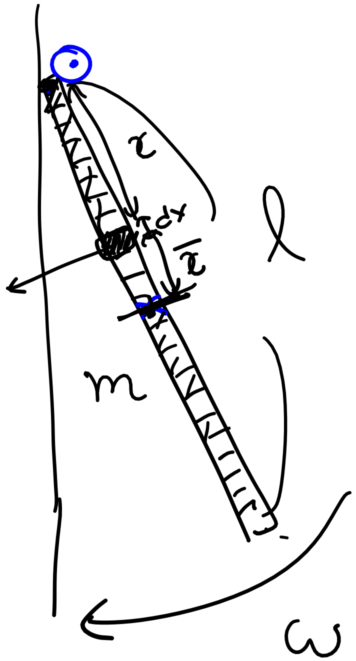
$$\vec{L} = \sum_i m_i\left(\vec{r}_{cm} \times \vec{v}_{cm} + \vec{r}\,'_i \times \vec{v}_{cm} + \vec{r}_{cm} \times \vec{v}\,'_i + \vec{r}\,'_i \times \vec{v}\,'_i\right)$$

$$= \sum_i m_i\left(\vec{r}_{cm} \times \vec{v}_{cm}\right) + \sum_i m_i \vec{r}\,'_i \times \vec{v}_{cm} + \sum_i m_i \vec{r}_{cm} \times \vec{v}\,'_i + \sum_i m_i \vec{r}\,'_i \times \vec{v}\,'_i$$

$$= \underbrace{m\, \vec{r}_{cm} \times \vec{v}_{cm}}_{= \vec{L}_{cm}} + \underbrace{\left(\sum_i m_i \vec{r}\,'_i\right)}_{= 0} \times \vec{v}_{cm} + \vec{r}_{cm} \times \underbrace{\sum_i m_i \vec{v}\,'_i}_{= 0} + \underbrace{\sum_i m_i \vec{r}\,'_i \times \vec{v}\,'_i}_{= \vec{L}_{rel}}$$

$$\boxed{\vec{L} = \vec{L}_{cm} + \vec{L}_{rel}}$$

[Ex 7.2.1]



① $dL = \underset{\rho\, dx}{\underset{\parallel}{dm}}\; x \cdot x\omega$

$\int dm = m = \rho \underbrace{\int dx}_{l}$

$\rho = \frac{m}{l}$

$dL = \frac{m}{l}\,\omega\, x^2\, dx$

$L = \int dL = \frac{m\omega}{l} \int_0^l x^2\, dx$

$L = \frac{1}{3} m \omega l^2 = I\omega$

② $L_{cm} = m \frac{l}{2} \frac{l}{2} \omega = \frac{1}{4} m \omega l^2$

$L_{cm} + L_{rel} = \frac{1}{3} m \omega l^2$

$dL_{rel} = dm\, \bar{x}\, \bar{x}\, \omega$, $dm = \rho\, d\bar{x}$

$L_{rel} = 2 \times \frac{m\omega}{l} \int_0^{\frac{l}{2}} \bar{x}^2\, d\bar{x} = \frac{2m\omega}{l} \frac{1}{3} \cdot \left(\frac{l}{2}\right)^3$

$= \frac{1}{12} m \omega l^2$

Kinetic Energy

$$T_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i \qquad \vec{v}_i = \vec{v}_{cm} + \vec{\bar{v}}_i$$

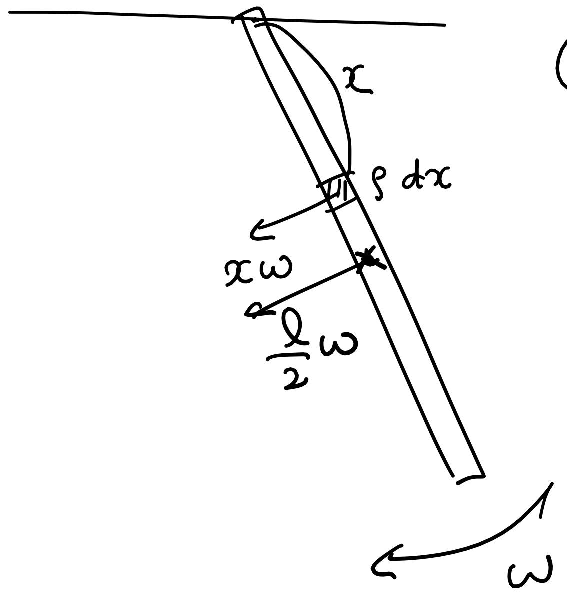
$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \vec{v}_i \cdot \vec{v}_i$$

$$= \frac{1}{2} \sum_{i=1}^{N} m_i \underbrace{\left(\vec{v}_{cm} + \vec{\bar{v}}_i\right) \cdot \left(\vec{v}_{cm} + \vec{\bar{v}}_i\right)}_{\vec{v}_{cm} \cdot \vec{v}_{cm} + 2\vec{v}_{cm} \cdot \vec{\bar{v}}_i + \vec{\bar{v}}_i \cdot \vec{\bar{v}}_i}$$

$$= \frac{1}{2} m v_{cm}^2 + \Big(\underbrace{\sum_{i=1}^{N} m_i \vec{\bar{v}}_i}_{=0}\Big) \cdot \vec{v}_{cm} + \frac{1}{2} \sum_{i=1}^{N} m_i \vec{\bar{v}}_i \cdot \vec{\bar{v}}_i$$

$$= T_{cm} + T_{rel}$$

[Ex 7.2.2.]



① $$dT = \frac{1}{2}\frac{m}{l}(x\omega)^2\, dx$$

$$T = \int dT = \frac{1}{2}\frac{m}{l}\omega^2 \int_0^l x^2\, dx$$

$$= \frac{1}{6} m\omega^2 l^2 \quad ✓$$

② $$T_{cm} = \frac{1}{2} m\left(\frac{l}{2}\omega\right)^2 = \frac{1}{8} m\omega^2 l^2$$

2개

$$T_{rel} = \frac{1}{2}\, 2\, \frac{m}{l}\omega^2 \int_0^{\frac{l}{2}} x^2\, dx = \frac{1}{2} 2 \cdot \frac{m}{l}\omega^2 \frac{1}{3}\left(\frac{l}{2}\right)^3$$

$$= \frac{1}{24} m\omega^2 l^2$$

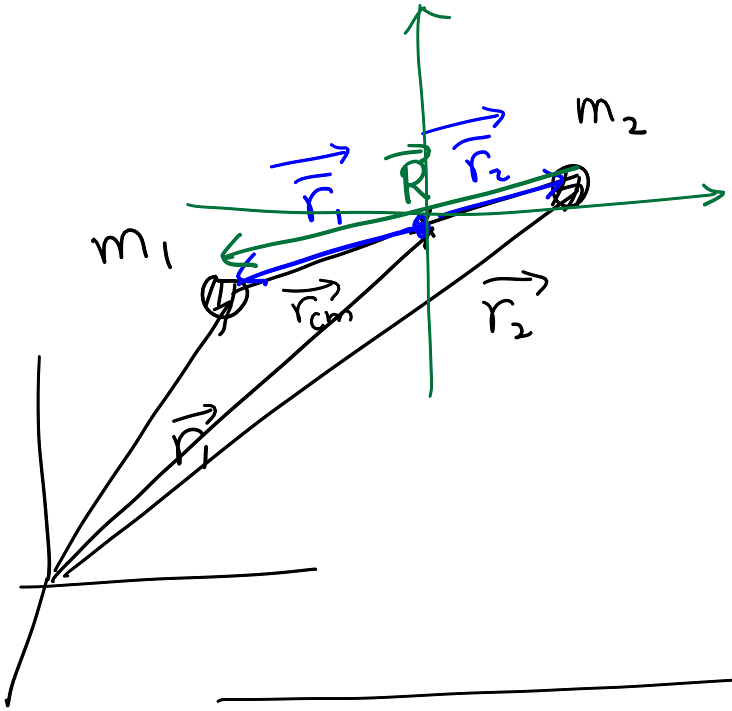
$$T_{cm} + T_{rel} = \frac{1}{6} m\omega^2 l^2$$

7.3. 2체

1학기 : 1체.

$$V(r) = -\frac{GMm}{r}$$

M.



$$\vec r_{cm} = \frac{m_1\vec r_1 + m_2\vec r_2}{m_1+m_2}$$

$$\begin{cases} \vec r_1 = \vec r_{cm} + \vec{\bar r}_1 \\ \vec r_2 = \vec r_{cm} + \vec{\bar r}_2 \end{cases} \Rightarrow m_1\vec{\bar r}_1 + m_2\vec{\bar r}_2 = 0$$

$$\vec r_1 - \vec r_2 = \vec{\bar r}_1 - \vec{\bar r}_2 = \vec R$$

$$m_1\vec{\bar r}_1 - m_1\vec{\bar r}_2 = m_1\vec R$$

$$\begin{cases} (m_1+m_2)\vec{\bar r}_2 = -m_1\vec R \\ \vec{\bar r}_1 = \vec R + \vec{\bar r}_2 = \end{cases}$$

$$\boxed{\begin{aligned} \vec{\bar r}_2 &= -\frac{m_1}{m_1+m_2}\vec R \\ \vec{\bar r}_1 &= \frac{m_2}{m_1+m_2}\vec R \end{aligned}}$$

$$\begin{aligned} m_1\ddot{\vec r}_1 &= \vec F_1 = m_1(\ddot{\vec r}_{cm} + \ddot{\vec{\bar r}}_1) = \cancel{\vec F_1^{(e)}} + \vec F_{12} \\ m_2\ddot{\vec r}_2 &= \vec F_2 = m_2(\ddot{\vec r}_{cm} + \ddot{\vec{\bar r}}_2) = \cancel{\vec F_2^{(e)}} + \vec F_{21} \end{aligned}$$

$$m_1\ddot{\vec{\bar r}}_1 + m_2\ddot{\vec{\bar r}}_2 = 0$$

$$\downarrow$$

$$(m_1+m_2)\ddot{\vec r}_{cm} = 0$$

$$\vec{r}_{cm} = \text{free motion} \longrightarrow \vec{r}_{cm} = 0$$

$\uparrow$ $\vec{r}_{cm}$ 을 원점.

$$m_1 \ddot{\vec{r}}_1 = \vec{F}_{12} = \underbrace{m_1 \frac{m_2}{m_1+m_2}}_{\mu\,(\text{환원질량})} \ddot{\vec{R}} = \vec{F}_{12}$$

$$m_2 \ddot{\vec{r}}_2 = \vec{F}_{21} = m_2 \left(-\frac{m_1}{m_1+m_2}\right) \ddot{\vec{R}} = \vec{F}_{21} = -\vec{F}_{12}$$

$$\therefore \quad \mu \ddot{\vec{R}} = \vec{F}_{12} \qquad (2\text{체} \longrightarrow 1\text{체})$$

