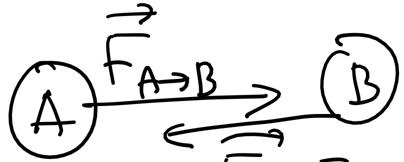
2장. Newtonian Mechanics

1. 관성의 법칙 (Galileo)

$\vec{F} = 0$ 이면 $\vec{v}, \vec{p}$ 는 일정.

2.* $\dfrac{d\vec{p}}{dt} = \vec{F}$ (운동법칙)

3. 작용과 반작용



$\vec{F}_{B\to A} = -\vec{F}_{A\to B}$

관성 (inertial) 좌표계 : 뉴턴의 법칙이 성립되는 좌표계 (정지 혹은 등속으로 움직이는 좌표계.)

$$\vec{p} = m\vec{v}$$

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2} = m\ddot{\vec{r}}$$

$$\downarrow$$

$$\vec{F}(\vec{r}, \vec{v})$$

$$m\ddot{\vec{r}} = \vec{F}\left(\vec{r}, \frac{d\vec{r}}{dt}\right) \Leftarrow \text{2차 미분방정식}$$

초기조건 (Initial Condition)

$$\begin{cases} \vec{r}(t=0) = \vec{r}_0 \leftarrow \\ \dot{\vec{r}} = \vec{v}(0) = \vec{v}_0 \end{cases}$$

질량 : m 관성질량

$=$

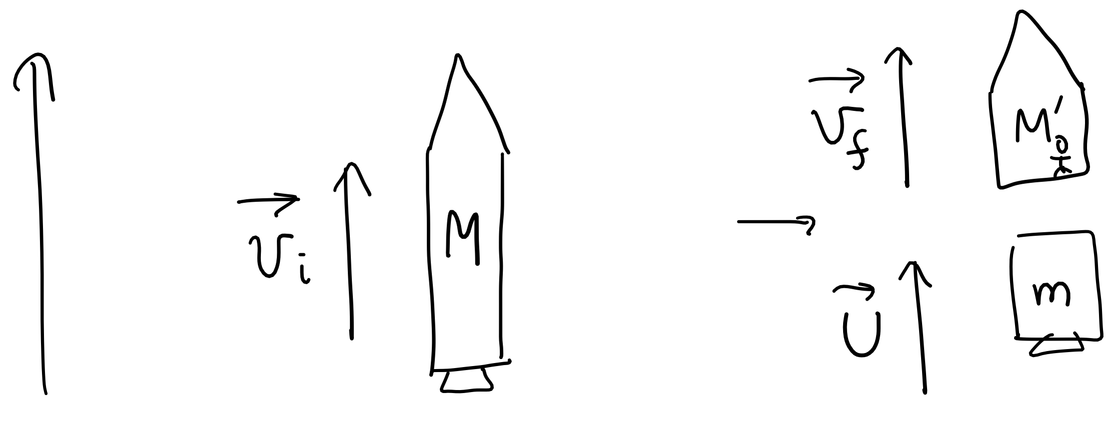
등가성 원리.

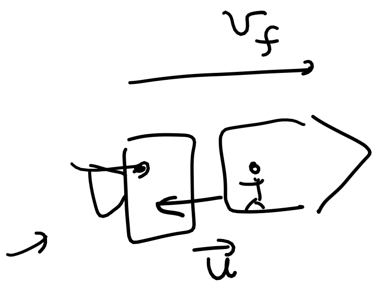
$$\vec{F}_g = G\frac{m_1 m_2}{r^2}\vec{e}_r = m\vec{g}$$

중력 질량

[Ex 2.1.2] $\quad \vec{F} = 0 \qquad \frac{d\vec{p}}{dt} = 0 \rightarrow \vec{p}(t_1) = \vec{p}(t_2)$

$$M = M' + m$$





$$\underset{\sim}{v_f} - u = U$$

$$\vec{p}_i = M\vec{v}_i = \vec{p}_f = M'\vec{v}_f + m\vec{U}$$

$$M v_i = M' v_f + m(v_f - u)$$

$$= (\underbrace{M' + m}_{M}) v_f - mu$$

$$mu = M(v_f - v_i) \rightarrow v_f = v_i + \frac{m}{M}\underset{\sim}{u}$$

2.2. 1차원 운동

$$F = m\underset{\uparrow\,\text{const}}{a} = ma = m\frac{d^2x}{dt^2} \longrightarrow \frac{d^2x}{dt^2} = a$$

