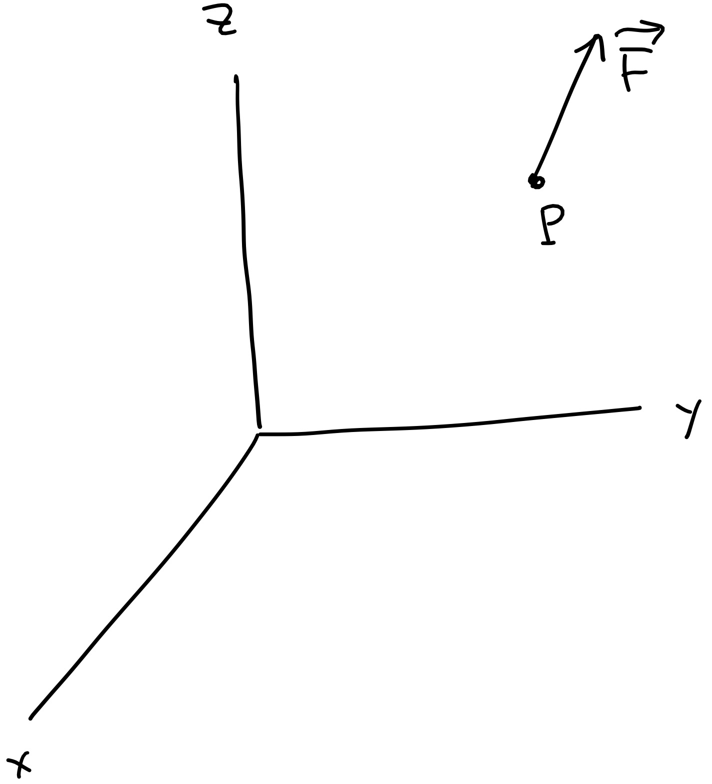
# Chap 4. 3차원 (2차원) 운동

$\vec{r}$ , $\vec{v}$ , $\vec{p} = m\vec{v}$ $\rightarrow$ $\vec{F} = \dfrac{d\vec{p}}{dt}$ (Newton 방정식)

$(F_x, F_y, F_z)$ $= m\dfrac{d^2\vec{r}}{dt^2}$ $\rightarrow$

$$\underbrace{\begin{aligned} m\ddot{x} &= F_x(x, y, z) \\ m\ddot{y} &= F_y(\,\text{''}\,) \\ m\ddot{z} &= F_z(\,\text{''}\,) \end{aligned}}$$




$\vec{F}$ 의 조건.

$\downarrow$

보존력 :

$T + V$ 가 일정한 힘

$$\vec{F}\cdot\vec{v} = \frac{d\vec{p}}{dt}\cdot\vec{v} = m\,\frac{d\vec{v}}{dt}\cdot\vec{v} = m\cdot\frac{1}{2}\,\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{d}{dt}\left(\frac{1}{2}mv^2\right)$$

$$= \frac{dT}{dt}$$

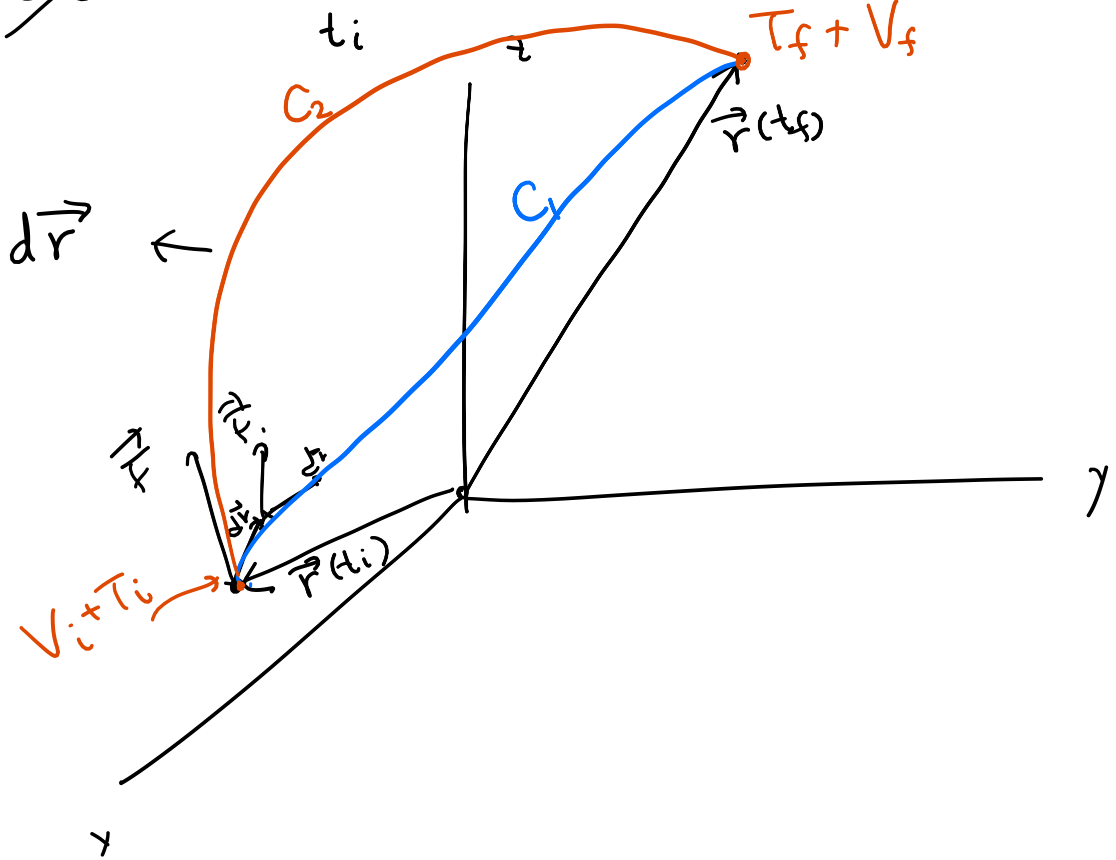
$$\parallel$$

$$\vec{F}\cdot\frac{d\vec{r}}{dt}$$

$$\int_{t_i}^{t_f} \cancel{dt}\,\vec{F}\cdot\frac{d\vec{r}}{\cancel{dt}} = \int_{t_i}^{t_f}\left(\frac{dT}{dt}\right)dt = T(t_f) - T(t_i) = \Delta T$$

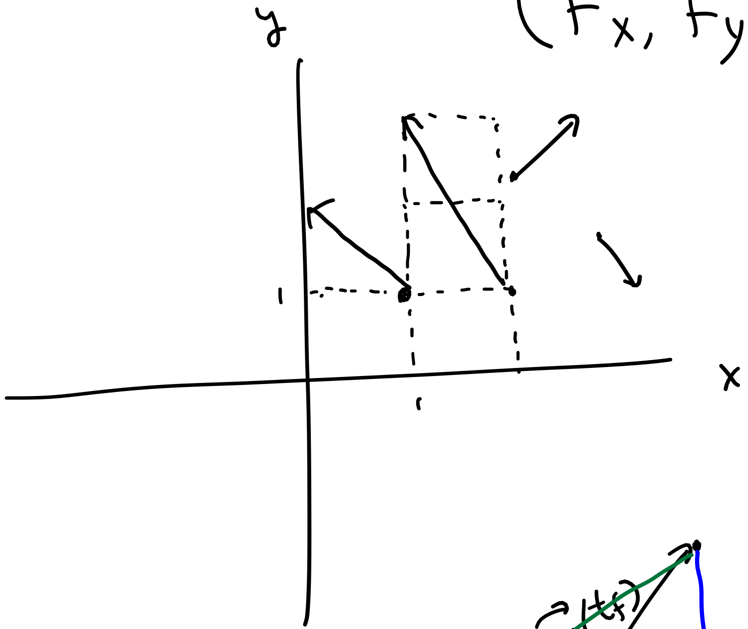
$$\parallel$$

$$\int_{C\ \vec{r}(t_i)}^{\vec{r}(t_f)} \vec{F}\cdot d\vec{r} \leftarrow$$



(ex) 2차원.