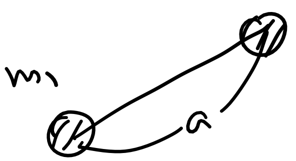
$$\frac{m_1 m_2}{m_1+m_2} \ddot{\vec{R}} = -\frac{G m_1 m_2}{R^2} \hat{R}$$

(ex) $M_\odot$

$$\left(\quad m_1 \ddot{\vec{R}} = -\frac{G\, m_1 (m_1 + \overset{M_\odot}{m_2})}{R^2} \hat{R} \right.$$

$$\left\{ \begin{aligned} \frac{M_J}{M_\odot} &= \frac{1}{1000} \\ \frac{M_E}{M_\odot} &\approx 3 \times 10^{-6} \end{aligned} \right.$$

$$\vec{r}_1 \longrightarrow \vec{R} \qquad m_2 \longrightarrow m_2 + m_1 \qquad (\text{ex})\ M_\odot \longrightarrow M_\odot + M_E \approx M_\odot$$



$$\tau \cong 2\pi \left(G M_\odot\right)^{-\frac{1}{2}} a^{\frac{3}{2}} \cong 2\pi \left(G(M_\odot + m)\right)^{-\frac{1}{2}} a^{\frac{3}{2}}$$

$$\tau = 2\pi \left(G(m_1+m_2)\right)^{-\frac{1}{2}} a^{\frac{3}{2}}$$

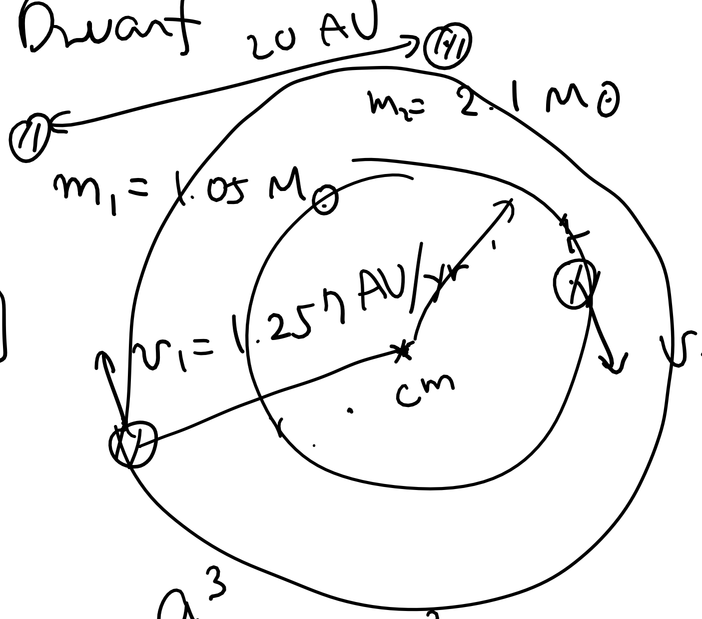
$$\tau = 2\pi\left(G(m_1+m_2)\right)^{-\frac{1}{2}} a^{\frac{3}{2}}$$

$$\div)\quad 1\,yr = 2\pi\left(G(M_\odot+\cancel{m_E})\right)^{-\frac{1}{2}} (1\,AU)^{\frac{3}{2}}$$

$$\underset{(\text{단위. }yr)}{\tau} = (\underset{M_\odot}{m_1} + \underset{}{m_2})^{-\frac{1}{2}}\ \underset{AU}{a^{\frac{3}{2}}} \Rightarrow \left(\frac{a^{\frac{3}{2}}}{\tau}\right)^2 = m_1+m_2$$

Sirius

White Dwarf



$$\tau = (1.05 + 2.1)^{-\frac{1}{2}}\ 20^{\frac{3}{2}} = 50\,yr.$$

$$r_1 = \frac{\tau \cdot v_1}{2\pi} = 1\,AU \qquad (\tau = 5,\ v_1 = 1.257)$$

[Ex. 7.3.1]

$$\tau_1 = \frac{2\pi r_1}{v_1} = \tau_2 = \frac{2\pi r_2}{v_2} = 5\,yr$$

$$r_2 = \frac{\tau\, v_2}{2\pi} = 4\,AU \qquad (v_2 = 5.027)$$

$m_1 = 4\,M_\odot$

$m_2 = 1\,M_\odot$

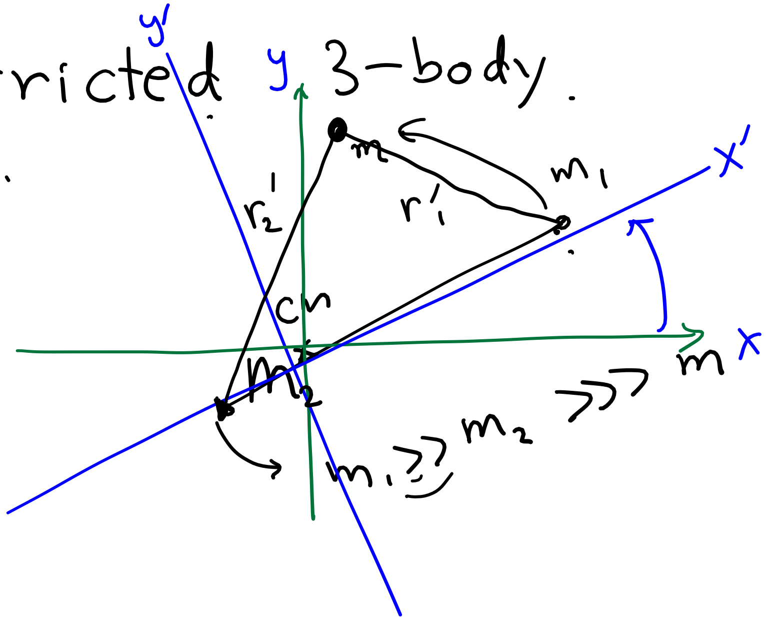
$$\begin{cases} m_1 + m_2 = \dfrac{a^3}{\tau^2} = \dfrac{5^3}{5^2} = 5\,M_\odot \\ m_1 = 4m_2 \end{cases}$$

$$a = 1 + 4 = 5\,AU$$

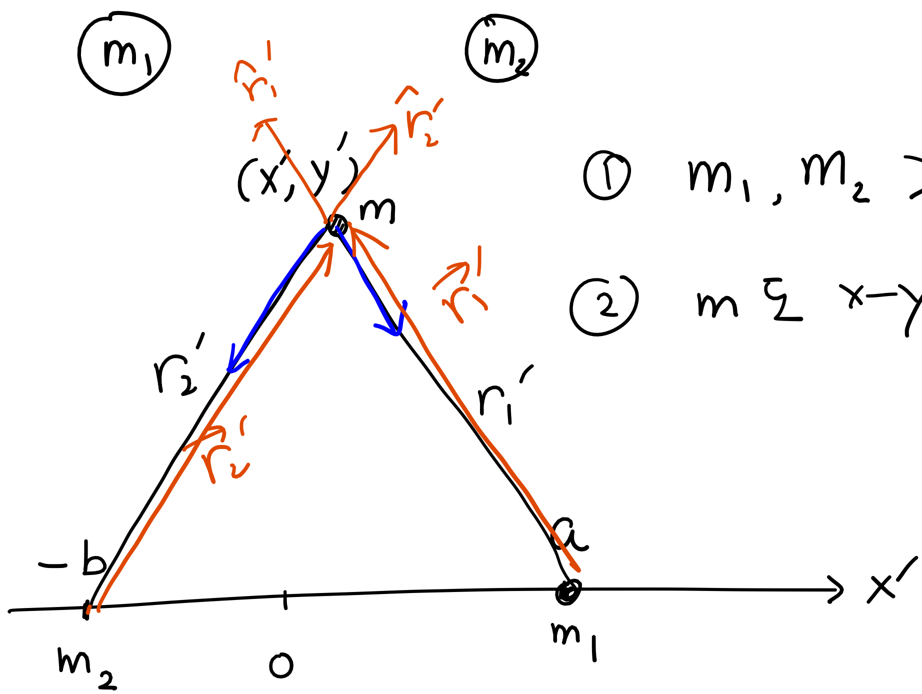
$$m_1\vec{v_1} + m_2\vec{v_2} = 0 \rightarrow \frac{|\vec{v_1}|}{|\vec{v_2}|} = \frac{m_2}{m_1} = \frac{1}{4}$$

7.4 3체문제 (Three-body problem)

restricted 3-body.



- $x'-y'$ 에서 m 의 운동



① $m_1, m_2 \gg m$

② m 도 $x-y$ 평면.

$$r_2' = \sqrt{(x'+b)^2 + y'^2}$$

$$r_1' = \sqrt{(x'-a)^2 + y'^2}$$

$$\vec{F} = -\hat{r}_1' \frac{Gm_1 m}{r_1'^2} - \hat{r}_2' \frac{Gm_2 m}{r_2'^2}$$

$$\hat{r}_1' = \frac{\vec{r}_1'}{r_1'} = \frac{1}{r_1'}(x'-a, y')$$

$$\hat{r}_2' = \frac{\vec{r}_2'}{r_2'} = \frac{1}{r_2'}(x'+b, y')$$

$$\vec{F}' = \vec{F} - 2\cancel{m}\,\vec{\omega}\times\vec{v}' - \cancel{m}\,\vec{\omega}\times(\vec{\omega}\times\vec{r}') = m\,\vec{a}' = \cancel{m}(\ddot{x}', \ddot{y}')$$

$$\vec{\omega} = \omega\,\vec{k}' \qquad 2\,\vec{\omega}\times\vec{v}' = 2\omega\,\vec{k}'\times(\dot{x}'\,\vec{i}' + \dot{y}'\,\vec{j}')$$

$$= 2\omega\,(\dot{x}'\,\vec{j}' - \dot{y}'\,\vec{i}')$$

$$-\vec{\omega}\times(\vec{\omega}\times\vec{r}') = -\omega\,\vec{k}'\times\big(\underbrace{\omega\,\vec{k}'\times(x'\,\vec{i}' + y'\,\vec{j}')}\big)$$

$$= -\omega^2\,\vec{k}'\times(x'\,\vec{j}' - y'\,\vec{i}') = \omega^2(x'\,\vec{i}' + y'\,\vec{j}')$$

$$(x'-a,\ y') \qquad\qquad (x'+b,\ y')$$

$$\vec{F} = -\vec{r}_1'\,\frac{Gm_1 m}{r_1'^3} - \vec{r}_2'\,\frac{Gm_2 m}{r_2'^3}$$