$$\frac{dx}{dt} = v \qquad \underbrace{\frac{dv}{dt} = a}$$

$$v = at + v_0$$

$$(v(0) = v_0 = c_1)$$

$$\frac{dx}{dt} = at + v_0$$

$$\downarrow$$

$$\boxed{x = \frac{1}{2}at^2 + v_0 t + x_0}$$

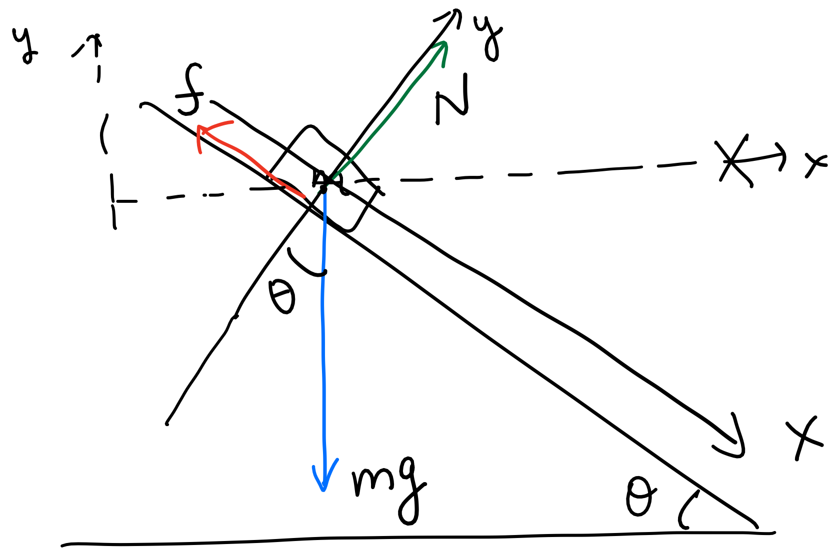
$$(x(0) = x_0)$$

$$x - x_0 = \frac{1}{2}at^2 + v_0 t \qquad t = \frac{v - v_0}{a}$$

$$= \frac{1}{2a}(v^2 - v_0^2)$$

$$\underline{v^2 - v_0^2 = 2a(x - x_0)}$$

[Ex 2.2.1]



$$\vec{F} = (F_x, F_y) = (mg\sin\theta - f,\ N - mg\cos\theta) = m\vec{a}$$
$$= m(\ddot{x}, \ddot{y})$$

$$f = \mu_k N$$

$$y = \dot{y} = \ddot{y} = 0 \Rightarrow N = mg\cos\theta \rightarrow f = \mu_k mg\cos\theta$$

$$\cancel{m}\ddot{x} = \cancel{m}\underbrace{g(\sin\theta - \mu_k\cos\theta)}_{a\,=\,\text{constant}} > 0 \rightarrow \sin\theta > \mu_k\cos\theta$$

$$\tan\theta > \mu_k$$

$$\theta > \underline{\tan^{-1}(\mu_k)} = \theta_1$$

$$f_s = \mu_s N$$

$$\mu_s mg\cos\theta = f_s > mg\sin\theta \rightarrow \text{안미끄러짐}$$

$$\mu_s > \mu_k$$

$$\tan\theta < \mu_s \rightarrow \theta < \underline{\tan^{-1}(\mu_s)}_{\theta_2} \qquad \theta_2 > \theta_1$$

$\theta > \theta_1 \longrightarrow \quad a = g(\sin\theta - \mu_k \cos\theta)$

$$x = \frac{1}{2} a t^2$$

F : const.

$\dot{x} = \dot{x}(t) = \dot{x}(x) = v(x)$

2.3. F : x 의 함수.

$$\underbrace{F(x) = m\ddot{x}}_{\text{2차 미방}} = m\frac{d}{dt}(\dot{x}) = m\frac{d\dot{x}}{dx}\underbrace{\frac{dx}{dt}}_{\dot{x}=v} = m\underbrace{\frac{dv}{dx}\cdot v}_{\frac{1}{2}\frac{d(v^2)}{dx}}$$

$$= \frac{d}{dx}\Big(\underbrace{\frac{1}{2}mv^2}_{T}\Big) = \underbrace{\frac{dT}{dx}}$$

dW : x 에서 x+dx 까지 한 일 (work)   T ← 운동에너지

$$W = \int_{x_0}^{x} dW = \int_{x_0}^{x} F(x)\,dx = \int_{x_0}^{x}\left(\frac{dT}{dx}\right)dx = T(x) - T(x_0) = T - T_0$$

보존력 : 힘들 중 $F = -\dfrac{dV}{dx}$ 로 써지는 힘

(ex) $F = c \rightarrow V = -cx$ $F = -kx \quad V = \frac{1}{2}kx^2, \cdots$

$$W = \int_{x_0}^{x} \left(-\frac{dV}{dx}\right) dx = -\left[V(x) - V(x_0)\right] = T(x) - T(x_0)$$

$$\Downarrow$$

$$\underbrace{T(x)}_{\text{운동}E} + \underbrace{V(x)}_{\text{위치}E} = T(x_0) + V(x_0) = \text{위치에 무관}$$

$$= \text{상수} = \text{보존량} = \text{역학적 에너지} = E$$

$$F \longleftrightarrow V$$

$$E = \frac{1}{2} m v^2 + V(x) \longrightarrow v^2 = \frac{2}{m}\left(E - V(x)\right)$$

$$\frac{dx}{dt} = v = \pm\sqrt{\frac{2}{m}\left(\underset{\uparrow \text{상수}}{E} - V(x)\right)} = f(x)$$