$(F_x, F_y)$

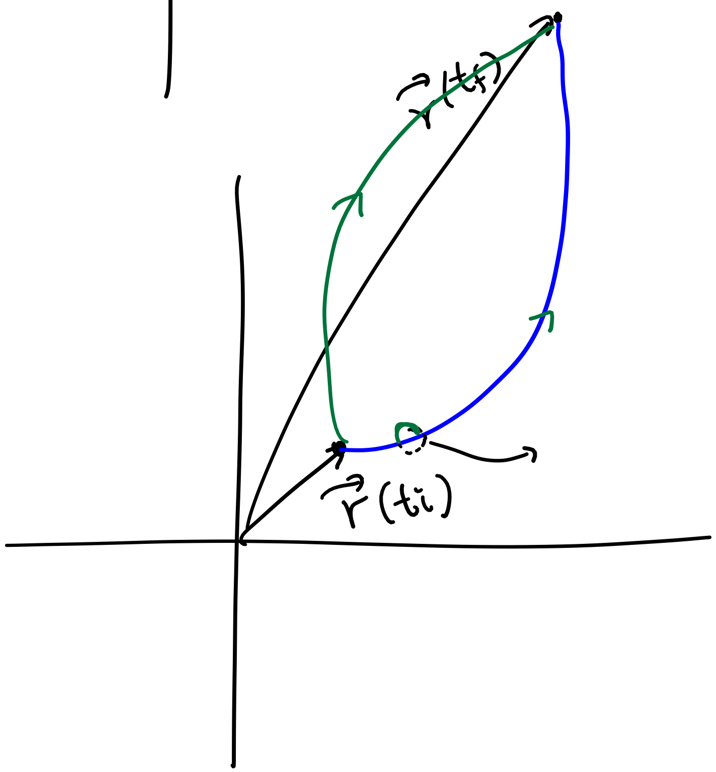



$$F_x = -by$$

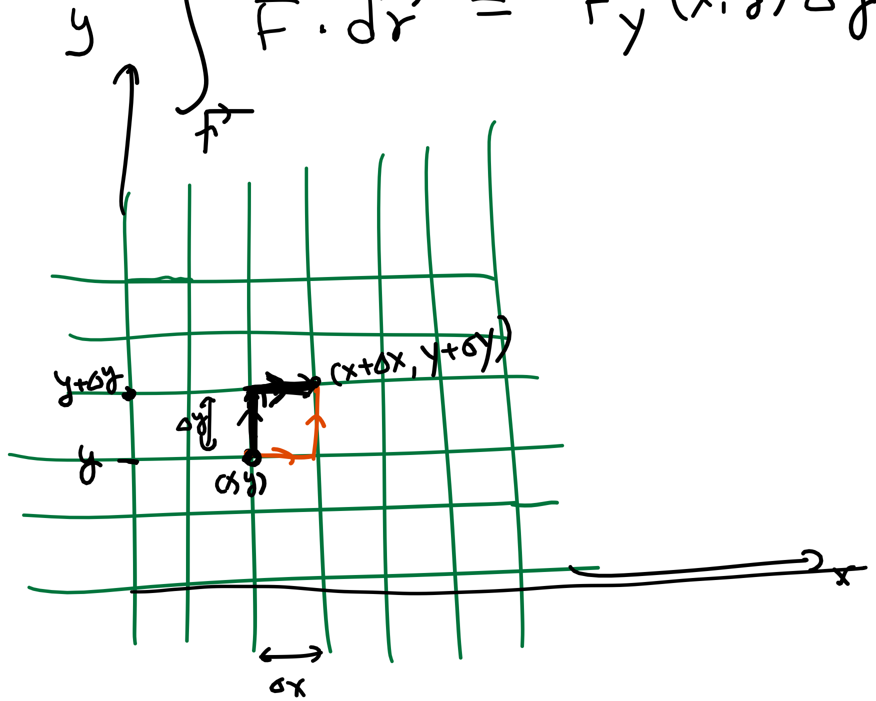
$$F_y = bx$$

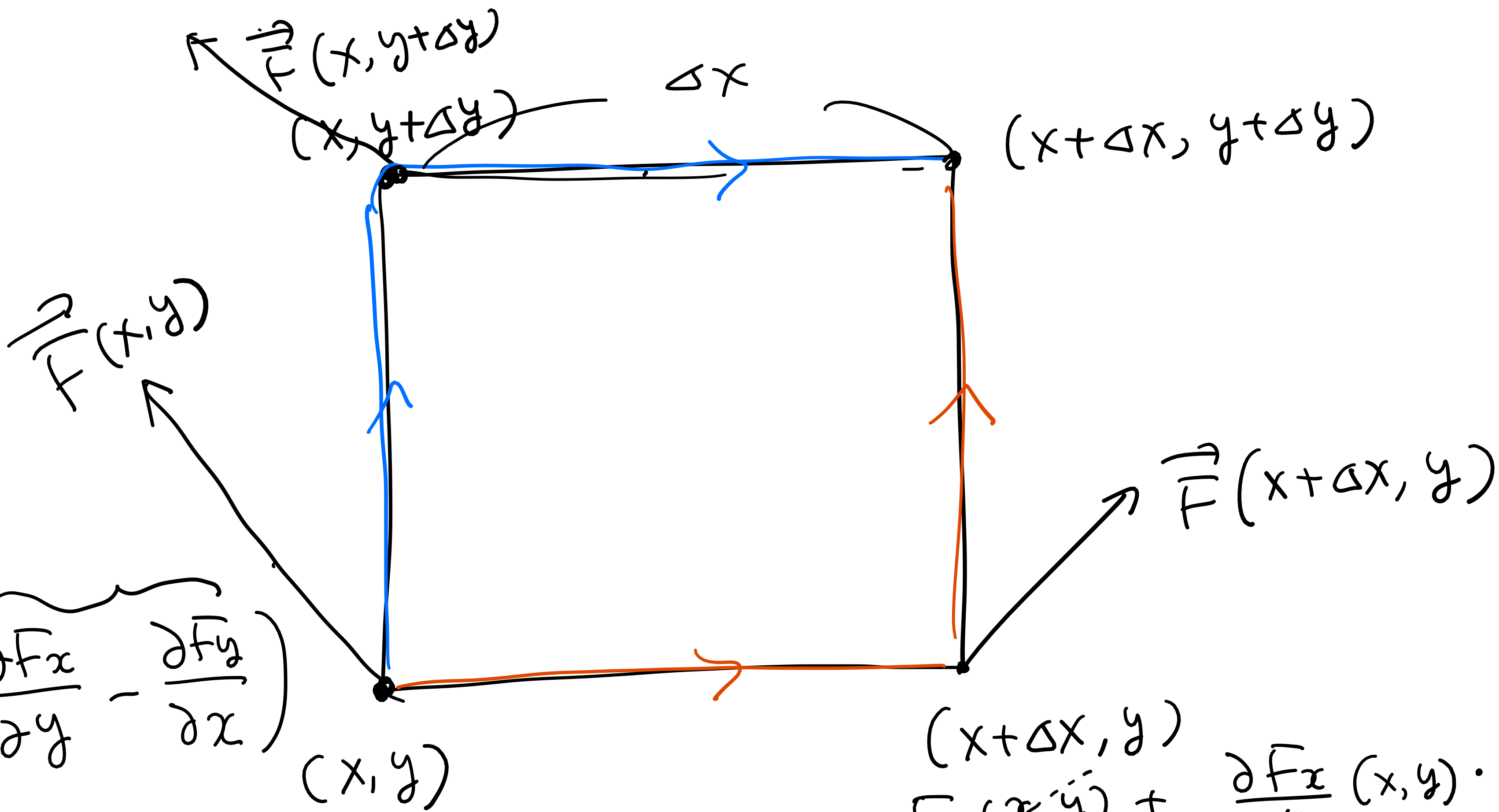
$$b=1$$

$$d\vec{r} = (\Delta x, \Delta y)$$




$$\int_{\vec{r}} \vec{F} \cdot d\vec{r} = F_y(x,y)\Delta y + F_x(x, y+\Delta y)$$



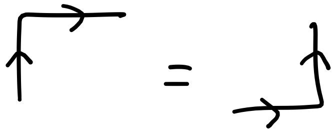



$$\Delta x \cdot \Delta y \left( \frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} \right)$$

$$= \cancel{F_y(x,y)}\,\Delta y + \underbrace{F_x(x, y+\Delta y)}_{\cancel{F_x(x,y)} + \frac{\partial F_x}{\partial y}(x,y)\cdot \Delta y} \Delta x$$

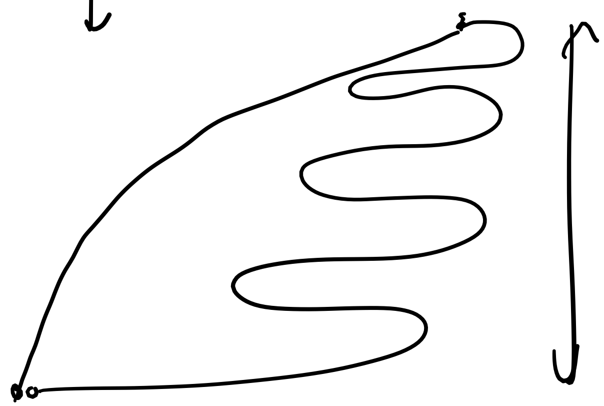
$$- \left( \cancel{F_x(x,y)}\,\Delta x + \underbrace{F_y(x+\Delta x, y)}_{\cancel{F_y(x,y)} + \frac{\partial F_y}{\partial x}(x,y)\,\Delta x} \Delta y \right) \overset{?}{=} 0$$

if $\dfrac{\partial F_x}{\partial y} - \dfrac{\partial F_y}{\partial x} = 0 \longrightarrow$



$\left( (\vec{\nabla} \times \vec{F})_z = 0 \right)$

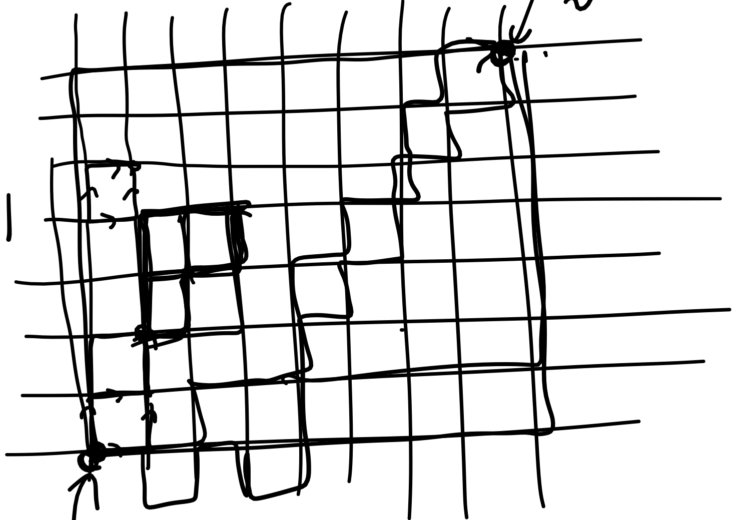
for all x, y



$\Delta T = T(t_f) - T(t_i)$

$\parallel$

$\displaystyle\int_{\vec{r}(t_i)}^{\vec{r}(t_f)} \vec{F} \cdot d\vec{r}$ = 항상 경로에 무관.