$x' = x(1) \quad y' = x(3)$

$\dot{x}' = x(2) \quad \dot{y}' = x(4)$

$$\boxed{\begin{aligned}
\ddot{x}' &= \underbrace{-\frac{Gm_1}{r_1'^3}(x'-a) - \frac{Gm_2}{r_2'^3}(x'+b) + \omega^2 x'}_{\vec{F}'_{x'}} + 2\omega\dot{y}' \\
\ddot{y}' &= \underbrace{-\frac{Gm_1}{r_1'^3}\,y' - \frac{Gm_2}{r_2'^3}\,y' + \omega^2 y'}_{\vec{F}'_{y'}} - 2\omega\dot{x}'
\end{aligned}}$$

$$\vec{F}'_{x'} = -\frac{\partial}{\partial x'} V(x',y') = \underbrace{-\frac{Gm_1}{r_1'^3}(x'-a) - \frac{Gm_2}{r_2'^3}(x'+b)} + \omega^2 x'$$

$$\vec{F}'_{y'} = -\frac{\partial}{\partial y'} V(x',y') = -\frac{Gm_1}{r_1'^3}\, y' - \frac{Gm_2}{r_2'^3}\, y' + \omega^2 y'$$

$$V(x',y') = -\frac{Gm_1}{r_1'} - \frac{Gm_2}{r_2'} - \frac{1}{2}\omega^2\left(x'^2 + y'^2\right)$$

$$r_2' = \sqrt{(x'+b)^2 + y'^2} \leftarrow$$

$$r_1' = \sqrt{(x'-a)^2 + y'^2} \leftarrow$$

$$-\frac{\partial V}{\partial x'} = -\frac{Gm_1}{r_1'^2}\underbrace{\frac{\partial r_1'}{\partial x'}}_{``\frac{x'-a}{r_1'}} - \frac{Gm_2}{r_2'^2}\underbrace{\frac{\partial r_2'}{\partial x'}}_{``\frac{x'+b}{r_2'}} + \omega^2 x' \quad ✓$$

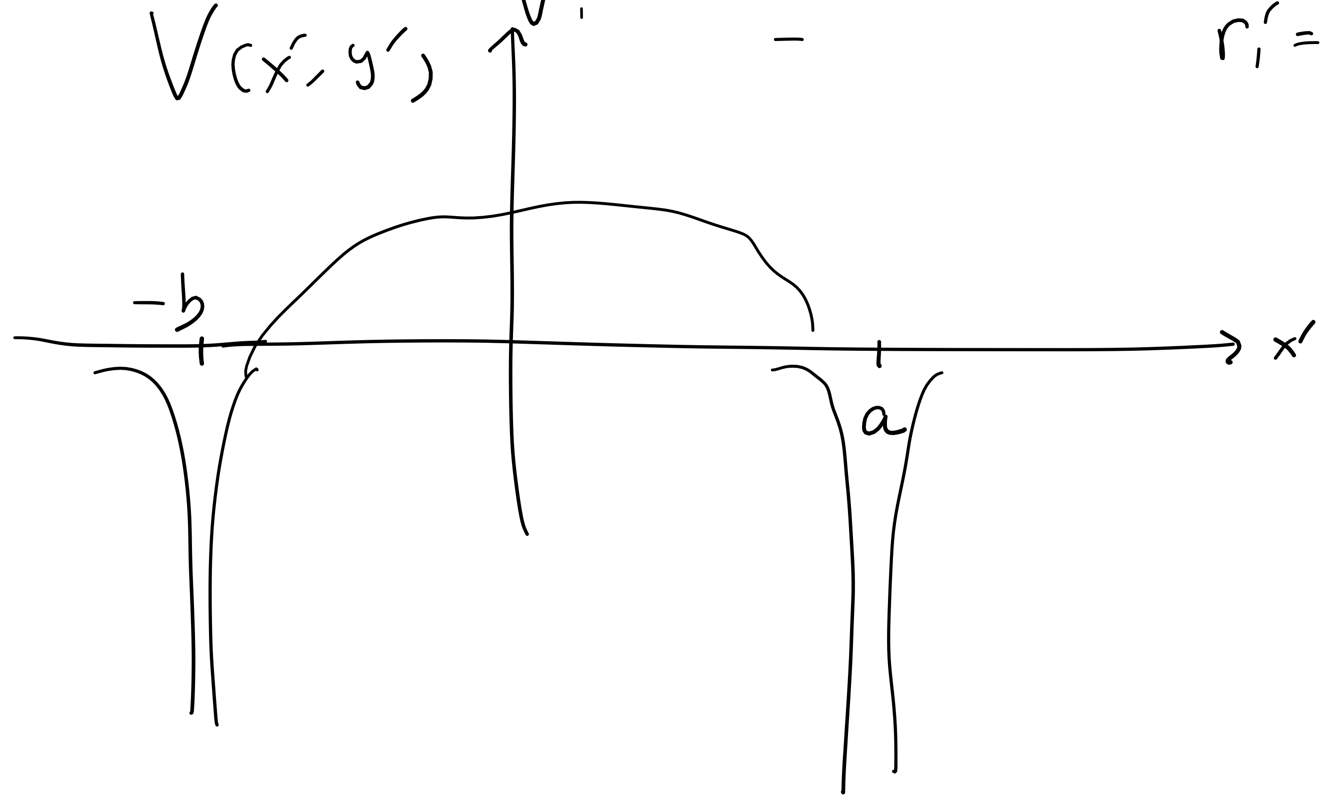
$$-\frac{\partial V}{\partial y'} = -\frac{Gm_1}{r_1'^2}\underbrace{\frac{\partial r_1'}{\partial y'}}_{``\frac{y'}{r_1'}} - \frac{Gm_2}{r_2'^2}\underbrace{\frac{\partial r_2'}{\partial y'}}_{``\frac{y'}{r_2'}} + \omega^2 y' \quad ✓$$

$$\vec{F}' = -\vec{\nabla}\underbrace{V(x',y')}_{V!} - \underbrace{2m\vec{\omega}\times\vec{v}'}_{-} = m\vec{a}'$$

$x' = a,\ y' = 0$

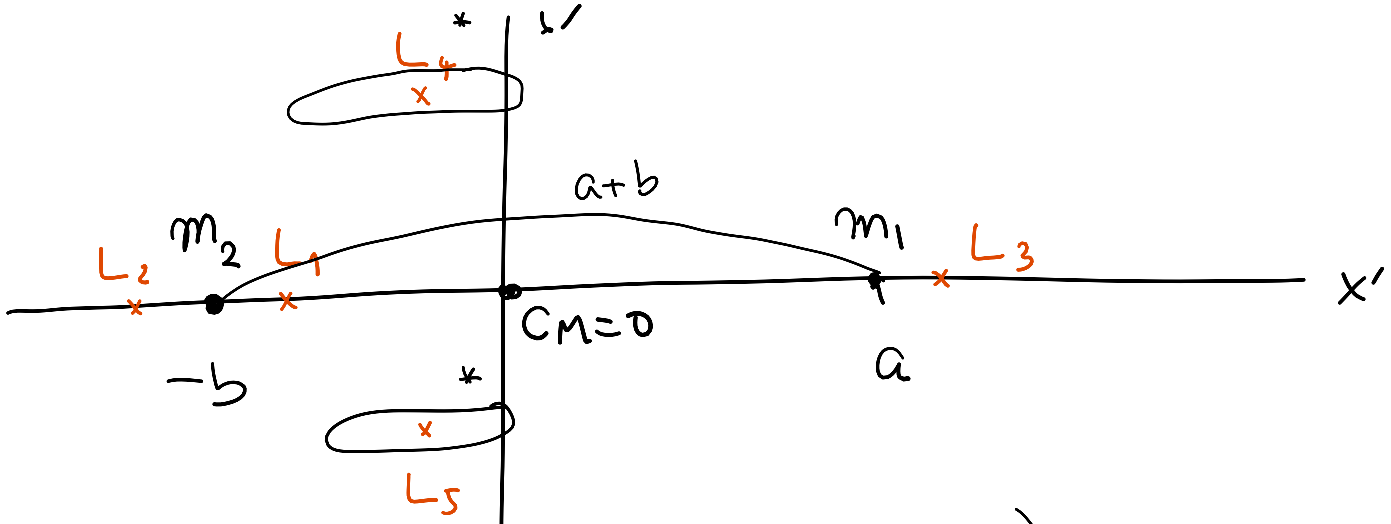
$r_1' = 0$

7.4.2

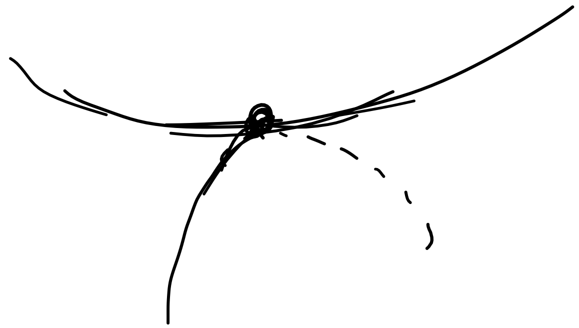


V flat $\rightarrow \dfrac{\partial V}{\partial x'} = \dfrac{\partial V}{\partial y'} = 0 \Rightarrow$ 5개 존재

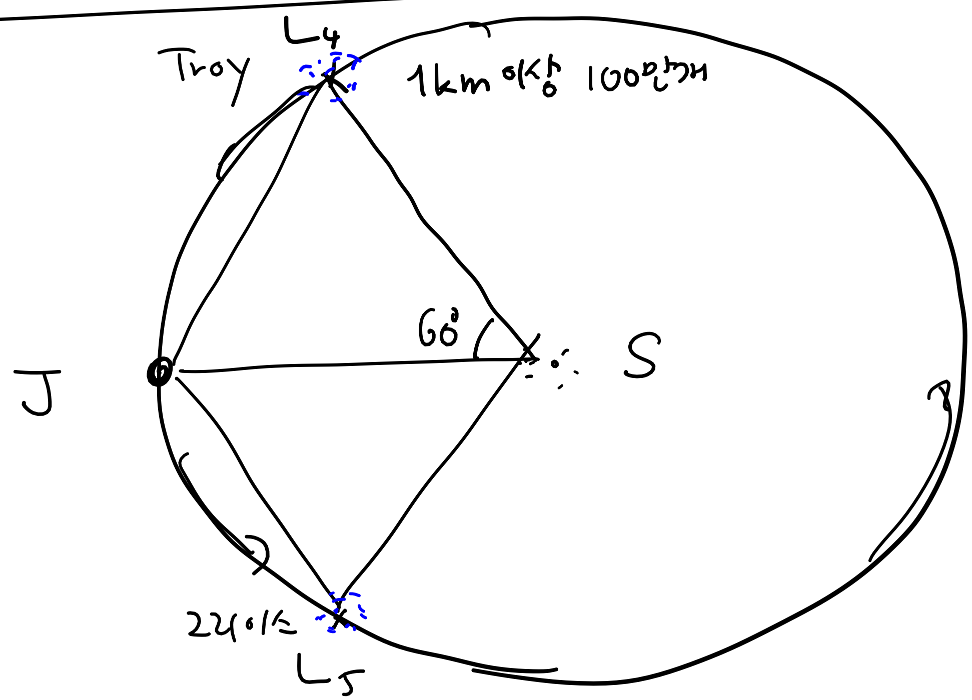
Lagrange points.



saddle points



Trojan Group



OE

Table 7.4.1 → 초기조건.