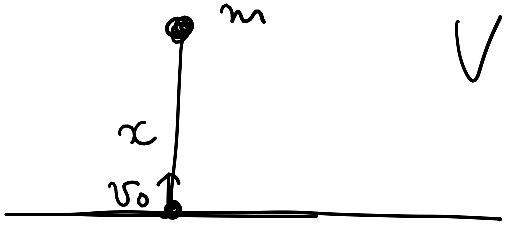
$$\frac{dx}{dt} = f(x)$$ 1차 미방 $\longrightarrow$ 적분.

$$\downarrow$$

$$dx = f(x)\,dt \longrightarrow \frac{dx}{f(x)} = dt$$

$$\longrightarrow \int_{x_0}^{x} \frac{dx}{f(x)} = \int_{t_0}^{t} dt = t - t_0$$

[Ex 2.3.1]



$$V(x) = mgx$$

$$E = \frac{1}{2}mv^2 + mgx = \frac{1}{2}mv_0^2 + mg\cancel{0}$$

$$v^2 - v_0^2 = -2gx$$

$$\frac{dx}{dt} = v = \pm\sqrt{\frac{2}{m}\left(\frac{mv_0^2}{2} - mgx\right)} = \oplus\sqrt{v_0^2 - 2gx}$$

$$\int_0^x \frac{dx}{\sqrt{\underset{\equiv z}{v_0^2 - 2gx}}} = \int_0^t dt = t$$

$$\|$$

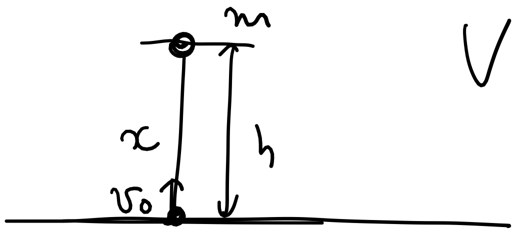
$$dz = -2g\,dx \to dx = -\frac{1}{2g}dz$$

$$-\frac{1}{2g}\int_{v_0^2}^{z}\frac{dz}{\sqrt{z}} \qquad -\frac{1}{2g}\int z^{-\frac{1}{2}}dz = 2\left(-\frac{1}{2g}\right)\left(\sqrt{z} - \sqrt{v_0^2}\right) = t$$

$$\sqrt{v_0^2 - 2gx} = \sqrt{z} = v_0 - gt$$

$$x = v_0t - \frac{1}{2}gt^2 \Leftarrow \text{제곱}\left(\cancel{v_0^2} - 2gx = \cancel{v_0^2} - 2v_0gt + g^2t^2 \right.$$

[Ex 2.3.1] (2)



$$V(x) = mgx$$

$$E = \frac{1}{2} m v^2 + mgx = 0 + mgh$$

$$v^2 - v_0^2 = -2gx$$

$$\frac{dx}{dt} = v = \pm\sqrt{\frac{2}{m}(mgh - mgx)} = \pm\sqrt{2g(h-x)}$$

$$\int_h^x \frac{dx}{\sqrt{h-x}} = -\sqrt{2g}\int_0^t dt = -\sqrt{2g}\, t$$

$$z = h - x$$

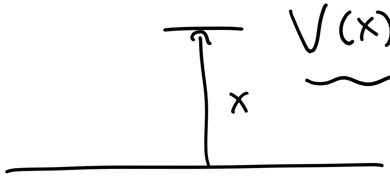
$$dz = -dx$$

$$-\int_0^z \frac{dz}{\sqrt{z}} \qquad -2\sqrt{z} = -\sqrt{2g}\, t \rightarrow z = \frac{1}{2} g t^2 = h - x$$

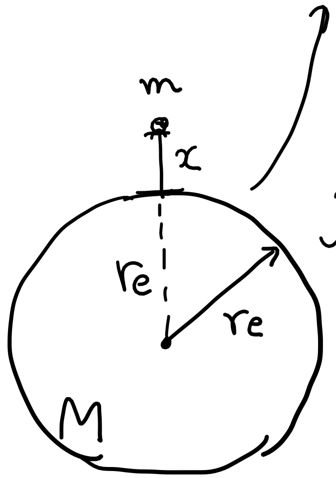
$$\therefore x = h - \frac{1}{2} g t^2$$