$\parallel$

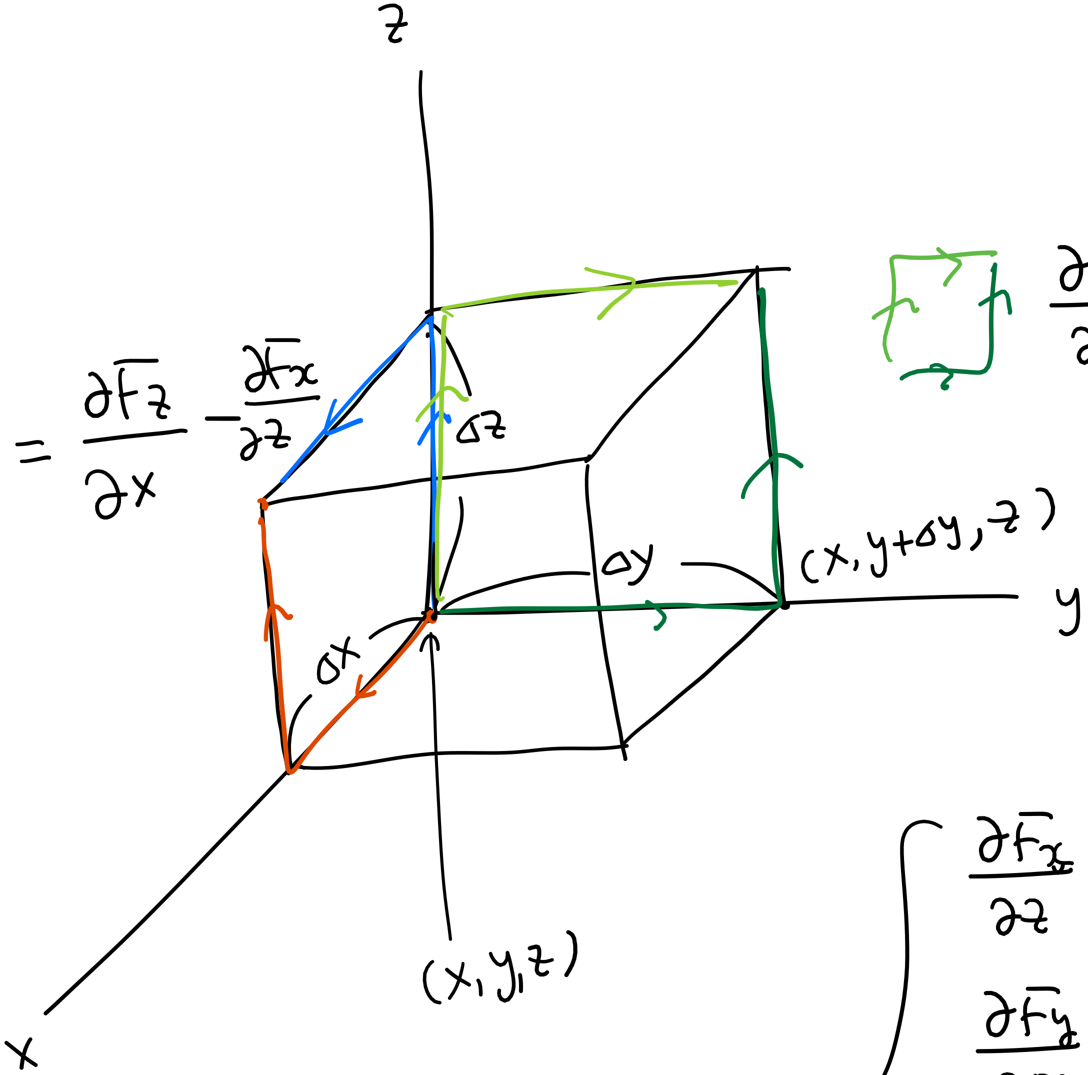
$V(\vec{r}(t_i)) - V(\vec{r}(t_f))$

위치에너지.

$T(t_f) + V(t_f) = T(t_i) + V(t_i)$

역학적 에너지 보존.

$$0 = \frac{\partial \bar{F}_z}{\partial x} - \frac{\partial \bar{F}_x}{\partial z}$$




$$\frac{\partial \bar{F}_y}{\partial z} - \frac{\partial \bar{F}_z}{\partial y} = 0$$

$$\begin{cases} \dfrac{\partial \bar{F}_x}{\partial z} - \dfrac{\partial \bar{F}_z}{\partial x} = 0 \\[2ex] \dfrac{\partial \bar{F}_y}{\partial x} - \dfrac{\partial \bar{F}_x}{\partial y} = 0 \\[2ex] \dfrac{\partial \bar{F}_y}{\partial z} - \dfrac{\partial \bar{F}_z}{\partial y} = 0 \end{cases}$$

Curl : $\vec{\nabla} \times \vec{F} = 0 \longrightarrow \vec{F}$는 보존력

미분 연산자이면서 vector

$$\vec{\nabla} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \quad \text{"del"}$$

divergence : $$\vec{\nabla} \cdot \vec{F} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (F_x, F_y, F_z)$$

$$= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \text{scalar}$$

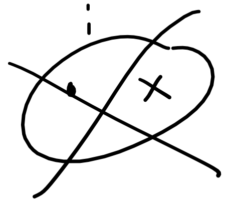
$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \vec{i}\left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \vec{j}\left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \vec{k}\left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

If $\quad F_x = f(x) \quad F_y = g(y) \quad F_z = h(z)$

$$\downarrow$$

$$\vec{\nabla} \times \vec{F} = 0$$

- Gradient ; $\vec{\nabla} f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$.



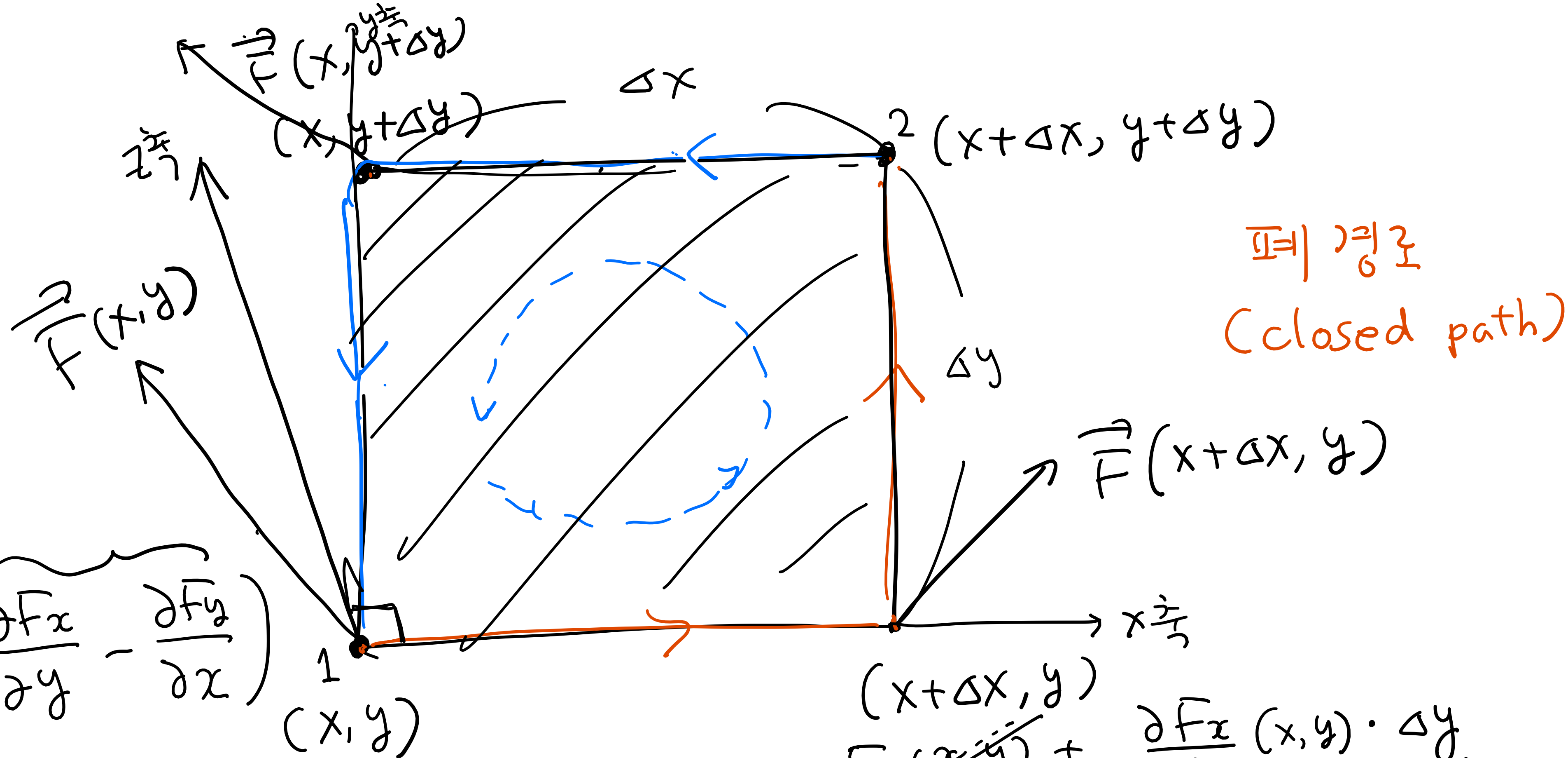
$(U = -V)$

$\vec{F}$ 보존력 $\Longleftrightarrow \vec{\nabla} \times \vec{F} = 0 \Longleftrightarrow \vec{F} = \vec{\nabla} U$

$$\vec{\nabla} \times (\vec{\nabla} U) = \underbrace{(\vec{\nabla} \times \vec{\nabla})}_{} U$$

$$= 0$$

$$\vec{A} \times \vec{A} = 0$$




$$\overbrace{\Delta x \cdot \Delta y \left(\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x}\right)}$$