MatLab code (cyber)

Fig. 7.4.6 , 7.4.7 , 7.4.8

단위 선택 : 질량 거리 시간

$\underline{G(m_1+m_2)=1}$ $\qquad a+b=1$ $\qquad \tau=2\pi$

$\omega=\frac{2\pi}{\tau}=1$

$\frac{m_2}{m_1}=\frac{a}{b}$

$\Uparrow$

$m_1a-m_2b=0 \rightarrow \underline{m_1a=m_2b}$

$\frac{a}{a+b}\equiv\alpha \qquad \frac{b}{a+b}=1-\alpha \rightarrow \underline{a=\alpha}, \ \underline{b=1-\alpha}$

$(a+b=1 \text{ unit})$

$= \frac{\frac{m_2}{m_1}}{\frac{m_2}{m_1}+1}=\frac{m_2}{m_1+m_2}\approx\frac{m_2}{m_1}\approx 0.000953875$

$Gm_1=\frac{\cancel{G}m_1}{\cancel{G}(m_1+m_2)}=\frac{m_1}{m_1+m_2}=\frac{1}{1+\frac{m_2}{m_1}}$

$=1$

$=\frac{1}{1+\frac{a}{b}}=\frac{b}{a+b}=1-\alpha$

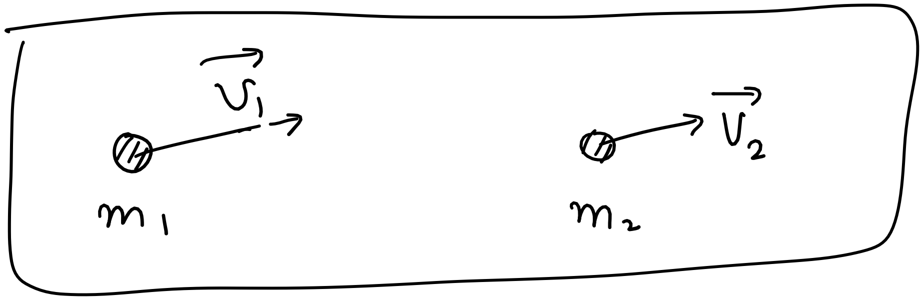
$Gm_2=\frac{Gm_2}{G(m_1+m_2)}=\frac{\frac{m_2}{m_1}}{1+\frac{m_2}{m_1}}=\frac{\frac{a}{b}}{1+\frac{a}{b}}=\frac{a}{a+b}=\alpha$

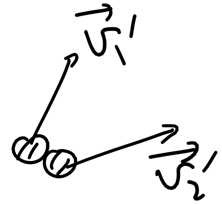
$$\boxed{\begin{aligned}\ddot{x}' &= -\frac{(1-\alpha)}{r_1'^3}(x'-\alpha) - \frac{\alpha}{r_2'^3}(x'+1-\alpha) + x' + 2\dot{y}' \\ \ddot{y}' &= -\frac{(1-\alpha)}{r_1'^3}\,y' - \frac{\alpha}{r_2'^3}\,y' + y' - 2\dot{x}'\end{aligned}}$$

$$r_2' = \sqrt{(x'+1-\alpha)^2 + y'^2} \leftarrow$$

$$r_1' = \sqrt{(x'-\alpha)^2 + y'^2} \leftarrow$$

7.5. (2체문제.) Collision





m_1, m_2는 유지.

총 운동량 보존 : $\vec{F}^{(e)} = 0$ $\quad \vec{F}_{12} + \vec{F}_{21} = 0$

항상 성립

총 운동에너지 보존 : 가끔. (성립되면 탄성충돌)

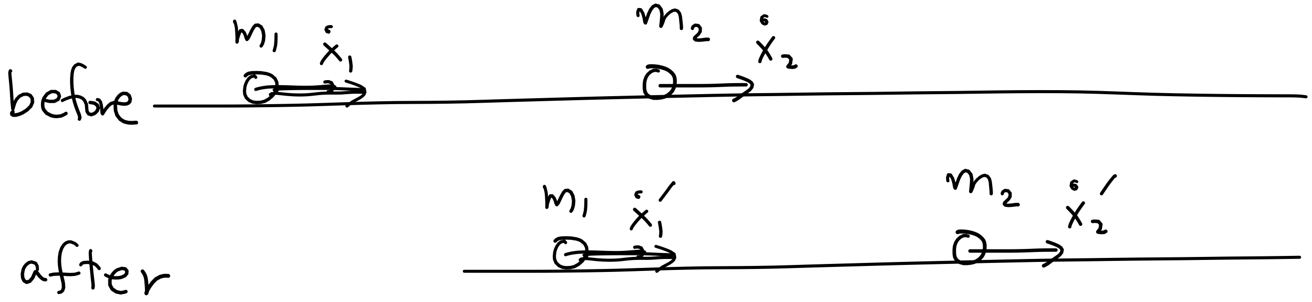
$$\vec{p}_1 + \vec{p}_2 = \vec{p}_1' + \vec{p}_2'$$

$$\vec{p}_i = m_i \vec{v}_i \qquad \vec{p}_i' = m_i \vec{v}_i' \quad , i=1,2$$

$$\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} = \frac{p_1'^2}{2m_1} + \frac{p_2'^2}{2m_2} + Q$$

Direct Collisions

1D



$$m_1 \dot{x}_1 + m_2 \dot{x}_2 = m_1 \dot{x}_1' + m_2 \dot{x}_2'$$

$$\epsilon \equiv \frac{\dot{x}_2' - \dot{x}_1'}{\dot{x}_1 - \dot{x}_2}$$

$$\left(\frac{\epsilon = 1 \quad \text{탄성} \rightarrow Q = 0}{\epsilon = 0} \right)$$

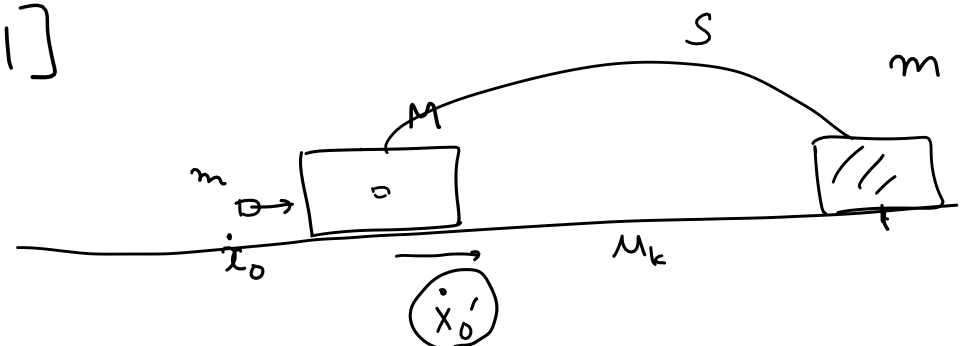
$$\dot{x}_1' = \frac{(m_1 - \epsilon m_2)\dot{x}_1 + (m_2 + \epsilon m_2)\dot{x}_2}{m_1 + m_2}$$

$$\dot{x}_2' = \frac{(m_1 + \epsilon m_1)\dot{x}_1 + (m_2 - \epsilon m_1)\dot{x}_2}{m_1 + m_2}$$

$$\frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 = \frac{1}{2} m_1 \dot{x}_1'^2 + \frac{1}{2} m_2 \dot{x}_2'^2 + Q$$

$$Q = \frac{1}{2} \mu v^2 (1 - \epsilon^2) \quad \begin{cases} \mu = \dfrac{m_1 m_2}{m_1 + m_2} \\ v = \dot{x}_1 - \dot{x}_2 \end{cases}$$

[Ex 7.5.1]



$$m \dot{x}_0 = (M+m) \dot{x}_0'$$

$$\dot{x}_0'^2 = 2 \overset{\mu_k g}{a} S \qquad (v^2 - v_0^2 = 2as)$$

$$F_f = \mu_k (M+m) g = (M+m) a$$

$$a = \mu_k g$$

$$\left(\frac{m}{M+m} \dot{x}_0\right)^2 = 2 \mu_k g S$$

7.6. 2차원 Oblique Collision (2체)

1, ② ← target

	LAB (fixed target)	CM
before collision	$\vec{v}_1 \ , \ \vec{v}_2 = 0$ $\vec{p}_1 = m_1\vec{v}_1 \ , \ \vec{p}_2 = m_2\vec{v}_2 = 0$	$\Leftrightarrow$ $\overline{\vec{v}}_1 , \overline{\vec{v}}_2$ $\overline{\vec{v}}_1 = \vec{v}_1 - \vec{v}_{cm} = \frac{m_2}{m_1+m_2}\vec{v}_1$ $\overline{\vec{v}}_2 = -\vec{v}_{cm} = -\frac{m_1}{m_1+m_2}\vec{v}_1$ $\overline{\vec{p}}_1 = m_1\overline{\vec{v}}_1 = \overbrace{\frac{m_1 m_2}{m_1+m_2}}^{\mu}\vec{v}_1$ $\overline{\vec{p}}_2 = m_2\overline{\vec{v}}_2 = -\frac{m_1 m_2}{m_1+m_2}\vec{v}_1$ $\overline{\vec{p}}_1 + \overline{\vec{p}}_2 = 0$
after collision	$\vec{v}_1{}' \ , \ \vec{v}_2{}'$ $\vec{p}_1{}' = m_1\vec{v}_1{}' \ , \ \vec{p}_2{}' = m_2\vec{v}_2{}'$	$\Leftrightarrow$ $\overline{\vec{v}}_1{}' \ , \ \overline{\vec{v}}_2{}'$ $\overline{\vec{p}}_1{}' = m_1\overline{\vec{v}}_1{}' \ , \ \overline{\vec{p}}_2 = m_2\overline{\vec{v}}_2{}'$ $\overline{\vec{v}}_1{}' = \vec{v}_1{}' - \frac{m_1\vec{v}_1}{m_1+m_2}$ $\overline{\vec{v}}_2{}' = \vec{v}_2{}' - \frac{m_1\vec{v}_1}{m_1+m_2}$

