11장. Spontaneous symmetry Breaking (SSB)

Weinberg-Salam : Higgs mechanism : SSB + $\underline{\text{gauge}}$ symmetry

SSB → Goldstone boson; gauge (local) ⇒ $W^{\pm}, Z^0$ become massive.

G. t' Hooft + M. Veltman → dimensional regularization

Benjamin Lee (이휘소)

} SSB renormalization

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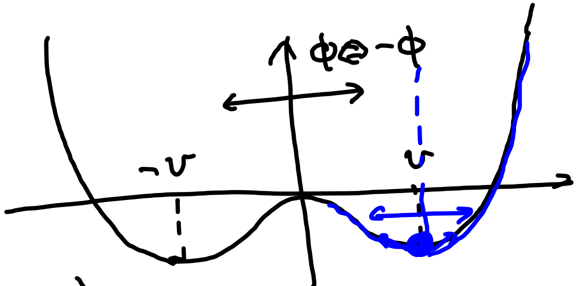
11. 1. SSB

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4$$

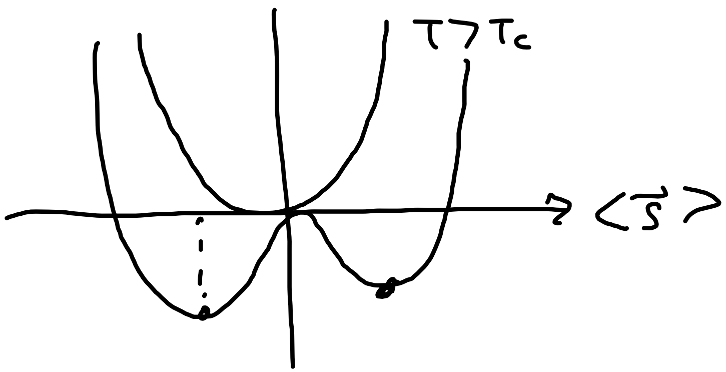
$m^2 = -\mu^2 < 0$



$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4 \rightarrow \phi_0 = \pm v = \pm\sqrt{\frac{6}{\lambda}}\,\mu$$

$$G = \frac{1}{2}\vec{S}\,\vec{\nabla}^2\vec{S}(\vec{x}) \overbrace{- A\vec{S}^2 - \frac{\lambda}{4!}\vec{S}^4}^{V(\vec{S})}$$

$A \leftarrow a_0 (T - T_c)$



__SSB__: $\mathcal{L}$ is still symmetric but Ground state (vacuum) is broken.

$\phi(x) = v + \sigma(x)$ $\qquad \sigma \to -\sigma(x)$ $\quad$ ($\phi \leftrightarrow -\phi$; Z_2 discrete)

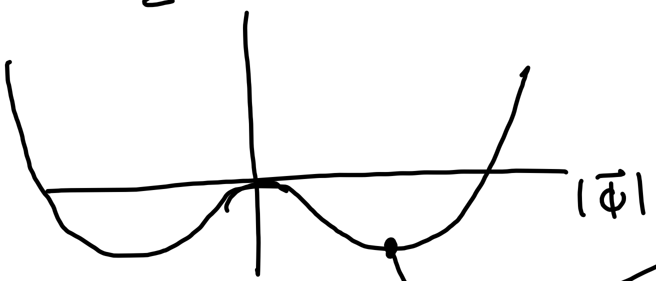
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 - \sqrt{\frac{\lambda}{6}}\,\mu\,\sigma^3 - \frac{\lambda}{4!}\sigma^4$$

Linear $\boxed{\sigma\text{-model}}$: $\phi^i \qquad i = 1, \dots, N$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}\left((\phi^i)^2\right)^2$$

Global sym: $O(N)$: $\phi^i \to R^{ij}\phi^j \to (\phi^i)^2 = \vec{\Phi}^T \cdot \vec{\Phi}$ invariant.