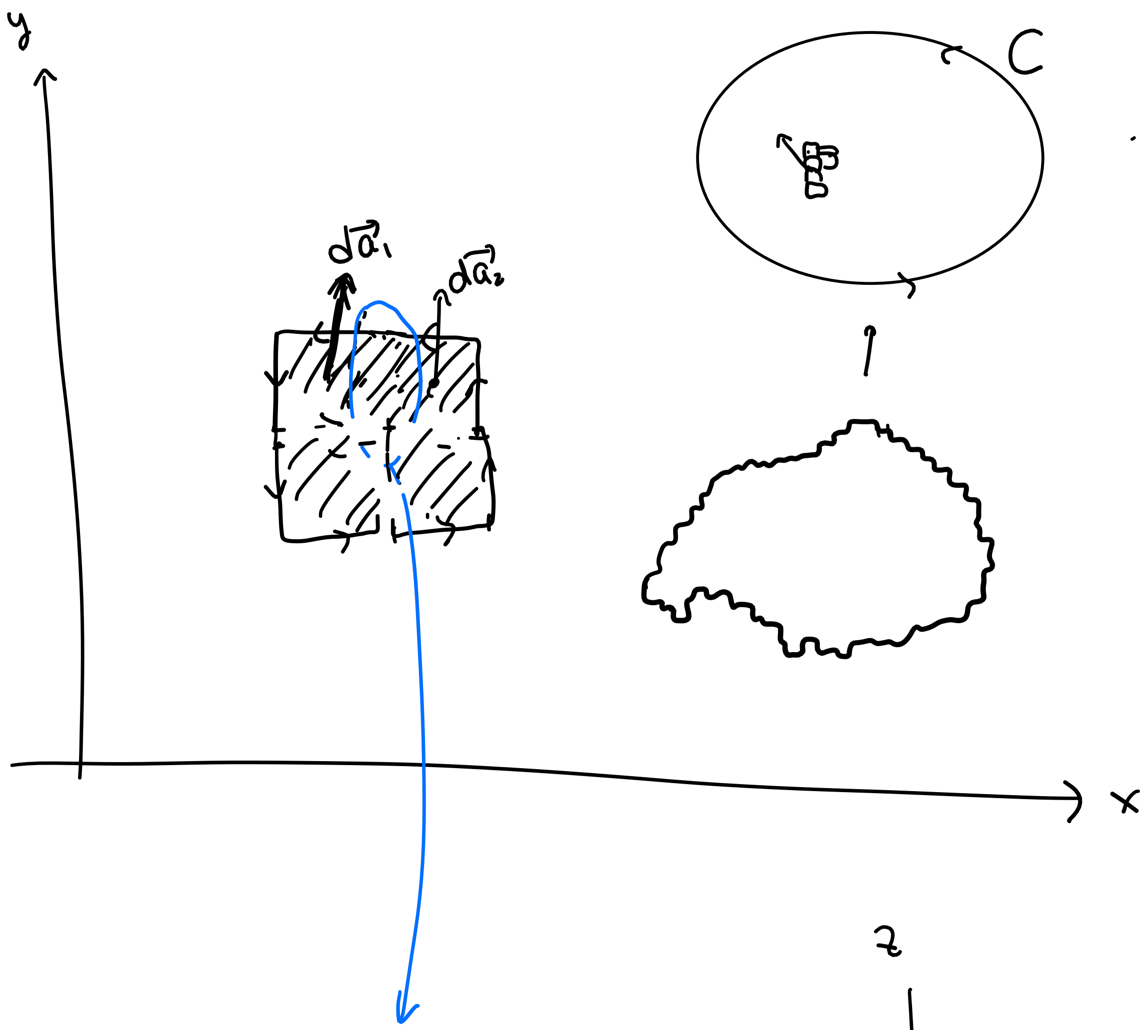
$$= -\cancel{F_y(x,y)}\,\Delta y - \overbrace{F_x(x, y+\Delta y)}^{\cancel{F_x(x,y)} + \frac{\partial F_x}{\partial y}(x,y)\cdot\Delta y}\,\Delta x$$

$$+\left(\cancel{F_x(x,y)}\,\Delta x + \underbrace{F_y(x+\Delta x, y)}_{\cancel{F_y(x,y)} + \frac{\partial F_y}{\partial x}(x,y)\,\Delta x}\,\Delta y\right) = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\underset{\Delta x \Delta y}{\overset{da}{\shortparallel}}$$

$$\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = (\vec{\nabla} \times \vec{F})^z$$

$$= (\vec{\nabla} \times \vec{F}) \cdot d\vec{a}$$

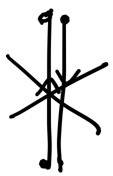
$$d\vec{a} = da\,\vec{k}$$

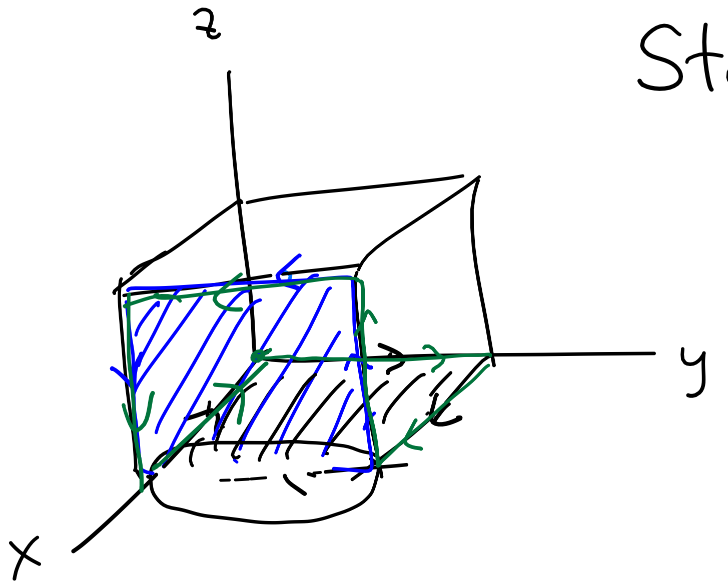



$$\int \vec{F}\cdot d\vec{r} = (\vec{\nabla}\times\vec{F})\cdot d\vec{a}_1 + (\vec{\nabla}\times\vec{F})\cdot d\vec{a}_2$$

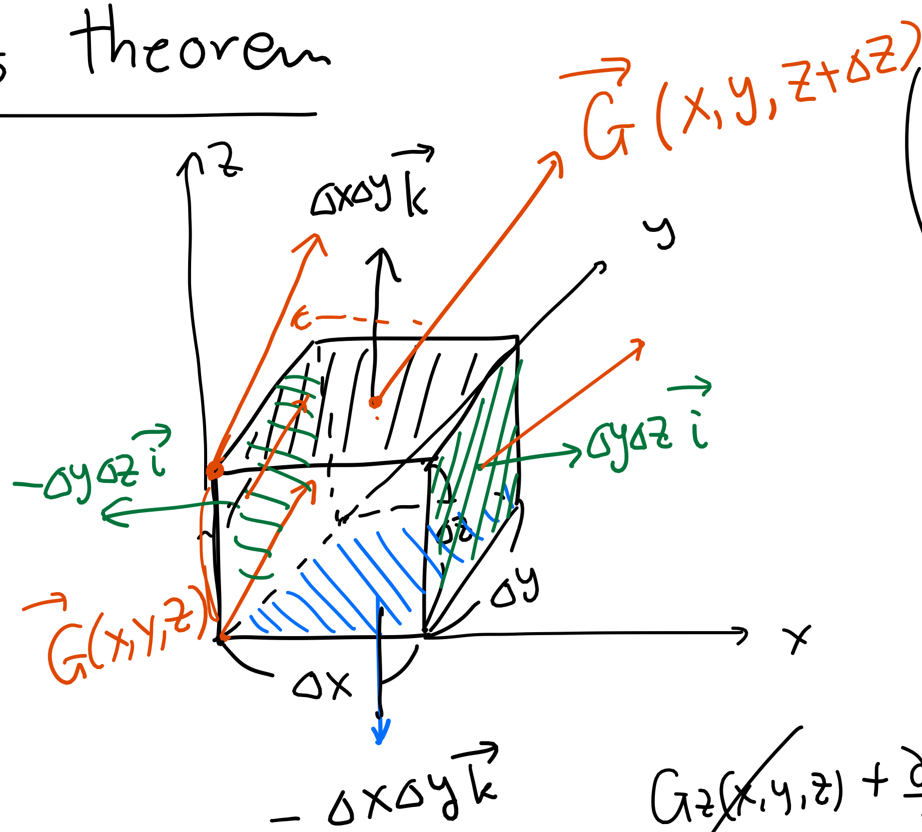
$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \int (\vec{\nabla}\times\vec{F})\cdot d\vec{a}}$$

Stokes theorem





# Gauss theorem




$$\oint_C \vec F\cdot d\vec r \quad \text{선적분}$$

$$\oint_{\square} \vec G\cdot d\vec a = \overbrace{G_z(x,y,z+\Delta z)}^{=\,\cancel{G_z(x,y,z)}+\frac{\partial G_z}{\partial z}\underbrace{\Delta z}}\underbrace{\Delta x\Delta y} - \cancel{G_z(x,y,z)\Delta x\Delta y}$$

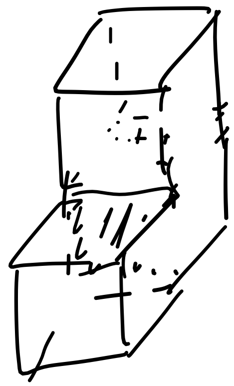
$$+\overbrace{G_x(x+\Delta x,y,z)}^{=\,\cancel{G_x(x,y,z)}+\frac{\partial G_x}{\partial x}\underbrace{\Delta x}}\underbrace{\Delta y\Delta z} - \cancel{G_x(x,y,z)\Delta y\Delta z}$$

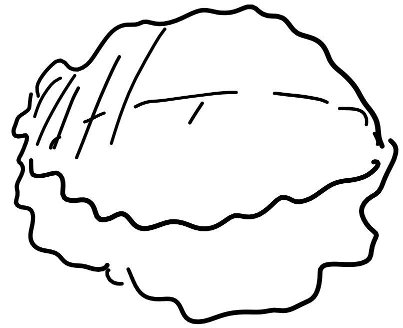
$$+\overbrace{G_y(x,y+\Delta y,z)}^{=\,\cancel{G_y(x,y,z)}+\frac{\partial G_y}{\partial y}\Delta y}\Delta x\Delta z - \cancel{G_y(x,y,z)\Delta x\Delta z}$$

$$=$$

$$\int_{\square} \vec\nabla\cdot\vec G\, dv$$

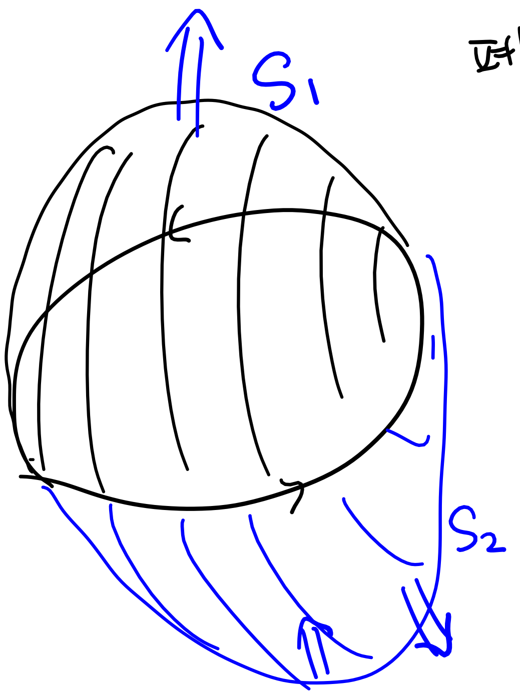
$$(\vec\nabla\cdot\vec G)\,\Delta V = \left(\frac{\partial G_z}{\partial z}+\frac{\partial G_x}{\partial x}+\frac{\partial G_y}{\partial y}\right)\underbrace{\Delta x\Delta y\Delta z}$$





$$\oint \vec{G}\cdot d\vec{a} = \int \vec{\nabla}\cdot\vec{G}\, dv$$

폐곡면




$S_1 \cup (-S_2)$

$$\Rightarrow \underbrace{\overbrace{\int_{S_1}(\vec{\nabla}\times\vec{F})\cdot d\vec{a}}^{=\oint_C \vec{F}\cdot d\vec{r}} + \overbrace{\int_{-S_2}(\vec{\nabla}\times\vec{F})\cdot d\vec{a}}^{-\oint_C \vec{F}\cdot d\vec{r}}}_{=\oint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{a}}$$