LAB

$$\vec{v}_{cm} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} = \underbrace{\frac{m_1 \vec{v}_1{}' + m_2 \vec{v}_2{}'}{m_1 + m_2}}_{} = \frac{m_1 \vec{v}_1}{m_1 + m_2}$$

$(\vec{v}_2 = 0)$

CM

$$\vec{\bar{v}}_i = \vec{v}_i - \vec{v}_{cm} \quad , \quad i = 1, 2$$

운동에너지

CM : $\underbrace{\dfrac{\bar{p}_1{}^2}{2m_1} + \dfrac{\bar{p}_2{}^2}{2m_2} = \dfrac{\bar{p}_1{}'^2}{2m_1} + \dfrac{\bar{p}_2{}'^2}{2m_2}}_{} + Q$

총운동량보존: $\vec{\bar{p}}_1 + \vec{\bar{p}}_2 = \vec{\bar{p}}_1{}' + \vec{\bar{p}}_2{}' = 0 \Rightarrow \vec{\bar{p}}_2 = -\vec{\bar{p}}_1 \ , \ \vec{\bar{p}}_2{}' = -\vec{\bar{p}}_1{}'$

$$\frac{\bar{p}_1{}^2}{2m_1} + \frac{\bar{p}_1{}^2}{2m_2} = \frac{\bar{p}_1{}^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \underbrace{\frac{\bar{p}_1{}^2}{2\mu}}_{}$$

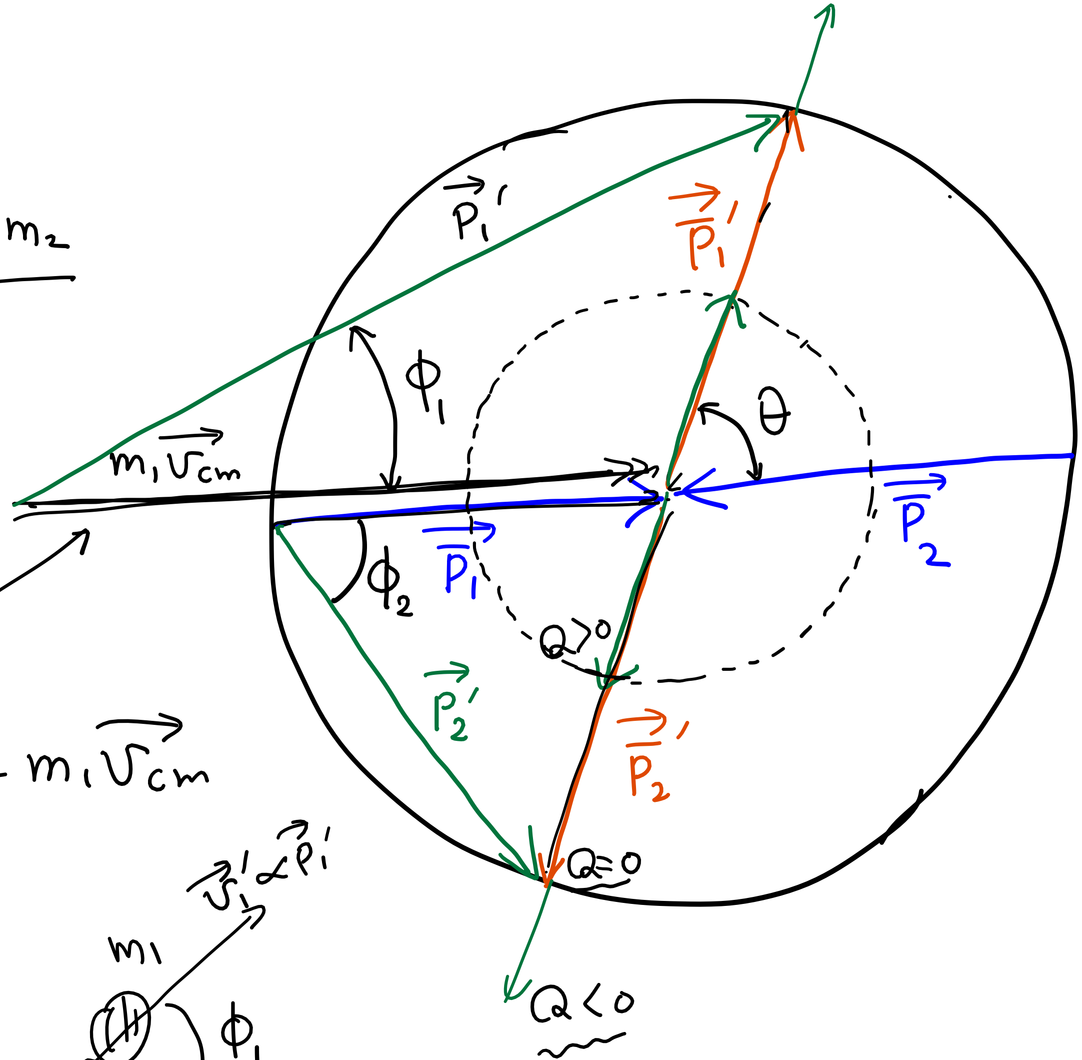
$$= \underbrace{\frac{\bar{p}_1{}'^2}{2\mu} + Q}_{}$$

$Q > 0 \qquad \bar{p}_1 > \bar{p}_1{}'$

$Q = 0 \qquad \bar{p}_1 = \bar{p}_1{}'$

$Q < 0 \qquad \bar{p}_1 < \bar{p}_1{}'$

$m_1 > m_2$



2 (운동량 보존)
1 (" ")
3개

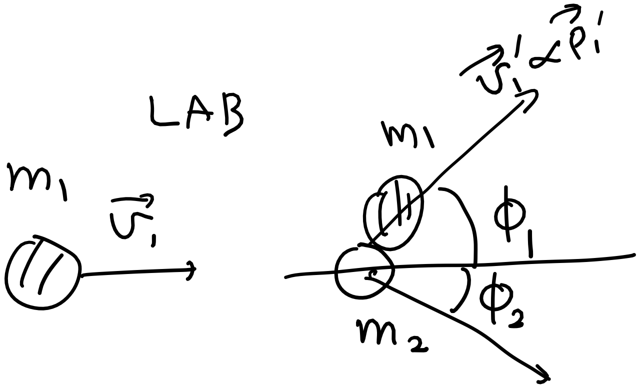
$E = E' + Q$

$\dfrac{\bar{p}_1^2}{2\mu} \qquad \dfrac{\bar{p}_1'^2}{2\mu}$

변수 2 + 2 = 4개

1개는 미정.

$$\vec{p}_1' = \vec{\bar{p}}_1' + m_1\vec{v}_{cm}$$



$$\vec{p}_2' = m_2\vec{v}_2' = m_2(\vec{\bar{v}}_2' + \vec{v}_{cm})$$
$$= \vec{\bar{p}}_2' + \underbrace{m_2\vec{v}_{cm}}_{=\frac{m_2 m_1}{m_1+m_2}\vec{v}_1 \,=\, m_1\vec{\bar{v}}_1 \,=\, \vec{\bar{p}}_1}$$

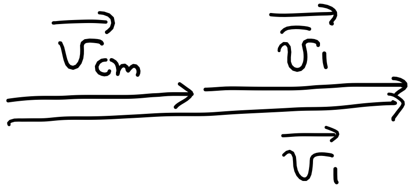
$$\vec{p}_1 = m_1\vec{v}_1$$

$$\vec{\bar{p}}_1 = m_1\vec{\bar{v}}_1$$
$$= m_1(\vec{v}_1 - \vec{v}_{cm})$$
$$= \vec{p}_1 - m_1\vec{v}_{cm}$$

$$\vec{v}_1 = \vec{\bar{v}}_1 + \vec{v}_{cm} = \vec{\bar{v}}_1 + \underbrace{\frac{m_1\vec{v}_1}{m_1+m_2}}_{\vec{v}_{cm}}$$

$$\vec{\bar{v}}_1 = \frac{m_2}{m_1+m_2}\vec{v}_1$$

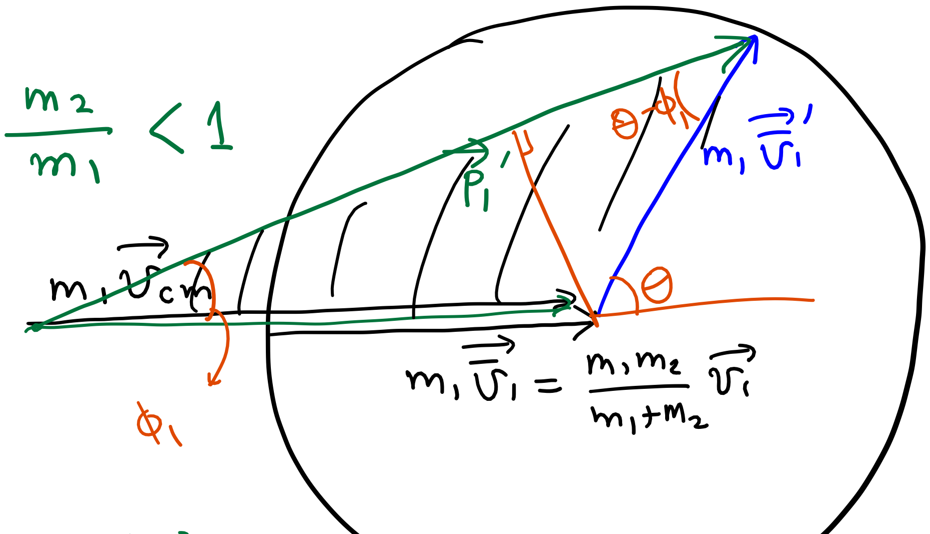
$$\frac{|\vec{\bar{v}}_1|}{|\vec{v}_{cm}|} = \frac{m_2}{m_1} < 1$$



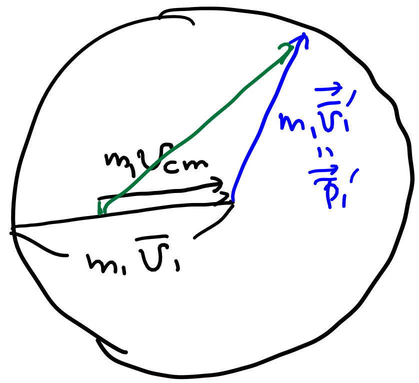
$$\vec{v}_1{}' = \vec{\bar{v}}_1{}' + \vec{v}_{cm}$$

$$m_1\vec{v}_1{}' = \vec{p}_1{}' = \underbrace{m_1\vec{\bar{v}}_1{}'}_{\vec{\bar{p}}_1{}'} + m_1\vec{v}_{cm}$$