$$V(\phi^i) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}(\phi^2)^2 \to |\vec{\Phi}| = \sqrt{\frac{\mu^2}{\lambda}} \equiv v$$



$$\vec{\phi} = (\overbrace{0, 0, \cdots, 0}^{N-1}, v)$$

$\phi_i = 0, \quad \phi_N = v$

$$\vec{\phi}(x) = (\overbrace{\underset{\phi_1}{\pi_1}, \cdots\cdots, \underset{\phi_{N-1}}{\pi_{N-1}}}^{N-1}, \underbrace{v+\sigma(x)}_{\phi_N})$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\pi})^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}\overset{m^2}{\underline{\underline{(2\mu^2)}}}\sigma^2 - \sqrt{\lambda}\,\mu\,\sigma^3 - \sqrt{\lambda}\,\mu(\vec{\pi})^2\sigma - \frac{\lambda}{4}\sigma^4$$
$$- \frac{\lambda}{2}\vec{\pi}^2\sigma^2 - \frac{\lambda}{4}((\vec{\pi})^2)^2$$

$$\pi^i \to \underbrace{R^{ij}}_{O(N-1)}\pi^j$$

(N−1) 7H

SSB; $O(N) \longrightarrow O(N-1)$ $\therefore\ \vec{\pi}$ massless

Goldstone mechanism.

$$\frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2} = \text{dim of broken sym.} = \boxed{N-1}$$

Goldstone theorem

If Continuous sym is broken (SSB), GB for each broken dir.

(pf) $\mathcal{L} = (\text{K.T.}) - V(\phi)$; vacuum

$$\left.\frac{\partial V}{\partial \phi^a}\right|_{\phi_0} = 0$$

$$V = V(\phi_0) + \frac{1}{2}(\phi-\phi_0)^a(\phi-\phi_0)^b \underbrace{m^2_{ab}}_{\left.\frac{\partial^2 V}{\partial\phi^a\partial\phi^b}\right|_{\phi=\phi_0}} + \cdots$$

if $\underline{\phi^a \to \phi^a + \alpha\,\Delta^a(\phi)}$ is a sym of $\mathcal{L}$

$$V\left(\phi^a + \alpha\,\Delta^a(\phi)\right) = V(\phi^a) \Rightarrow \sum_a \Delta^a(\phi)\frac{\partial V}{\partial \phi^a} = 0$$

$$\left.\frac{\partial}{\partial\phi^b}\left(\sum_a \Delta^a(\phi)\frac{\partial V}{\partial\phi^a}\right)\right|_{\phi=\phi_0} = 0 = \cancel{\left.\frac{\partial\Delta^a}{\partial\phi^b}\right|_{\phi_0}\underbrace{\left(\frac{\partial V}{\partial\phi^a}\right)_{\phi_0}}_{=0}} + \Delta^a(\phi_0)\overbrace{\left.\frac{\partial^2 V}{\partial\phi^b\phi^a}\right|_{\phi=\phi_0}}^{m^2_{ab}}$$

$$(m^2)\left(\vec{\Delta}\right) = 0$$

if vac. is sym: $\phi^a_0 \to \phi^a_0 + \underbrace{\alpha\,\Delta^a(\phi_0)}_{=0} \to \Delta^a(\phi_0) = 0$

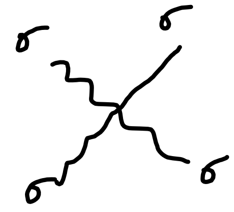
if vac is SSB: $\Rightarrow \Delta^a(\phi_0) \neq 0 \Rightarrow \det m^2_{ab} = 0$