//

[Ex 2.3.2]



$$V(x) = mgx$$



$x \ll r_e$

$$F(x) = -\frac{GMm}{r^2} = -\frac{GMm}{(r_e+x)^2}$$

$$r = r_e + x$$

$$F(x) = m\ddot{x} = m v \frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}mv^2\right)$$

$$-\left(V(x)-V(x_0)\right) = \int_{x_0}^{x} \overset{= -\frac{dV}{dx}}{F(x)}\, dx = \int_{x_0}^{x} \frac{d}{dx}\left(\frac{1}{2}mv^2\right) dx = \frac{1}{2}m\underset{v(x)}{v^2} - \frac{1}{2}m\underset{v(x_0)}{v_0^2}$$

$$= -GMm \int_{x_0}^{x} \frac{1}{(r_e+x)^2}\, dx = GMm \left.\frac{1}{x+r_e}\right|_{x_0}^{x}$$

$$= \underbrace{\frac{GMm}{x+r_e}}_{=-V(x)} - \frac{GMm}{x_0+r_e}$$

$$V(x) = -\frac{GMm}{r}$$

$$V(x) = -\frac{GMm}{r_e + x} = -\frac{GMm}{r_e} \cdot \frac{1}{1+\frac{x}{r_e}}$$

$r_e \gg x \quad \frac{x}{r_e} \ll 1$

근사

$$\frac{1}{1+\alpha} \approx 1-\alpha \quad (\text{Taylor 전개})$$

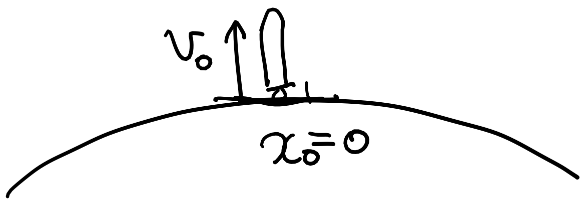
$\alpha \ll 1$

$$V(x) \cong -\frac{GMm}{r_e}\left(1-\frac{x}{r_e}\right) = \underbrace{-\frac{GMm}{r_e}}_{\text{상수}} + \underbrace{\frac{GM}{r_e^2}}_{= g}mx$$

$= g = 9.8\,m/s^2$

$$\rightarrow \quad E = \frac{1}{2}mv^2 - \frac{GMm}{r_e + x} = \frac{1}{2}mv_0^2 - \frac{GMm}{r_e + x_0} = \text{상수}$$

$$= \frac{1}{2}mv_0^2 - \frac{GMm}{r_e}$$

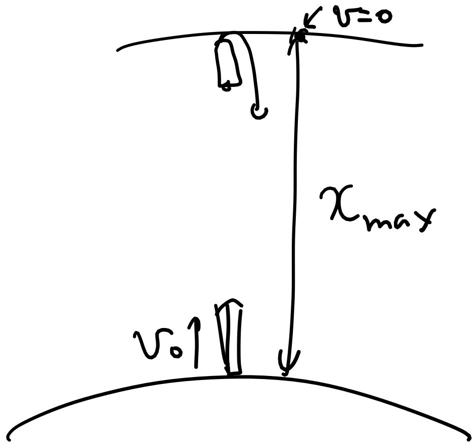


$$v^2 - v_0^2 = 2\underbrace{GM}_{g r_e^2}\left(\frac{1}{r_e + x} - \frac{1}{r_e}\right)$$

$$= 2g\left(r_e^{\cancel{2}} \cdot \frac{(-x)}{\cancel{r_e}(r_e + x)}\right)$$

$$v^2 - v_0^2 = 2g\left(r_e^{\cancel{2}} \cdot \frac{(-x)}{\cancel{r_e}(r_e+x)} \right) = -2g \frac{x}{1+\frac{x}{r_e}} = -2gx\left(1+\frac{x}{r_e}\right)^{-1}$$

$$v^2 = v_0^2 - 2gx\left(1+\frac{x}{r_e}\right)^{-1}$$



$v=0 \rightarrow x = x_{max}$

$$= \underline{v_0^2 - 2g x_{max}\left(1+\frac{x_{max}}{r_e}\right)^{-1} = 0}$$

$$\frac{x_{max}}{r_e} \equiv y \qquad v_0^2 - 2gr_e \underbrace{\left(\frac{y}{1+y}\right)}_{1-\frac{1}{1+y}} = 0$$

$$\therefore \quad \frac{v_0^2}{2gr_e} - 1 + \frac{1}{1+y} = 0$$

$$\frac{1}{1+y} = 1 - \frac{v_0^2}{2gr_e} \leftarrow$$

$$\frac{x_{max}}{r_e} = y = \left(1 - \frac{v_0^2}{2gr_e}\right)^{-1} - 1$$

$$\therefore \quad x_{max} = r_e\left[\left(1 - \frac{v_0^2}{2gr_e}\right)^{-1} - 1\right] \; //$$