$\frac{m_2}{m_1} < 1$



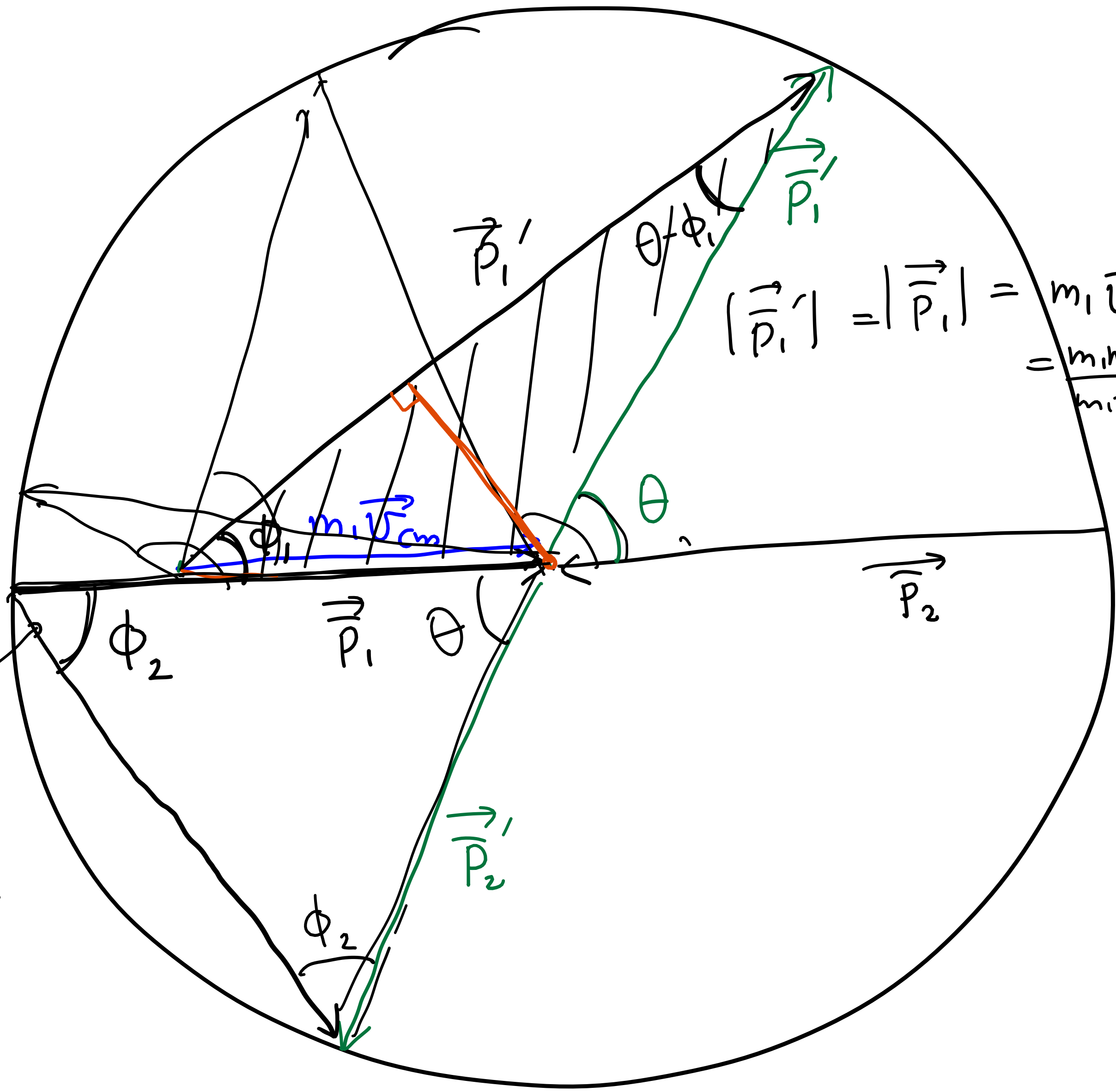
$m_1|\vec{v}_{cm}| > m_1\vec{\bar{v}}_1$



$m_1 v_{cm} < m_1\bar{v}_1$

$\rightarrow m_2 > m_1$

$Q = 0$



$$\vec{p}_1{}' = m_1\vec{v}_1{}'$$
$$= m_1(\vec{\bar v}_1{}' + \vec{v}_{cm})$$
$$= \vec{\bar p}_1{}' + m_1\vec{v}_{cm}$$

$$\vec{p}_2{}' = m_2\vec{v}_2{}'$$
$$= m_2(\vec{\bar v}_2{}' + \vec{v}_{cm})$$
$$= \vec{\bar p}_2{}' + \underbrace{m_2\vec{v}_{cm}}_{\vec{\bar p}_1}$$

$$|\vec{\bar p}_1{}'| = |\vec{\bar p}_1| = m_1\bar{v}_1 = \frac{m_1 m_2}{m_1+m_2}v_1$$

$$|\vec{\bar p}_1| = \frac{m_1 m_2}{m_1+m_2}v_1$$

$$|m_1\vec{v}_{cm}| = \frac{m_1^2}{m_1+m_2}v_1$$

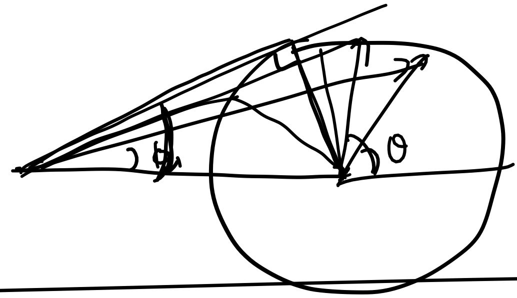
$m_2 > m_1$

$$2\phi_2 + \theta = \pi$$
$$\underline{\phi_2 = \frac{1}{2}(\pi - \theta)}$$

$$m_1 \frac{\cancel{m_1 v_1}}{\cancel{m_1+m_2}} \sin\phi_1 = \frac{\cancel{m_1} m_2}{\cancel{m_1+m_2}} \cancel{v_1} \sin(\theta-\phi_1)$$