Gauss 정리

$$\oint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{a} = \int \vec{\nabla}\cdot(\vec{\nabla}\times\vec{F})\, dv = 0$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0 \quad \text{always}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \longrightarrow \vec{B} = \vec{\nabla} \times \underbrace{\vec{A}}$$

vector potential

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$$\vec{\nabla} \times \vec{F} = 0 \longrightarrow \vec{F} = \vec{\nabla} U$$

$-U$ : potential

$\vec{F}$ : 보존력

$\equiv$

경로에 무관

$\equiv$

$\vec{\nabla} \times \vec{F} = 0$

$\equiv$

$\vec{F} = -\vec{\nabla} V$

$$\vec{F} = -\vec{\nabla} V \qquad (U = -V)$$

$$\Delta T = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r} = -\int_{\vec{r}_i}^{\vec{r}_f} (\vec{\nabla} V) \cdot d\vec{r}$$

$$\Delta T = T_f - T_i$$

$$(\vec{\nabla} V) \cdot d\vec{r} = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right) \cdot (dx, dy, dz) = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

(ex)

$$f(x_f) - f(x_i) = \int_{x_i}^{x_f} \frac{\partial f}{\partial x} dx$$

$$= -\left( V(\vec{r}_f) - V(\vec{r}_i) \right)$$

$\therefore$

$$T(f) + V(f) = T(i) + V(i)$$

$$\Rightarrow \quad T + V = 보존.$$

[Ex. 4.2.1]

$$V(\vec{r}) = V_0 - \frac{1}{2} k\delta^2 e^{-r^2/\delta^2}$$

$\vec{F} = -\vec{\nabla} V$

$r^2 = x^2 + y^2$

$\downarrow$

$r = \sqrt{x^2+y^2}$

$\vec{r} = \vec{i}\, x + \vec{j}\, y$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$$

$$F_x = -\frac{\partial V}{\partial x} = -k\cancel{r}\, e^{-r^2/\delta^2} \underbrace{\frac{\partial r}{\partial x}}_{\frac{x}{\cancel{r}}}$$

$$F_y = -k\cancel{r}\, e^{-r^2/\delta^2} \frac{y}{\cancel{r}}$$

$$\underline{\vec{F} = -k\, e^{-r^2/\delta^2}\, \vec{r}}$$

$\Delta \ll \delta$

[Ex 4.2.2.]

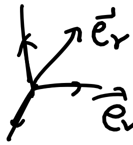
$\vec{r} = 0 \quad v = v_0 \longrightarrow \underbrace{\vec{r} = \hat{e}_r \Delta\ , \ v = ?}$

$$E = \frac{1}{2} m v_0^2 + \cancel{V_0} - \cancel{\frac{1}{2} k\delta^2} = E = \frac{1}{2} m v^2 + \cancel{V_0} - \frac{1}{2} k\delta^2 e^{-\Delta^2/\delta^2}$$

$e^{-\Delta^2/\delta^2}$: $\cancel{1} - \frac{\Delta^2}{\delta^2} + \cdots$

$$\frac{1}{2} m v_0^2 = \frac{1}{2} m v^2 + \frac{1}{2} k \cancel{\delta^2} \frac{\Delta^2}{\cancel{\delta^2}} \rightarrow \underline{\underline{v^2 = v_0^2 - \frac{k}{m}\Delta^2}}$$

# [Ex. 4.2.5] Curl in 구면좌표계.




$$\vec{e}_r = \frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}$$

(cf) 직교좌표계

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

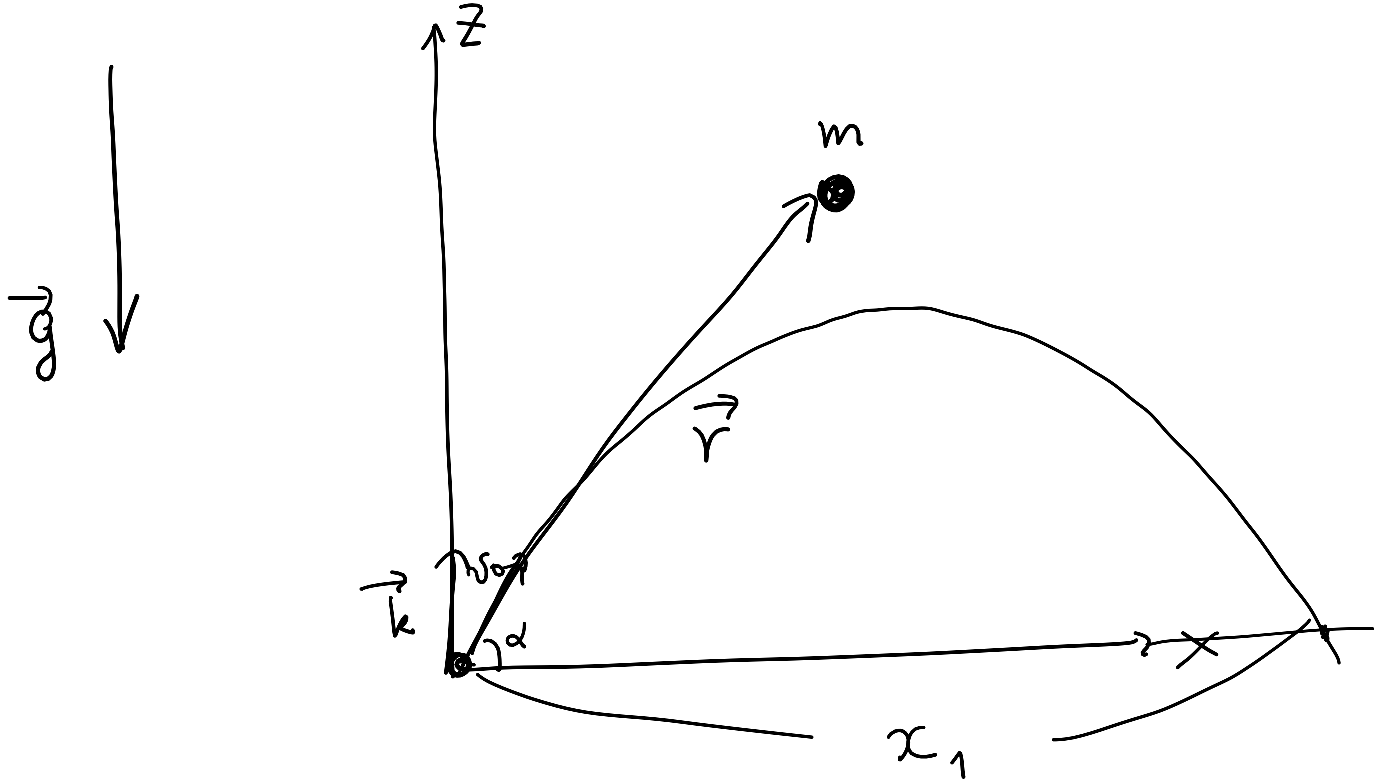
$$\vec{F} = \vec{e}_r \frac{1}{r^2}$$

$$= \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}(x, y, z)$$

구면좌표계

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{e}_r & \vec{e}_\theta r & \vec{e}_\phi r\sin\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & F_\theta r & F_\phi r\sin\theta \end{vmatrix}$$

# 2차원 수직면의 운동




$v_{0x} = v_0 \cos\alpha$

$v_{0z} = v_0 \sin\alpha$

$$\frac{v_0^2 \sin(2\alpha)}{g} = \frac{2 v_0^2 \cos\alpha \sin\alpha}{g} = x_1 = v_{0x} t_1$$

$\uparrow$

$z = 0 \rightarrow t_1 = \frac{2 v_{0z}}{g}$

$\uparrow$

if $\vec{r_0} = 0$

$$\boxed{\begin{aligned} z &= v_{0z} t - \frac{1}{2} g t^2 \\ x &= v_{0x} t \end{aligned}}$$

$=$

① 마찰이 없을 때

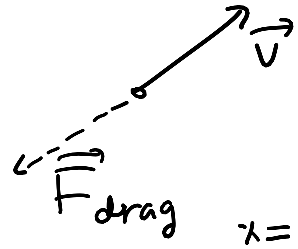
$$m \frac{d^2 \vec{r}}{dt^2} = \vec{F} = m \vec{g} = -mg\vec{k}$$

$$\frac{d^2 \vec{r}}{dt^2} = -g\vec{k} = \text{상수 벡터}$$

$$\frac{d\vec{r}}{dt} = \vec{v} = \vec{v_0} - g\vec{k}t \rightarrow \underline{\vec{r} = \vec{r_0} + \vec{v_0} t - \frac{1}{2} g t^2 \vec{k}}$$