$\hookrightarrow m^2$ eigenvalue $= 0$

$\vec{\phi} = (\vec{\pi}, \sigma + v)$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})^2 + \frac{1}{2}\mu^2 (\vec{\phi})^2 - \frac{\lambda}{4}\left((\vec{\phi})^2\right)^2$$

$$+ \frac{1}{2}\delta_Z \underbrace{(\partial_\mu \vec{\phi})^2}_{(\partial_\mu \vec{\pi})^2 + (\partial_\mu \sigma)^2} - \frac{1}{2}\delta_\mu \underbrace{\vec{\phi}^2}_{\vec{\pi}^2 + (\sigma+v)^2} - \frac{\delta_\lambda}{4}\underbrace{(\vec{\phi}^2)^2}_{(\vec{\pi}^2 + (\sigma+v)^2)^2}$$



$-6i\lambda$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\pi})^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}\overbrace{(2\mu^2)}^{m^2}\sigma^2 - \sqrt{\lambda}\,\mu\,\sigma^3 - \sqrt{\lambda}\,\mu(\vec{\pi})^2\sigma - \frac{\lambda}{4}\sigma^4$$

$$- \frac{\lambda}{2}\vec{\pi}^2\sigma^2 - \frac{\lambda}{4}\left((\vec{\pi})^2\right)^2$$

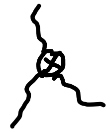
$$+ \frac{\delta_Z}{2}(\partial_\mu \pi^k)^2 + \frac{\delta_Z}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(\delta_\mu + v^2\delta_\lambda)\vec{\pi}^2 - \frac{1}{2}(\delta_\mu + 3\delta_\lambda v^2)\sigma^2$$

$$- (\delta_\mu v + \delta_\lambda v^3)\sigma - \delta_\lambda v \sigma \vec{\pi}^2 - \delta_\lambda v\, 3!\, \frac{\sigma^3}{3!}$$

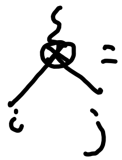
$$- \frac{\delta_\lambda}{4}(\vec{\pi}^2)^2 - \frac{\delta_\lambda}{2}\sigma^2\vec{\pi}^2 - \frac{3!\,\delta_\lambda}{4!}\sigma^4$$

σ : ~~~~

π^i : i —— i



$= -6iv\delta_\lambda$

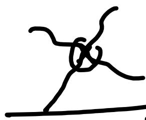


$= -2iv\delta^{ij}\delta_\lambda$

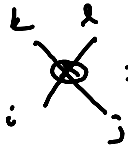
$\otimes$~~ : $-i(\delta_\mu v + \delta_\lambda v^3)$

~~$\otimes$~~ : $-i(\delta_\mu + 3\delta_\lambda v^2 + \delta_Z p^2)$

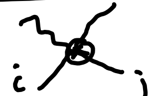
—$\otimes$— : $-i(\delta_\mu + v^2\delta_\lambda + p^2\delta_Z)$



$= -i\,6\,\delta_\lambda$



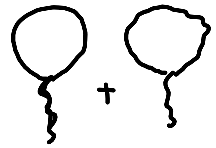
$= -2i\delta_\lambda(\delta^{il}\delta^{jk} + \text{cyclic})$



$= -2i\delta^{ij}\delta_\lambda$

Renormalization condition

$$\langle \phi^N \rangle = v \longrightarrow \phi^N = v + \sigma \to \langle \sigma \rangle = 0$$



$= 0$

$m^2 = 2\mu^2 \to$

$$= \frac{i}{p^2-m^2} + \frac{-i\Sigma(p^2)\cdot i}{p^2-m^2} + \cdots = \frac{i}{p^2-m^2}\,\frac{1}{1-\frac{\Sigma}{p^2-m^2}}$$