$$\boxed{m_1 \sin\phi_1 = m_2 \sin(\theta-\phi_1)}$$

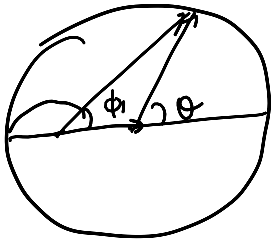
$m_1 > m_2$



$\Leftarrow$ 최대각 ϕ_1가 존재

$0 \leq \phi_1 \leq \sin^{-1}\left(\frac{m_2}{m_1}\right)$

$m_2 > m_1$



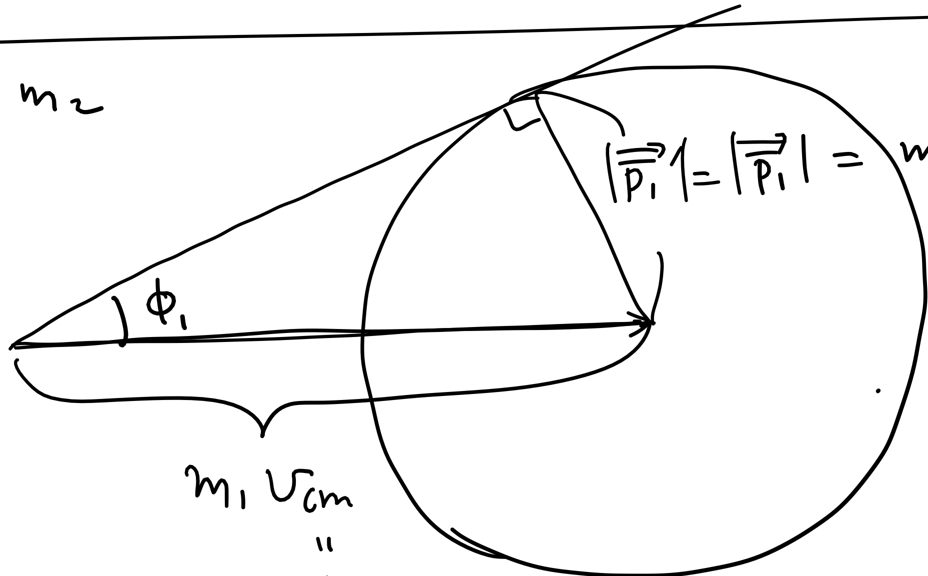
$0 \leq \phi_1 \leq \pi$

$m_1 \gg m_2$

$\phi_1 \ll 1$

$Q=0$

$m_1 > m_2$



$|\vec{p}_1'| = |\vec{p}_1| = m_1 |\vec{v}_1'|$

$= m_1 \frac{m_2}{m_1+m_2} v_1$

$m_1 v_{cm} = \frac{m_1}{m_1+m_2} v_1$

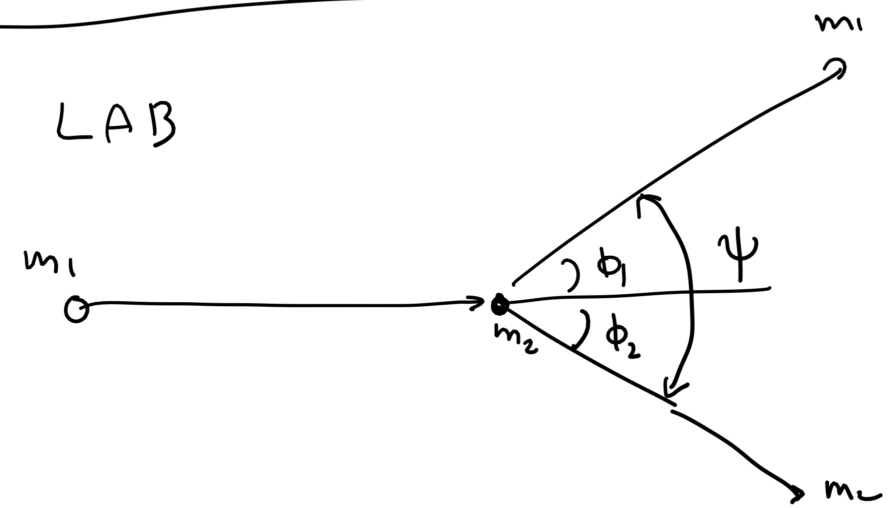
$\sin\phi_1 = \frac{m_2}{m_1}$

$$\theta - \phi_1 = \sin^{-1}\left(\frac{m_1}{m_2}\sin\phi_1\right)$$

$$\phi_2 = \frac{1}{2}(\pi - \theta)$$

$$m_1 \sin\phi_1 = m_2 \sin(\theta - \phi_1)$$

LAB

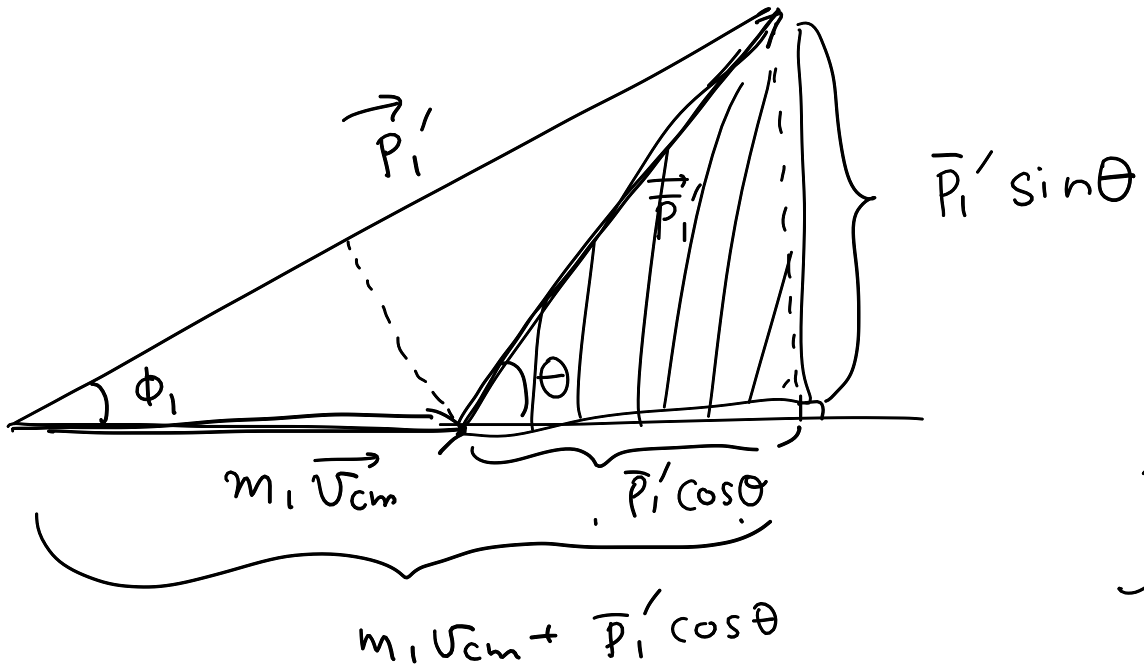


$$\psi = \phi_1 + \phi_2$$

if $m_1 = m_2 \rightarrow \sin\phi_1 = \sin(\theta - \phi_1) \longrightarrow \phi_1 = \theta - \phi_1$

$$\therefore \phi_1 = \frac{\theta}{2}$$

$$\psi = \frac{\theta}{2} + \frac{1}{2}(\pi - \theta) = \frac{\pi}{2} = 90^\circ$$



$$\tan\phi_1 = \frac{\sin\theta}{\gamma + \cos\theta}$$

$$\Uparrow$$

$$\tan\phi_1 = \frac{\bar{p}_1' \sin\theta}{m_1 v_{cm} + \bar{p}_1' \cos\theta} = \frac{\sin\theta}{\underbrace{\frac{m_1 v_{cm}}{\bar{p}_1'}}_{\gamma \equiv} + \cos\theta}$$

$$\gamma \equiv \frac{m_1 v_{cm}}{\bar{p}_1'} \overset{Q=0}{=} \frac{m_1 v_{cm}}{\bar{p}_1} = \frac{m_1 v_{cm}}{m_1 \bar{v}_1} = \frac{\frac{m_1^2}{m_1+m_2} v_1}{\frac{m_1 m_2}{m_1+m_2} v_1} = \frac{m_1}{m_2}$$

$$\overset{Q \neq 0}{\neq} \frac{m_1 v_{cm}}{\bar{p}_1}$$

$Q > 0 \quad \bar{p}_1' < \bar{p}_1 \rightarrow \gamma > \frac{m_1}{m_2}$

$Q < 0 \quad \bar{p}_1' > \bar{p}_1 \rightarrow \gamma < \frac{m_1}{m_2}$

$Q=0 \quad \gamma = \frac{m_1}{m_2}$ & if $m_1 = m_2 \quad \gamma = 1$

$$\tan\phi_1 = \frac{\sin\theta}{1+\cos\theta} = \frac{\cancel{2}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}{\cancel{2}\cos^{\cancel{2}}\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$$\therefore \quad \phi_1 = \frac{\theta}{2}$$

$$\gamma^2 = \frac{m_1^2 v_{cm}^2}{\bar{p}_1'^2} = \frac{\left(m_1 \frac{m_1 v_1}{m_1+m_2}\right)^2}{\frac{\bar{p}_1'^2}{2\mu}\cdot 2\mu} = \frac{\left(\frac{m_1}{m_2}\mu v_1\right)^2}{2\mu\left(\frac{\bar{p}_1^2}{2\mu} - Q\right)}$$

$\frac{m_1 v_1}{m_1+m_2}$

$''$

$$\bar{p}_1 = m_1 \bar{v}_1 = m_1(v_1 - v_{cm})$$
$$= \mu v_1$$

$$= \frac{\left(\frac{m_1}{m_2}\right)^2 \mu^2 v_1^2}{2\mu\left(\frac{1}{2}\mu \underbrace{v_1^2}_{\frac{2T}{m_1}} - Q\right)} \qquad v_1^2 \leftarrow \frac{2T}{m_1}$$

$$T = \frac{1}{2}m_1 v_1^2 + 0$$

$$v_1^2 = \frac{2T}{m_1}$$

$$\mu = \frac{m_1 m_2}{m_1+m_2}$$

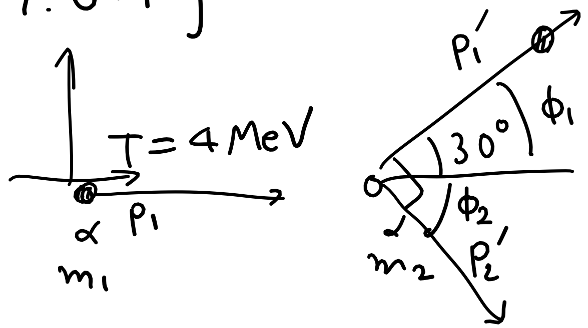
$$\frac{m_1}{\mu} = \frac{m_1+m_2}{m_2} = 1 + \frac{m_1}{m_2}$$

$$= \frac{\left(\frac{m_1}{m_2}\right)^2 \mu^2 \frac{2}{m_1}}{2\mu\left(\underset{\sim}{\frac{\mu}{m_1}} - \frac{Q}{T}\right)} = \frac{\left(\frac{m_1}{m_2}\right)^2 \cancel{\mu^2} \frac{\cancel{2}}{\cancel{m_1}}}{\cancel{2}\cancel{\mu}\left(\frac{\cancel{\mu}}{\cancel{m_1}}\right)\left(1 - \frac{Q}{T}\underset{\sim}{\frac{m_1}{\mu}}\right)}$$

$'' \left(1+\frac{m_1}{m_2}\right)$

$$\gamma = \frac{\left(\frac{m_1}{m_2}\right)}{\sqrt{\left(1 - \frac{Q}{T}\left(1+\frac{m_1}{m_2}\right)\right)}} \rightarrow \tan\phi_1 = \frac{\sin\theta}{\gamma + \cos\theta}$$

[Ex 7.6.1]



$$T_1' + T_2' = T = 4$$

$$m_1 = m_2 \rightarrow \underset{30^\circ}{\underset{\parallel}{\phi_1}} + \phi_2 = 90^\circ \rightarrow \phi_2 = 60^\circ$$

$$p_1 = p_1' \overbrace{\cos\phi_1}^{\frac{\sqrt{3}}{2}} + p_2' \cos\overset{\frac{1}{2}}{\phi_2}$$

$$0 = p_1' \underbrace{\sin\phi_1}_{\frac{1}{2}} - p_2' \underbrace{\sin\phi_2}_{\frac{\sqrt{3}}{2}}$$

$$\Rightarrow \begin{array}{l} p_1' = \frac{\sqrt{3}}{2} p_1 \rightarrow p_1'^2 = \frac{3}{4} p_1^2 \\ p_2' = \frac{1}{2} p_1 \rightarrow p_2'^2 = \frac{1}{4} p_1^2 \end{array}$$

$$T_1' = 3 \underset{1\,\mathrm{MeV}}{\underset{\parallel}{T_2'}} = 3\,\mathrm{MeV}$$

[Ex 7.6.2] 60°

[Ex 7.6.3] $\psi = \phi_1 + \phi_2 = \phi_1 + \frac{1}{2}(\pi - \theta)$

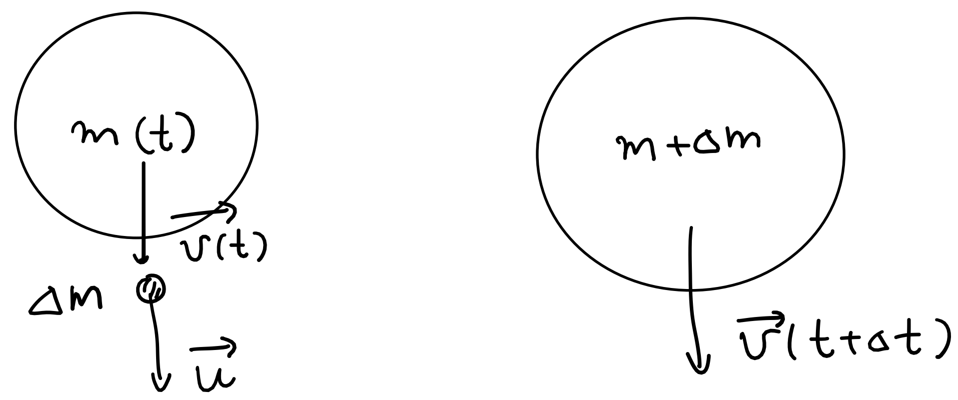
$\theta = \phi_1 + \sin^{-1}\left(\frac{m_1}{m_2}\sin\phi_1\right)$

$= \frac{\phi_1}{2} + \frac{\pi}{2} - \frac{1}{2}\sin^{-1}\left(\frac{m_1}{m_2}\sin\phi_1\right)$