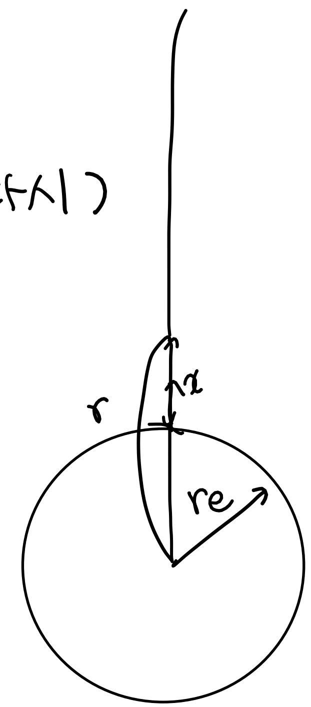
$$x_{max} = \infty \longrightarrow 1 - \frac{v_0^2}{2gr_e} = 0 \longrightarrow v_e =$$ 탈출속도

$$v_e = \sqrt{2gr_e} = \sqrt{\frac{2MG}{r_e}}$$

g: 9.8 m/s, r_e: 6.4×10^6 m

$$= 11 \times 10^3 \text{ m/s}$$

Ex (다시)



$$F(r) = -\frac{GMm}{r^2} = m\ddot{x} = m\ddot{r}$$

$$r = r_e + x \longrightarrow \ddot{r} = \ddot{x}$$

$$\dot{r} = v = \dot{x}$$

$$\ddot{r} = \frac{dv}{dt} = \frac{dr}{dt}\frac{dv}{dr} = v\frac{dv}{dr}$$

$$v\frac{dv}{dr} = \ddot{r} = -\frac{GM}{r^2}$$

$$\parallel$$

$$\frac{1}{2}\frac{d}{dr}(v^2)$$

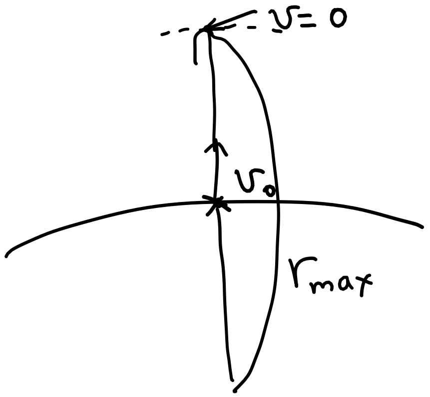
$$\longrightarrow \int_{r_0}^{r} \left(\frac{dv^2}{\cancel{dr}}\right)\cancel{dr} = -2GM \underbrace{\int_{r_0}^{r} \frac{1}{r^2}\,dr}_{-\frac{1}{r}+\frac{1}{r_0}}$$

$$\parallel$$

$$\int_{v_0^2}^{v^2} 1\, d(v^2) = v^2 - v_0^2$$

$$\therefore\ v^2 - v_0^2 = -2GM\left(\frac{1}{r_0} - \frac{1}{r}\right)$$

$$v^2 - v_0^2 = -2GM\left(\frac{1}{r_0} - \frac{1}{r}\right)$$



$r = r_e + \underset{\sim}{x}$

$r_0 = r_e$

$$0 - v_0^2 = -2GM\left(\frac{1}{r_e} - \frac{1}{r_{max}}\right)$$

$$\frac{1}{r_{max}} = \frac{1}{r_e} - \frac{v_0^2}{2GM}$$

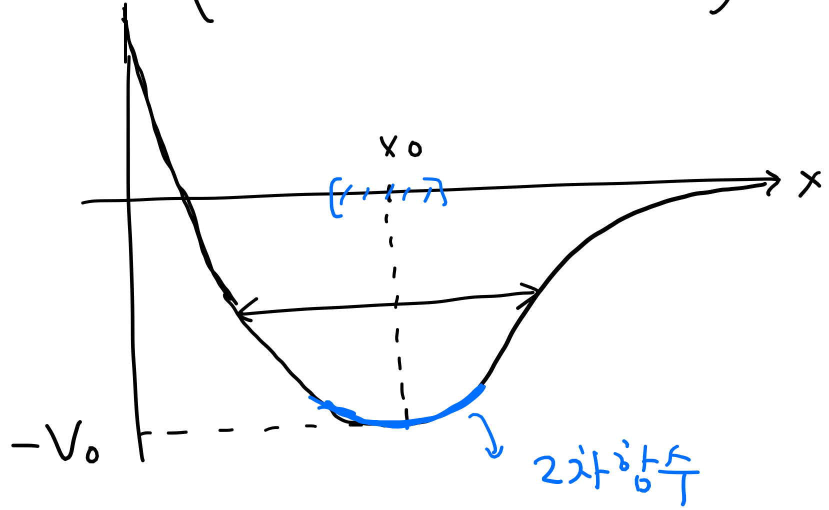
$$r_{max} = \Big(\underbrace{\frac{1}{r_e} - \frac{v_0^2}{2GM}}\Big)^{-1} = r_e + x_{max}$$