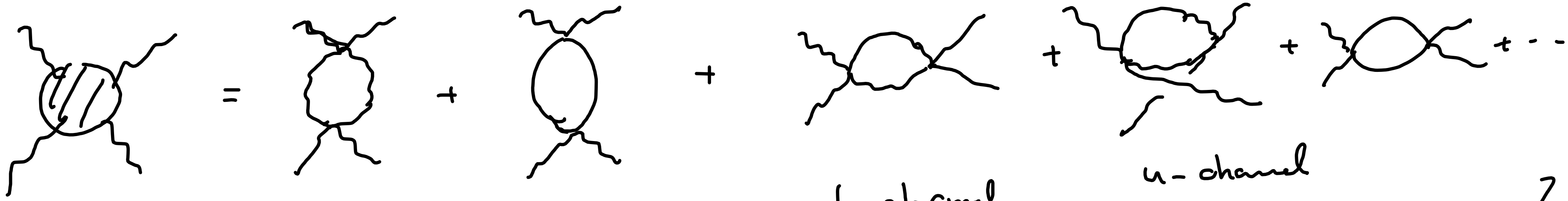
$$= \frac{i}{p^2-m^2-\Sigma(p^2)} \qquad \Sigma(p^2) = \overset{0}{\overset{\shortparallel}{\Sigma(m^2)}} + \Sigma'(m^2)(p^2-m^2)$$

$$\frac{d}{dp^2}\left(\sim \text{1PI} \sim\right)\Big|_{p^2=m^2} = -i\Sigma'(p^2)\Big|_{p^2=m^2} = 0$$

$$= \frac{i}{p^2-m^2}\cdot\underbrace{\frac{1}{1-\Sigma'(m^2)}}_{=1}$$

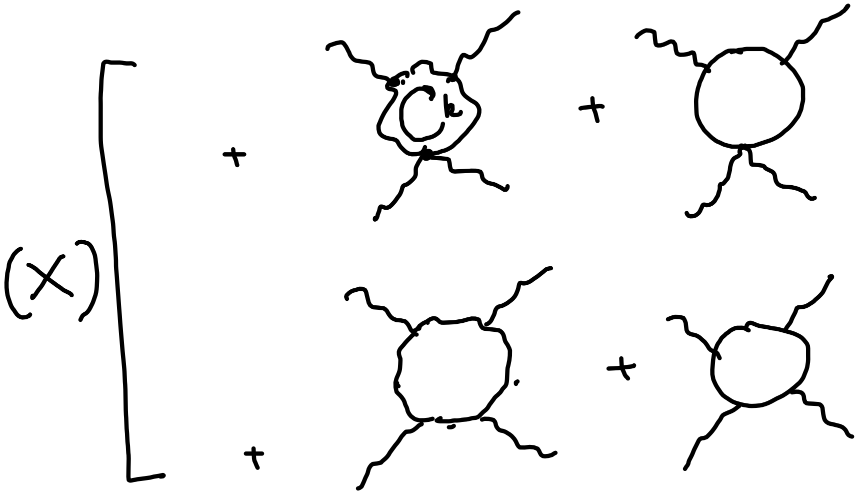
Vertex counter

(1)



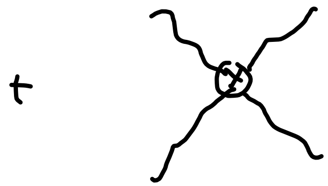
t- channel　　u- channel

s - channel　　cross channel



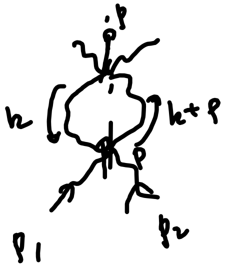
$$\to \int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2+\cdots}\,\frac{i}{k^2+\cdots}\,\frac{i}{k^2+\cdots} \sim \Omega_4 \int_0^\infty \frac{k^3\,dk}{k^6} \sim \text{finite}$$

$$\sim \int \frac{k^3\,dk}{k^8} \sim \text{finite}$$



$$-6i\,\delta\lambda \quad + \quad 6i\lambda^2(N+8)\,\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2} = 0$$

$$\therefore\quad \underline{\delta\lambda = \lambda^2(N+8)\,\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2} + \text{finite.}}\ \checkmark$$



$$= \frac{1}{2}(-6i\lambda)^2 \int \frac{d^dk}{(2\pi)^d}\, \frac{i}{k^2-m^2}\, \frac{i}{(k+p)^2-m^2}$$