② 마찰이 있을 때 (속도에 비례할 경우)

$$m\frac{d^2\vec{r}}{dt^2} = -mg\vec{k} - c\vec{V}$$




$$\frac{d\vec{r}}{dt} = \vec{V} \qquad \frac{c}{m} \equiv \gamma$$

$$\frac{d\vec{V}}{dt} = -g\vec{k} - \gamma\vec{V}$$

$\downarrow$

$$\frac{dV_x}{dt} = -\gamma V_x \longrightarrow \underline{V_x = V_{x0}e^{-\gamma t}} = \dot{x}$$

$\Uparrow$

$$x = V_{x0}\left(-\frac{1}{\gamma}e^{-\gamma t}\right)\Big|_0^t$$

$$x = \frac{1}{\gamma}V_{x0}\left(1 - e^{-\gamma t}\right)$$

$$\frac{dV_z}{dt} = -g - \gamma V_z = -\gamma\left(\underbrace{V_z + \frac{g}{\gamma}}_{W_z}\right)$$

$$\frac{dW_z}{dt} = \frac{dV_z}{dt}$$

$$\frac{dW_z}{dt} = -\gamma W_z \longrightarrow W_z = \underbrace{W_{z0}}_{V_{z0}+\frac{g}{\gamma}}e^{-\gamma t}$$

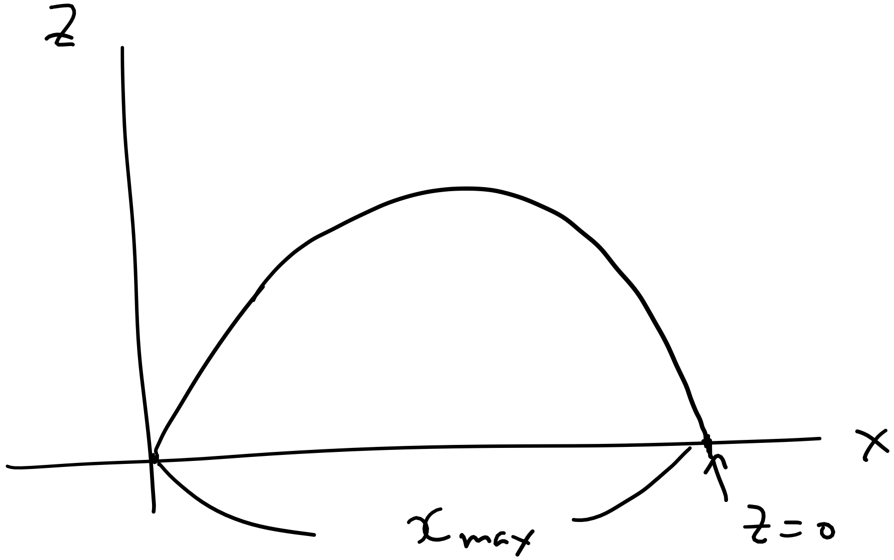
$$W_z = V_z + \frac{g}{\gamma}$$

$$\dot{z} = V_z = -\frac{g}{\gamma} + \left(V_{z0} + \frac{g}{\gamma}\right)e^{-\gamma t}$$

$$\boxed{z = -\frac{g}{\gamma}t + \left(V_{z0} + \frac{g}{\gamma}\right)\frac{1}{\gamma}\left(1 - e^{-\gamma t}\right)}$$

$$x = \frac{1}{\gamma} v_{x0}\left(1 - e^{-\gamma t}\right) \Rightarrow -\frac{1}{\gamma}\log\left(1 - \frac{\gamma}{v_{x0}} x\right) = t$$

$$z = -\frac{g}{\gamma} t + \left(v_{z0} + \frac{g}{\gamma}\right)\underbrace{\frac{1}{\gamma}\left(1-e^{-\gamma t}\right)}_{= \frac{x}{v_{x0}}} = \frac{g}{\gamma^2}\log\left(1-\frac{\gamma}{v_{x0}}x\right) + \frac{v_{z0}+\frac{g}{\gamma}}{\cancel{v_{x0}}} \underbrace{x}_{= \frac{\cancel{v_{x0}}}{\gamma}u} = 0$$




$x = x_{max} \to z = 0$

$\gamma$ 가 작으면 $u \equiv \frac{\gamma x}{v_{x0}} \ll 1$

$$\log(1-u) \approx -u - \frac{u^2}{2} - \frac{u^3}{3} + \cdots$$

$$\frac{g}{\gamma}\left(-u - \frac{u^2}{2} - \frac{u^3}{3}\right) + \left(v_{z0} + \frac{g}{\gamma}\right) u = 0$$

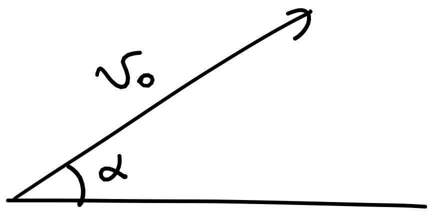
$$\frac{1}{u} \times \quad \frac{g}{\gamma}\left(-\cancel{1} - \frac{u}{2} - \cancel{\frac{u^2}{3}}\right) + \left(v_{z0} + \cancel{\frac{g}{\gamma}}\right) = 0$$

$$\frac{g}{\gamma}\left( - \frac{u}{2} - \frac{u^2}{3}\right) + v_{z0} = 0$$

$$-\frac{u}{2} - \frac{u^2}{3} + \frac{\gamma v_{z0}}{g} = 0 \;\longrightarrow\; u^2 + \frac{3}{2}u - \underbrace{\frac{3\gamma v_{z0}}{g}}_{\ll 1} = 0$$

$$\frac{\gamma x_{max}}{v_{x0}} = u = \sqrt{\frac{9}{16} + \frac{3\gamma v_{z0}}{g}} - \frac{3}{4}$$

$$= \sqrt{\frac{9}{16}}\sqrt{1 + \underbrace{\frac{16\gamma v_{z0}}{3g}}_{\epsilon}} - \frac{3}{4}$$



$$\left(\sqrt{1+\epsilon} \approx 1 + \frac{\epsilon}{2} - \frac{1}{8}\epsilon^2\right)$$

$$= \frac{3}{4}\left[\cancel{1} + \frac{8\gamma v_{z0}}{3g} - \frac{1}{8}\left(\frac{16\gamma v_{z0}}{3g}\right)^2\right] \cancel{-\frac{3}{4}}$$

$$x_{max} = \frac{2\cancel{\gamma} v_{z0} v_{x0}}{g\,\cancel{\gamma}} - \frac{\cancel{3}}{\cancel{4}}\cdot\frac{1}{\cancel{4}}\cdot\frac{\cancel{16}^{\,2}}{\cancel{9}_{\,3}}\,\frac{\gamma^{\cancel{2}} v_{z0}^2}{2g^2}\,\frac{v_{x0}}{\cancel{\gamma}}$$

$$= \frac{2 v_{x0} v_{z0}}{g} - \frac{16\gamma v_{z0}^2 v_{x0}}{6g^2} \qquad v_{x0} = v_0\cos\alpha, \quad v_{z0} = v_0\sin\alpha$$

$$= \frac{v_0^2 \sin 2\alpha}{g} - \frac{4\gamma v_0^3 \sin 2\alpha \cos\alpha}{3g^2}$$

(Ex. 4.3.1)

$$\vec F(\vec v) = -\vec v\,\big(\underbrace{c_1 + c_2|\vec v|}\big)$$

$$\simeq$$

$$c_1 + c_2 v_0 = m\gamma \longrightarrow x_{max}.$$

(Ex. 4.3.2)

$$\vec F = -c_2\,\vec v\,\underbrace{|\vec v|}$$

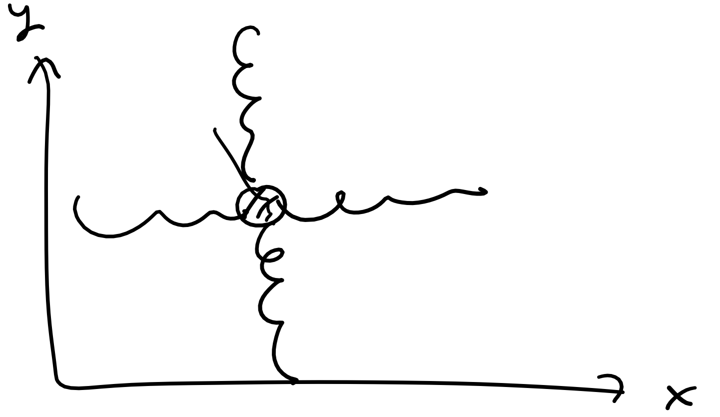
$$(v_x, v_z)\sqrt{v_x^2 + v_z^2}$$

$$F_{x,z} = -c_2\, v_{x,z}\sqrt{v_x^2 + v_z^2}$$

$$\left.\begin{aligned} \ddot z &= -g - \gamma\, v_z\sqrt{v_x^2 + v_z^2} \\ \ddot x &= -\gamma\, v_x\sqrt{v_x^2 + v_z^2} \end{aligned}\right\} \rightarrow \begin{aligned} x &= y(1) \\ z &= y(2) \\ \dot x &= v_x = y(3) \\ \dot z &= v_z = y(4) \end{aligned}$$

(예1) $\ddot x = \dfrac{d\,y(3)}{dt} = -\gamma\, y(3)\sqrt{y(3)^2 + y(4)^2}$

4.4. 2차원 H.O.




$$\vec{F} = -k\vec{r}$$

$$m\frac{d^2\vec{r}}{dt^2} = -k\vec{r}$$

$$\downarrow$$

$$\ddot{x} = -\omega^2 x \longrightarrow x = A\cos(\underbrace{\omega t + \alpha}_{\psi}) \qquad x = A\cos\psi$$

$$\ddot{y} = -\omega^2 y \longrightarrow y = B\cos(\omega t + \beta)$$

$$\beta - \alpha \equiv \Delta \qquad \beta = \underbrace{\alpha + \Delta}$$

$$y = B\cos(\underbrace{\omega t + \alpha}_{\equiv\psi} + \Delta)$$

$$= B(\underbrace{\cos\psi}_{\frac{x}{A}}\cos\Delta - \underbrace{\sin\psi}_{\sqrt{1-\cos^2\psi}\, = \,\sqrt{1-\left(\frac{x}{A}\right)^2}}\sin\Delta)$$