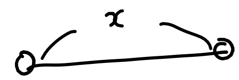
탈출속도 : $r_{max} = \infty$

$$v_e = \sqrt{\frac{2GM}{r_e}} = \sqrt{2gr_e} \quad //$$

$$GM = gr_e^2$$

[Ex 2.3.3] $V(x) = V_0 \left(1 - e^{-(x-x_0)/\delta}\right)^2 - V_0$





$F(x) = -\frac{dV}{dx} = 0$ 평형

$= -\cancel{V_0}\,\cancel{2}\left(1-\underbrace{e^{-(x-x_0)/\delta}}_{=1}\right)\cancel{\frac{1}{\delta}}\,\cancel{e^{-(x-x_0)/\delta}} = 0$

$1 \longrightarrow x = x_0$

[Ex 2.3.4] $|x - x_0| \ll \delta$ $\qquad e^{-\varepsilon} \approx 1 - \varepsilon \left(+ \frac{\varepsilon^2}{2} + \cdots\right)$

$\frac{x-x_0}{\delta} \equiv \varepsilon \qquad 1 - e^{-\varepsilon} \approx \varepsilon$

$V(x) \approx V_0 \frac{(x-x_0)^2}{\delta^2} - V_0 \rightarrow F = -\frac{2V_0}{\delta^2}(x-x_0)$

① $$\ddot{x} = \frac{F(x)}{m} \longrightarrow v\frac{dv}{dx} = \frac{1}{m}F(x) = \frac{1}{2}\frac{dv^2}{dx} = -\frac{1}{m}\frac{dV}{dx}$$

$$\ddot{x} = \frac{d}{dt}(\dot{x}) = \frac{d}{dt}(v) = \frac{dx}{dt}\frac{dv}{dx}, \qquad \frac{dx}{dt} = v$$

② $F(v)$ 인 경우

$$\underline{\frac{dv}{dt} = \ddot{x} = \frac{1}{m}F(v)} \longrightarrow dv = \frac{1}{m}F(v)\,dt$$

$$\downarrow$$

$$G(v) \cancel{-G(v_0)} = \int_{v_0}^{v} \frac{dv}{F(v)} = \frac{1}{m}\int_0^t dt = \frac{t}{m} \longleftarrow \frac{dv}{F(v)} = \frac{1}{m}\,dt$$

$$G = \int \frac{dv}{F(v)}$$

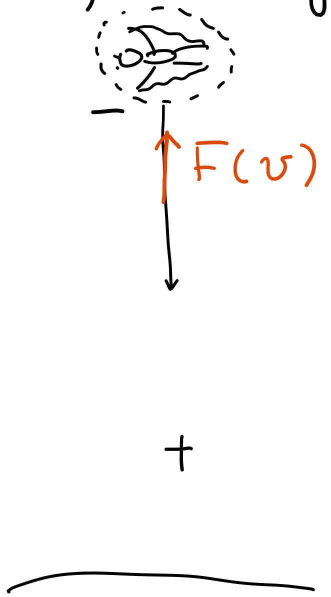
$$\downarrow$$

$$G(v) = G(v_0) + \frac{t}{m}$$

$$\downarrow$$

$$v = G^{-1}\left(G(v_0) + \frac{t}{m}\right) = \frac{dx}{dt}$$

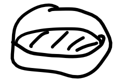
공기중 마찰력 (drag force)



$$\frac{dv}{dt} = a = g \rightarrow v = gt + 0$$

종단 속도 (terminal velocity)

$$m\frac{dv}{dt} = F(v) = -(c_1 v + c_2 \underbrace{v^2}_{v|v|})$$



$c_1 = 1.55 \times 10^{-4} D$

m단위

$c_2 = 0.22 D^2$

$$\frac{c_2 v^2}{c_1 v} = 1.4 \times 10^3 D v$$

(1) $\ll 1 \rightarrow c_1 v$ 만 고려 가능
$D \cdot v \ll 10^{-3}$ (ex) 이슬)

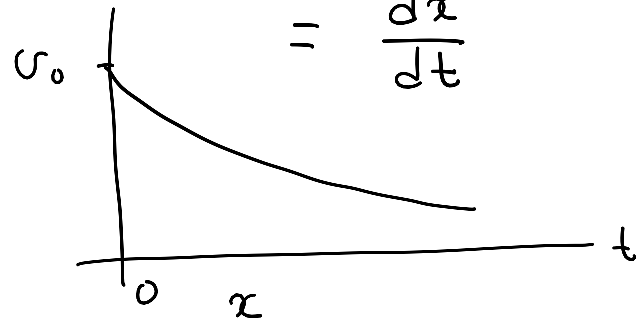
(2) $\gg 1 \rightarrow c_2 v^2$ 〃 $Dv \gg 10^{-3}$ (ex) 야구공)