$\phi_1 = 30^\circ \quad \frac{m_1}{m_2} = \frac{1}{4}$

$\psi = \frac{\pi}{12} + \frac{\pi}{2} - \frac{1}{2}\sin^{-1}\left(\frac{1}{8}\right) \approx 101^\circ$

7.7. mass가 변할 때



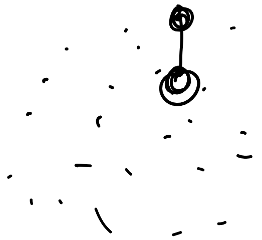
at t at $t+\Delta t$

$$\Delta\vec p = (m+\Delta m)\underbrace{\vec v(t+\Delta t)}_{\vec v(t)+\Delta\vec v} - \left(m\vec v(t)+\Delta m\vec u\right)$$

$$= \cancel{m\vec v(t)} + \Delta m\vec v + m\Delta\vec v - \cancel{m\vec v} - \Delta m\vec u$$

$$\therefore \vec F^{(e)} = \frac{\Delta\vec p}{\Delta t} = m\frac{\Delta\vec v}{\Delta t} - \frac{\Delta m}{\Delta t}\big(\underbrace{\vec u-\vec v}_{=\vec V}\big) \Rightarrow \boxed{m\dot{\vec v} - \dot m\vec V = \vec F^{(e)}}$$

(ex)



$\vec{u}=0$

$$\vec{V} = \vec{u} - \vec{v} = -\vec{v}$$

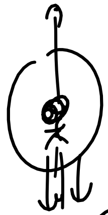
$$m\dot{\vec{v}} + \dot{m}\vec{v} = m\vec{g}$$

$$\| $$

$$\frac{d}{dt}(m\vec{v})$$

(ex)

$$\vec{V} = \vec{u} - \vec{v} = \text{일정}$$



$$\vec{F}^{(e)} = 0$$

$$\dot{\vec{v}} = \frac{\dot{m}}{m}\vec{V} \qquad \int \dot{\vec{v}}\,dt = \int \frac{\dot{m}}{m}\vec{V}\,dt$$

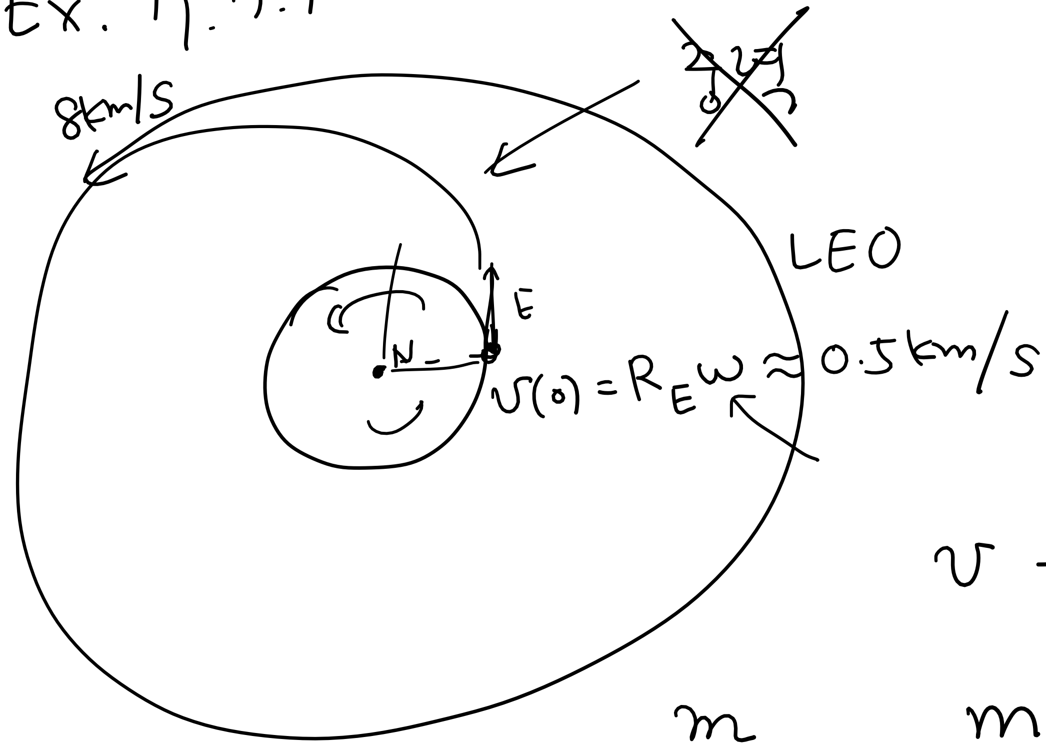
$$dt\frac{\dot{m}}{m} = \frac{\frac{dm}{\cancel{dt}}\cancel{dt}}{m} = \frac{dm}{m}$$

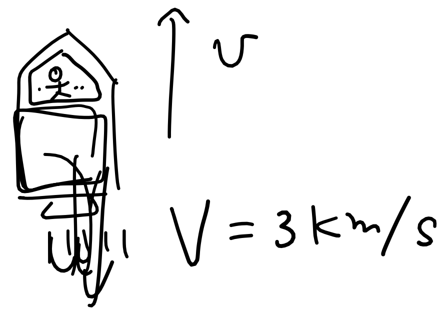
$$\|$$

$$\vec{v}(t) - \vec{v}(0) = \vec{V}\log m\Big|_{m(0)}^{m(t)}$$

$$\therefore \vec{v}(t) = \vec{v}(0) + \vec{V}\log\frac{m(t)}{m(0)}$$

Ex. 7.7.1





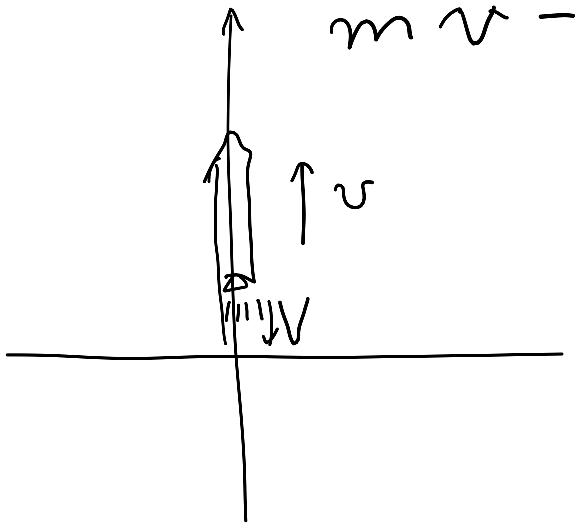
$$v - v(0) = -V \ln \frac{m}{m(0)}$$

$$\frac{m}{m+m_F} = \frac{m}{m(0)} = e^{-\frac{v - v(0)}{V}} = e^{-2.5}$$

$$m(0) = m + m_F$$

$$\frac{m_F}{m} = \underbrace{e^{2.5}}_{12.2} - 1 \approx 11.2$$

중력이 있으면



$$m\dot{\vec{v}} - \vec{V}\dot{m} = \vec{F}^{(e)} = m\vec{g}$$

$$m\underbrace{\dot{v}}_{\frac{dv}{dt}} + V\underbrace{\dot{m}}_{\frac{dm}{dt}} = -m\underset{\sim}{g}$$

$$\times\frac{dt}{mV}:\quad \frac{dv}{V} + \frac{dm}{m} = -\frac{g}{\underbrace{V}} dt = -\frac{dt}{\tau_s}$$

$$\frac{V}{g} = \tau_s \quad V \leftarrow 3000\ \mathrm{m/s}$$

τ_s = specific impulse = 300 s

$$\Rightarrow \text{적분} \quad \frac{1}{V}\int_0^{v_f} dv = -\int_{m_0}^{m_f}\frac{dm}{m} - \frac{1}{\tau_s}\int_0^{\tau_B} dt$$

$\tau_B = 600$ s

$$1 + \frac{\overbrace{m_F}^{105}}{m_R + m_P} \qquad \frac{v_f}{V} = \underbrace{-\ln\frac{m_f}{m_0}}_{=\ln\left(\frac{m_0}{m_f}\right)\ \mu} - \frac{\tau_B}{\tau_s}$$

$v_f = 8$ km/s

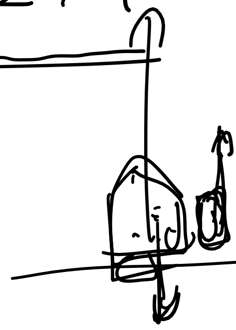
$V = 3$ km/s

$$\underbrace{\frac{m_R + m_P + m_F}{m_R + m_P}}_{\mu} = \frac{m_0}{m_f} = e^{\underset{=\frac{8}{3}}{\frac{v_f}{V}} + \underset{=2}{\frac{\tau_B}{\tau_s}}} = e^{\frac{14}{3}} \approx 106$$

m_P ← payload

Saturn V $\quad 32 : 1 = m_F : m_R + m_p$

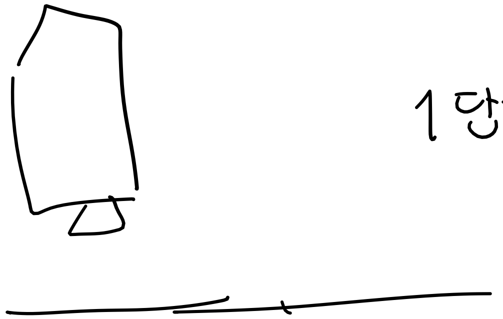
2단계 로켓



$v - V$

2단계 μ

$$\frac{m_R + m_p + m_F}{m_R + m_p} \equiv \mu$$



1단계 μ

$$\frac{1}{V}\int_0^{v_{f1}} dv = \ln\mu - \frac{\tau_B}{\tau_S} = \frac{v_{f1}}{V} \qquad \text{1단계}$$

$$+\quad \frac{1}{V}\int_{v_{f1}}^{v_{f2}} dv = \ln\mu - \frac{\tau_B}{\tau_S} = \frac{v_{f2} - v_{f1}}{V}$$

$$\frac{m_F}{m_R+m_p} \approx 27 \Leftarrow 2\ln\mu - \frac{2\tau_B}{\tau_S} = \frac{v_{f2}}{V} \rightarrow \underbrace{\mu}_{=\,1+\frac{m_F}{m_R+m_p}} = e^{\frac{v_{f2}}{2V} + \frac{\tau_B}{\tau_S}}$$

$$\frac{v_{f2}}{2V} = \frac{8\,\mathrm{km/s}}{2\cdot 3\,\mathrm{km/s}}, \qquad \frac{\tau_B}{\tau_S} = 2$$

$$= e^{\frac{10}{3}} \approx 28$$