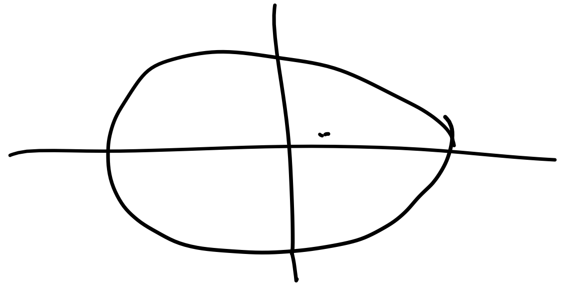
$$\underbrace{y - \frac{B}{A}\cos\Delta\, x} = -B\sin\Delta\sqrt{1-\left(\frac{x}{A}\right)^2}$$

$$\xrightarrow{\text{제곱}} \quad y^2 - \frac{2B\cos\Delta}{A}\underbrace{xy} + \frac{B^2}{A^2}\cos^2\Delta\, x^2 = B^2\sin^2\Delta\left(1 - \frac{x^2}{A^2}\right)$$

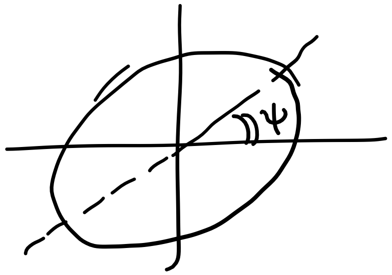
$$y^2 - \frac{2B\cos\Delta}{A}xy + \frac{B^2}{A^2}x^2 = B^2\sin^2\Delta$$

$$\left(ax^2 + bxy + cy^2 = f\right)$$

$b=0$



$b \neq 0$ 회전

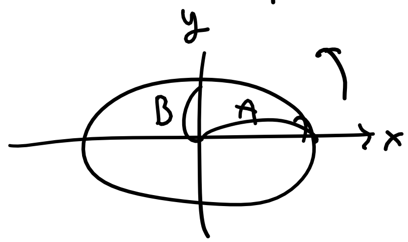


$$\tan 2\psi = \frac{2AB\cos\Delta}{A^2 - B^2} \Rightarrow \frac{\text{유도}}{\text{다음 페이지}}$$

$$\frac{x^2}{A^2\sin^2\Delta} + \frac{y^2}{B^2\sin^2\Delta} - \frac{2\cos\Delta}{AB\sin^2\Delta}xy = 1$$

$\cos\Delta = 0 \rightarrow \sin\Delta = 1$

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$



$$y^2 - \frac{2B\cos\Delta}{A}xy + \frac{B^2}{A^2}x^2 = B^2\sin^2\Delta$$

$$\downarrow$$

$$\frac{1}{B^2}y^2 - \frac{2\cos\Delta}{AB}xy + \frac{1}{A^2}x^2 = \sin^2\Delta = (x, y)\begin{pmatrix} \frac{1}{A^2} & -\frac{\cos\Delta}{AB} \\ -\frac{\cos\Delta}{AB} & \frac{1}{B^2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

2차원 회전

$$\begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$(x', y')\underbrace{\begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}\begin{pmatrix} \overset{\alpha}{\frac{1}{A^2}} & \overset{\gamma}{-\frac{\cos\Delta}{AB}} \\ \overset{\gamma}{-\frac{\cos\Delta}{AB}} & \overset{\beta}{\frac{1}{B^2}} \end{pmatrix}\begin{pmatrix} \overset{c}{\cos\psi} & -\overset{s}{\sin\psi} \\ \sin\psi & \cos\psi \end{pmatrix}}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix}\begin{pmatrix} c & -s \\ s & c \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} \alpha c + \gamma s & -\alpha s + \gamma c \\ \gamma c + \beta s & -\gamma s + \beta c \end{pmatrix} = \begin{pmatrix} \alpha c^2 + 2\gamma cs + \beta s^2, & \overset{0}{\boxed{\gamma(c^2 - s^2) + (\beta - \alpha)cs}} \\ \boxed{\gamma(c^2 - s^2) + (\beta - \alpha)cs} & \alpha s^2 - 2\gamma cs + \beta c^2 \end{pmatrix}$$

$$\frac{1}{2}\tan 2\psi = \frac{\cos\psi\sin\psi}{\cos^2\psi - \sin^2\psi} = \frac{cs}{c^2 - s^2} = -\frac{\gamma}{\beta - \alpha} = \frac{\frac{\cos\Delta}{AB}}{\frac{1}{B^2} - \frac{1}{A^2}} = \frac{AB\cos\Delta}{A^2 - B^2} \Rightarrow \underline{\underline{\tan 2\psi = \frac{2AB\cos\Delta}{A^2 - B^2}}}$$

비등방 H.O.

$$m\ddot{x} = -k_x x \qquad k_x \neq k_y$$

$$m\ddot{y} = -k_y y$$

$$\begin{cases} \ddot{x} = -\omega_1^2 x \\ \ddot{y} = -\omega_2^2 y \end{cases} \qquad \omega_1^2 \equiv \frac{k_x}{m} \qquad \omega_2^2 \equiv \frac{k_y}{m}$$

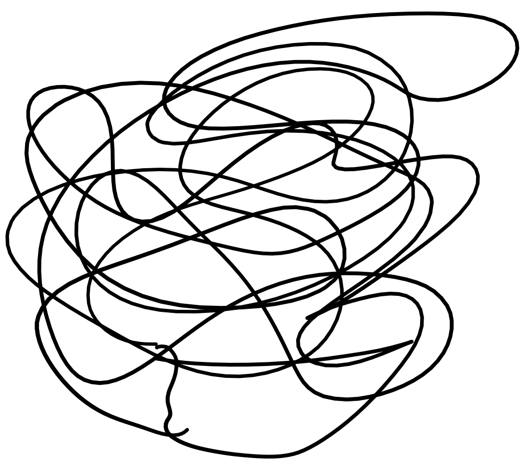
$$x = A\cos(\omega_1 t + \alpha) \leftarrow T_x = \frac{2\pi}{\omega_1}$$

$$y = B\cos(\omega_2 t + \beta) \leftarrow T_y = \frac{2\pi}{\omega_2}$$

$$\frac{n_1}{\omega_1} = \frac{n_2}{\omega_2}$$

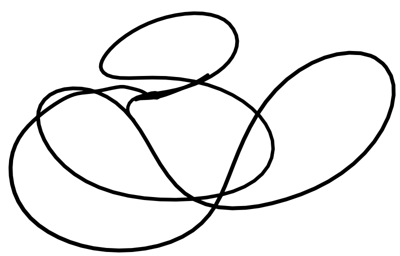
$$\Uparrow \qquad \frac{\omega_1}{\omega_2} = \frac{n_1}{n_2} = \text{유리수}$$

commensurate : $\underset{\sim\sim\sim}{n_1 T_x = n_2 T_y}$



$x \Leftarrow$ in 〃 $n_1 T_x = n_2 T_y$

$n_1, n_2$ ~~존재~~



[Ex. 4.4.1]

$$V = \frac{1}{2}k\,(x^2 + 4y^2)$$

$$F_x = -\frac{\partial V}{\partial x} = -kx = m\ddot{x}, \quad F_y = -\frac{\partial V}{\partial y} = -4ky = m\ddot{y}$$

$$\frac{k}{m} = \omega^2$$

$$\ddot{x} = -\omega^2 x$$

$$\ddot{y} = -4\omega^2 y \longrightarrow \omega_1 = \omega, \quad \omega_2 = 2\omega$$

$$x = A\cos(\omega t + \alpha)$$

$$y = B\cos(2\omega t + \beta)$$

$t=0 \quad x = a,\ y = 0,\ \dot{x} = 0 \quad \dot{y} = v_0$

$$A\cos\alpha = a \longrightarrow A = a$$

$$-A\omega\sin\alpha = 0 \longrightarrow \alpha = 0$$

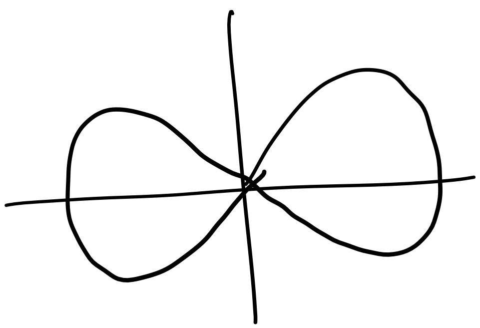
$$B\cos\beta = 0 \longrightarrow \beta = \frac{\pi}{2}$$

$$\dot{y} = -B\,2\omega\sin\beta = v_0 \longrightarrow B = -\frac{v_0}{2\omega}$$

$$\therefore\ x = a\cos\omega t$$

$$y = -\frac{v_0}{2\omega}\cos\left(2\omega t + \frac{\pi}{2}\right)$$

$$y = \frac{v_0}{2\omega}\sin(2\omega t)$$



4.5. Electric & Magnetic field.

$$\vec{F} = q\vec{E} \rightarrow \begin{aligned} \ddot{x} &= \frac{q}{m}E_x = 0 \\ \ddot{y} &= \frac{q}{m}E_y = 0 \\ \ddot{z} &= \frac{q}{m}\underset{\overset{\parallel}{E}}{E_z} \rightarrow z = z_0 + v_{z0}t + \frac{1}{2}\frac{qE}{m}t^2 \end{aligned} \qquad \vec{E} = (0, 0, E)$$

---

$\vec{B}$

$$\vec{F} = q(\vec{v} \times \vec{B})$$

[Ex 4.5.1] $\vec{B} = (0, 0, B)$, $\vec{F} = q(\vec{v} \times \vec{B}) = q\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & B \end{vmatrix}$

$$m\ddot{\vec{r}} = qB(\dot{y}, -\dot{x}, 0)$$

$$\ddot{x} = \frac{qB}{m}\dot{y} \xrightarrow{t\text{로적분}} \dot{x} = \frac{qB}{m}y + C_1$$

$$\ddot{y} = -\frac{qB}{m}\dot{x} \xrightarrow{\;''\;} \underbrace{\dot{y} = -\frac{qB}{m}x + C_2}$$

$$\ddot{z} = 0 \longrightarrow \underline{\dot{z} = v_{z0}} \rightarrow z = v_{z0}t$$