(3) 그 외 → 수치적

[Ex 2.4.1] ① 경우 $m\frac{dv}{dt} = -c_1 v$

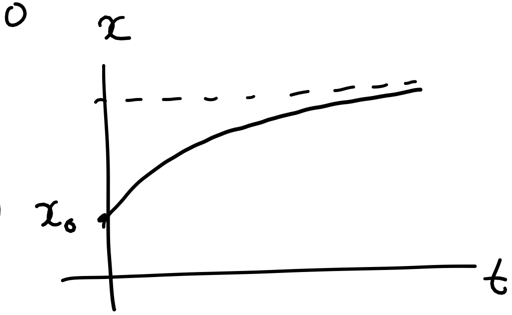
$$\log\frac{v}{v_0} = \underbrace{\ln}_{\log_e} v\Big|_{v_0}^{v} = \int_{v_0}^{v}\frac{dv}{v} = -\frac{c_1}{m}\int_0^t dt = -\frac{c_1}{m}t \longrightarrow v(t) = v_0 e^{-\frac{c_1}{m}t} = \frac{dx}{dt}$$

(~~Log~~)



$$x - x_0 = v_0\int_0^t e^{-\frac{c_1}{m}t}\,dt$$

$$x = x_0 + \frac{v_0 m}{c_1}\left(1-e^{-\frac{c_1}{m}t}\right)$$



[Ex 2.4.2] ② 경우 $m\frac{dv}{dt} = -c_2 v^2$

$$\frac{1}{v_0}-\frac{1}{v} = \int_{v_0}^{v}\frac{dv}{v^2} = -\frac{c_2}{m}\int_0^t dt = -\frac{c_2}{m}t \longrightarrow \frac{1}{v} = \frac{1}{v_0}+\frac{c_2}{m}t$$

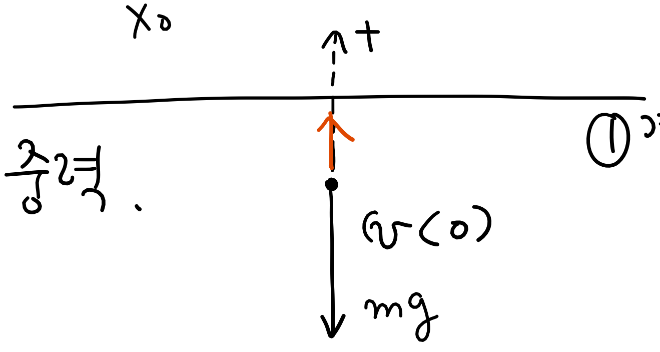
$$v = \frac{1}{\frac{1}{v_0}+\frac{c_2}{m}t} = \frac{v_0}{1+\underbrace{\frac{c_2 v_0}{m}}_{k}t}$$

$$v = \frac{v_0}{1+kt} = \frac{dx}{dt} \longleftarrow$$

$$x - x_0 = \int_{x_0}^{x} \frac{dx}{\cancel{dt}}\cancel{dt} = \int_0^t \frac{v_0}{1+kt}\,dt = \frac{v_0}{k}\ln(1+kt)$$

$$\therefore\ x = x_0 + \frac{v_0}{k}\ln(1+kt)$$

중력.



① 경우

$$m\frac{dv}{dt} = \underbrace{-c_1 v}_{+}\ \underbrace{-\,mg}_{-} = -c_1\left(v + \frac{mg}{c_1}\right)$$

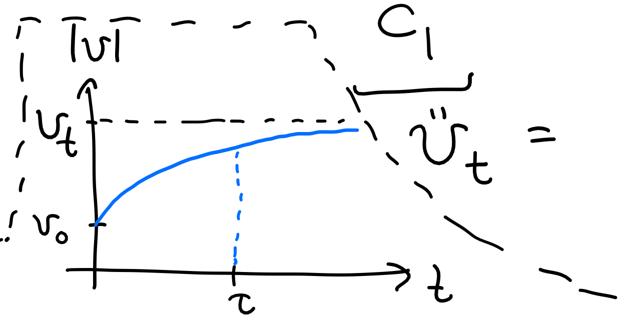
$$\log\frac{v + \frac{mg}{c_1}}{v_0 + \frac{mg}{c_1}} = \int_{v_0}^{v}\frac{dv}{v + \frac{mg}{c_1}} = -\frac{c_1}{m}\int_0^t dt = -\frac{c_1}{m}t$$

$$\therefore\ \boxed{v = -\frac{mg}{c_1} + \left(v_0 + \frac{mg}{c_1}\right)e^{-\frac{c_1}{m}t}}$$