$\therefore$

$$\ddot{x} = \frac{qB}{m}\left(-\frac{qB}{m}x + C_2\right)$$

$$\ddot{x} = -\underline{\left(\frac{qB}{m}\right)^2}x + \frac{qBC_2}{m}$$

$$\ddot{x} = -\underbrace{\left(\frac{qB}{m}\right)^2}_{\omega}\underbrace{\left(x - \frac{mC_2}{qB}\right)}_{\tilde{x}} \longrightarrow \ddot{\tilde{x}} = -\omega^2 \tilde{x}$$

$$\ddot{\tilde{x}} = \ddot{x}$$

$$\tilde{x} = A_x \cos(\omega t + \alpha)$$

$$\tilde{x} = x - \frac{mC_2}{qB}$$

$$x = \frac{mC_2}{qB} + A_x \cos(\omega t + \alpha) \longrightarrow \dot{x} = -A_x \omega \sin(\omega t + \alpha)$$

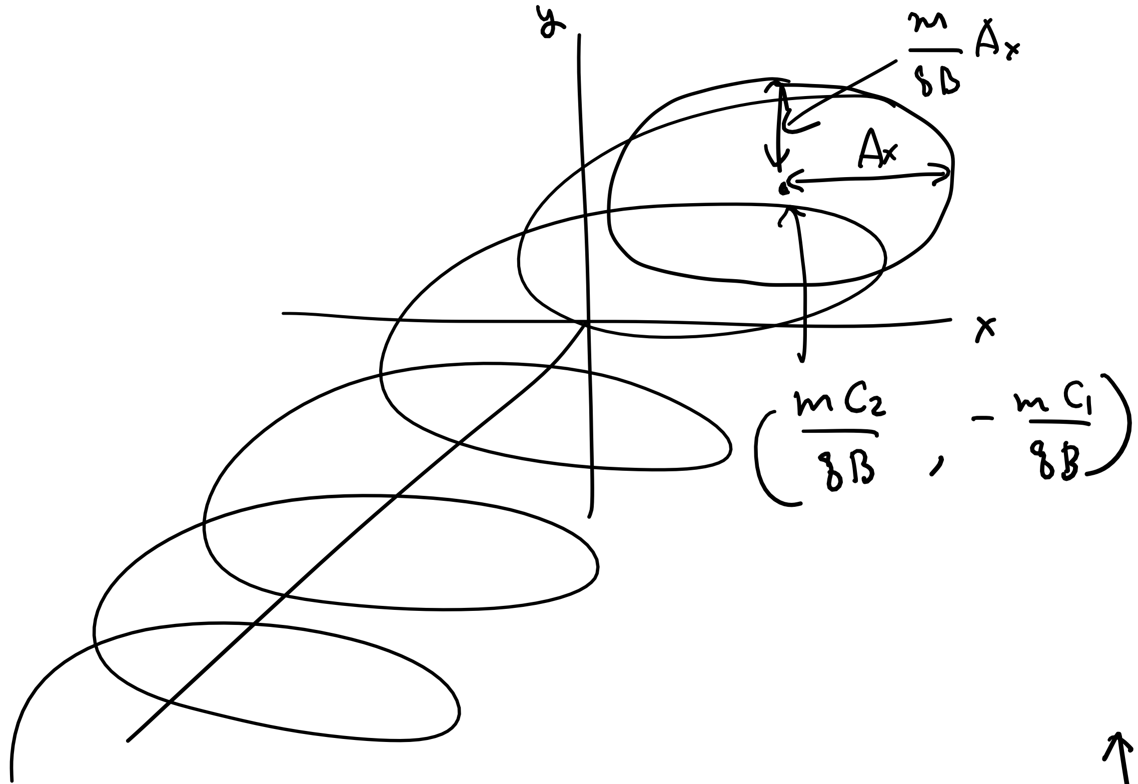
$$\dot{x} = \frac{qB}{m} y + C_1 \longrightarrow y = \frac{m}{qB}(\dot{x} - C_1)$$

$$= \frac{m}{qB}\left(-\omega A_x \sin(\omega t + \alpha) - C_1\right)$$

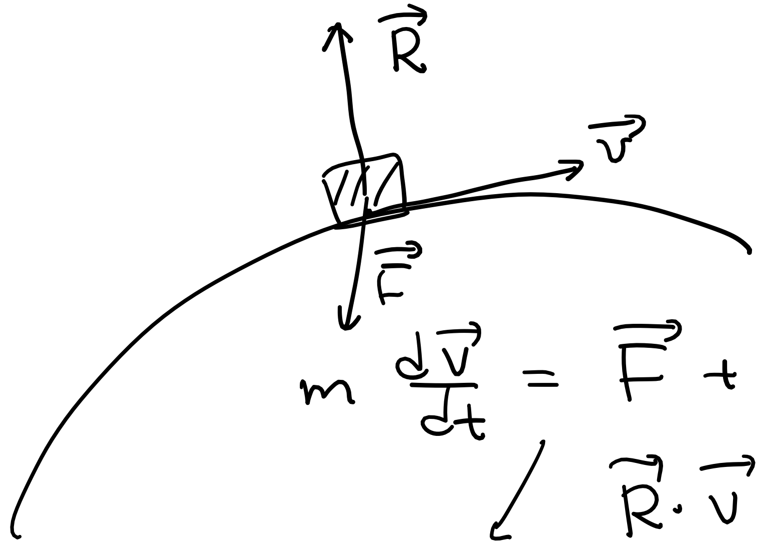
$$y + \frac{mC_1}{qB} = -\frac{mA_x}{qB}\sin(\omega t + \alpha)$$

$$x - \frac{mC_2}{qB} = A_x \cos(\omega t + \alpha)$$

$$\therefore \frac{\left(y + \frac{mC_1}{qB}\right)^2}{\left(\frac{m}{qB}A_x\right)^2} + \frac{\left(x - \frac{mC_2}{qB}\right)^2}{A_x^2} = 1$$




## 4.6. Constrained.



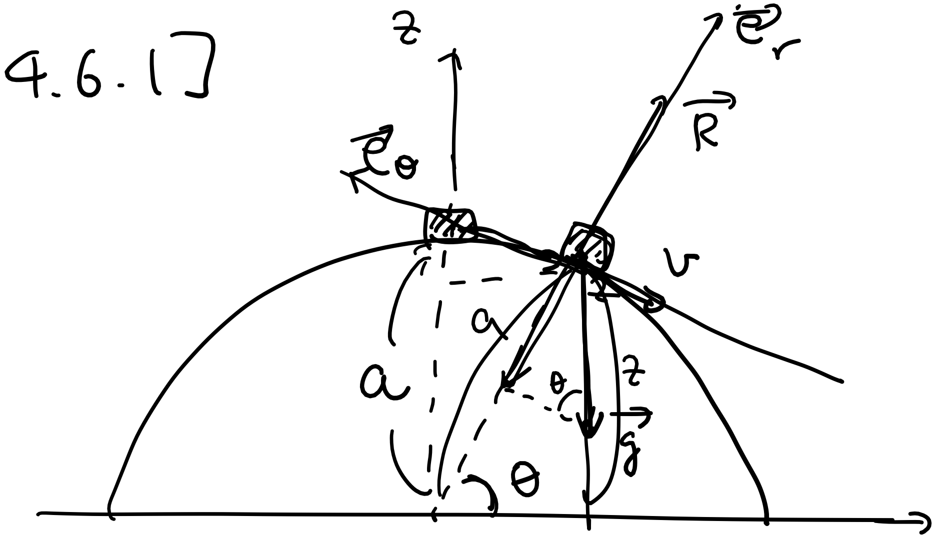

$$m\frac{d\vec v}{dt} = \vec F + \vec R$$

$$\vec R \cdot \vec v = 0$$

$$\frac{1}{2}m\frac{dv^2}{dt} = m\vec v\cdot\frac{d\vec v}{dt} = \underbrace{\vec v\cdot\vec F}_{\frac{dW}{dt}} + \cancel{\vec v\cdot\vec R}^{\,0}$$

$$T + V = E \leftarrow$$

[Ex. 4.6.1]




$$m\frac{d\vec{v}}{dt} = \vec{F} + \vec{R} = m\vec{g} + \vec{R}$$

$$m \quad \vec{a} \overset{\text{극좌표}}{=} (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta$$

$\downarrow$

$\vec{e}_r$ 성분 : $m(\ddot{r} - r\dot{\theta}^2) = -mg\sin\theta + R$

$\downarrow$

$$\underbrace{-a\dot{\theta}^2}_{= -\frac{v^2}{a}} = -g\sin\theta + \frac{R}{m}$$

$r = a = \text{const}$

$\dot{r} = \ddot{r} = 0$

$v = a\dot{\theta} \rightarrow \dot{\theta} = \frac{v}{a}$

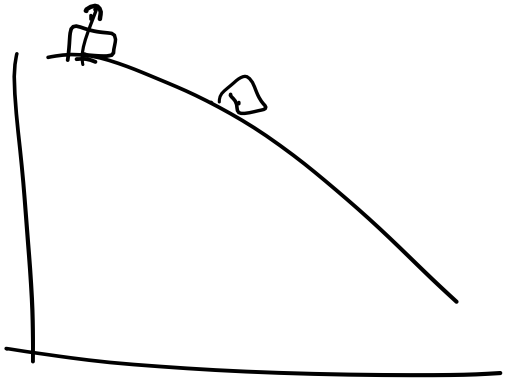
$$\frac{v^2}{a} = g\overbrace{\sin\theta}^{=\frac{z}{a}} - \frac{R}{m} = \frac{1}{a}\left(2g(a-z)\right)$$

$E = mga = \frac{1}{2}mv^2 + mgz \rightarrow v^2 = 2g(a-z)$

$$\frac{g}{a}z - \frac{R}{m} = \frac{2g}{a}(a-z) \Rightarrow \frac{R}{m} = \frac{gz}{a} - \frac{2g}{a}(a-z)$$

$$\frac{R}{m} = \frac{g}{a}(3z - 2a)$$

$R=0$



$z = \frac{2a}{3}$ 인 지점.

시험범위.

(土) 3시반 ~

· 푸는 문제 ; 3문제

- 숙제 그대로
- 유도
- 숙제가 아닌 연습문제의 "약간" 변형.

· MatLab 1문제.

(火) : 3시 30분.