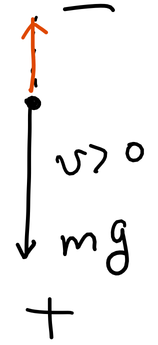
$$t \gg \frac{m}{c_1} = \tau \qquad e^{-\frac{c_1 t}{m}} \approx 0 \qquad \frac{c_1 t}{m} \gg 1$$

$$v \cong -\frac{mg}{c_1} = \underbrace{\qquad}_{}\ \ v_t = \frac{mg}{1.55\times10^{-4}D}$$

$$v = -v_t + (v_0 + v_t)e^{-\frac{t}{\tau}}$$



중력 + ② 경우



$$m\frac{dv}{dt} = mg - C_2 v^2 = -C_2\left(v^2 - \underbrace{\frac{mg}{C_2}}_{\equiv\, v_t^2}\right)$$

$$\rightarrow v_t = \sqrt{\frac{mg}{C_2}}$$

$C_2 = 0.22 \times D^2$

$v_t \propto \frac{1}{D}$

$$\int_{v_0}^{v} \frac{dv}{v^2 - v_t^2} = -\frac{C_2}{m}\int_0^t dt = -\frac{C_2}{m}\, t$$

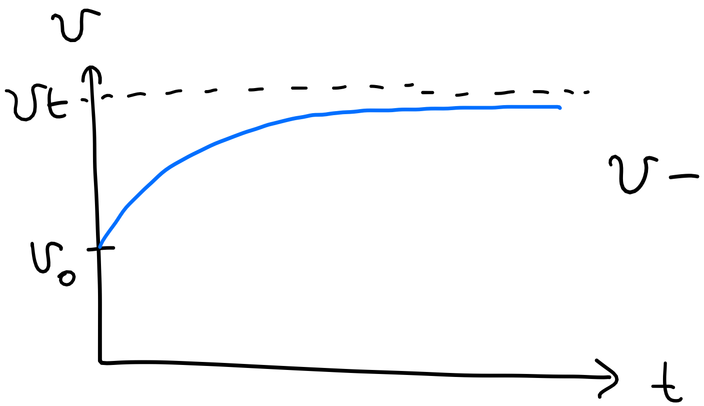
$$\frac{1}{(v-v_t)(v+v_t)} = \frac{1}{2v_t}\left(\frac{1}{v-v_t} - \frac{1}{v+v_t}\right)$$

$$\ln \frac{v-v_t}{v+v_t}\Bigg|_{v_0}^{v} = -\frac{2C_2}{m} v_t\, t$$

$$\therefore \frac{v-v_t}{v+v_t} = \underbrace{\frac{v_0-v_t}{v_0+v_t}}_{C_0}\, e^{-\frac{2C_2 v_t}{m} t} = C_0\, e^{-\frac{2t}{\tau}} \equiv \alpha, \qquad \frac{2C_2 v_t}{m} \equiv \frac{2}{\tau}$$

$$v - v_t = \alpha(v + v_t) \rightarrow (1-\alpha)v = (1+\alpha)v_t$$

$$\therefore v = \frac{1 + C_0\, e^{-2t/\tau}}{1 - C_0\, e^{-2t/\tau}}\, v_t$$



$v_0 = 0 \longrightarrow C_0 = -1$

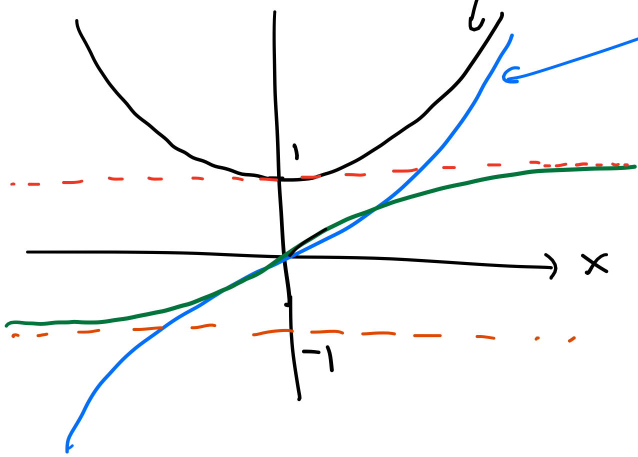
$$v = \frac{1 - e^{-2t/\tau}}{1 + e^{-2t/\tau}}\, v_t$$

$$= \frac{e^{2t/\tau} - 1}{e^{2t/\tau} + 1}\, v_t = \frac{e^{t/\tau} - e^{-t/\tau}}{e^{t/\tau} + e^{-t/\tau}}\, v_t$$

$v_t = v_t \tanh\left(\frac{t}{\tau}\right) = v$

"hyperbolic"

$$\boxed{\frac{e^x + e^{-x}}{2} = \cosh x, \quad \frac{e^x - e^{-x}}{2} = \sinh x, \quad \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x} = \tanh x}$$



$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sqrt{-1} = i$$

$$x \longrightarrow i\theta$$

$$\frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta = \cosh(i\theta)$$

$$\frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta = \frac{1}{i}\sinh(i\theta)$$

$$F(v) = -c_1 v - c_2 v|v|$$

MATLAB

$$\frac{dx}{dt} = f(x)$$

$$x(t=0) = x_0$$

(ex) $$\frac{dv}{dt} = g - c_1 v$$

$$v(0) = v_0$$

[t, x] = ode45(@name, t, x0)

ordinary differential equation