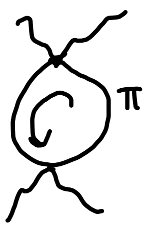
$$= 18\lambda^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\, \frac{1}{(k^2-\Delta^2)^2} \qquad \left(\Delta^2 = m^2(1-x) + 2x(1-x)\right)$$

$$i\int \frac{d^dk_E}{(2\pi)^d}\, \frac{1}{(k_E^2+\Delta^2)^2}$$

$$= 18\lambda^2\, i\int_0^1 dx\, \frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{d/2}} \underbrace{\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}} \approx \underline{18 i\lambda^2 \frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^2}} + \text{finite}$$

$$e^{-\underbrace{\left(2-\frac{d}{2}\right)}_{\varepsilon}\ln\Delta} \approx \underline{\underline{1}} - \underline{2\varepsilon \ln\Delta + \ldots}$$

$$\Gamma(\varepsilon) \sim \frac{1}{\varepsilon}$$



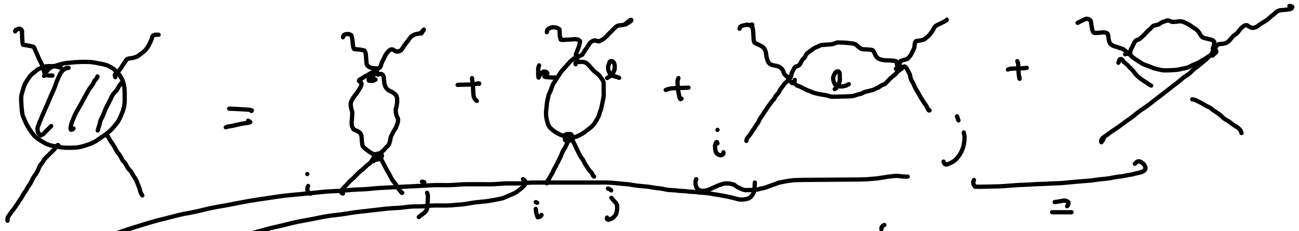
$$\pi^{i=1-(N-1)} = \underbrace{\frac{1}{2}(-2i\lambda)^2}_{2\lambda^2 i (N-1)} \overset{i\cdot i^2}{(N-1)} \frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^2} + \text{finite}$$

+)

$$= 2i\lambda^2(N+8)\, \frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^2} + \text{finite}$$

+

crosschenels $\times 3$

② 

$$\frac{1}{2}(-6i\lambda)(-2i\lambda\,\delta^{ij})\frac{-i}{(4\pi)^2}\Gamma\left(2-\frac{d}{2}\right) = 6i\lambda^2\delta^{ij}\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^2}$$

$$\frac{1}{2}(-2i\lambda\,\underbrace{\delta^{kl}})(-2i\lambda(\underbrace{\delta^{ik}\delta^{jl}}_{\delta^{ij}}+\underbrace{\delta^{ij}\delta^{kl}}_{\delta^{ij}\delta^{kl}\delta^{kl}}+\underbrace{\delta^{il}\delta^{kj}}_{\delta^{ij}}))\frac{-i}{(4\pi)^2}\Gamma\left(2-\frac{d}{2}\right) = 2i\lambda^2(N+1)\delta^{ij}\times\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^2}$$

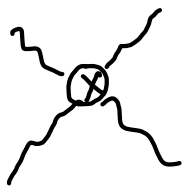
$$\sum_{k=1}^{N-1}\delta^{kk} = N-1$$

$$(N+1)\,\delta^{ij}$$

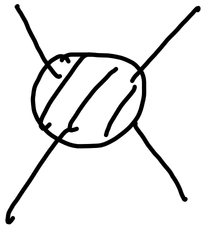
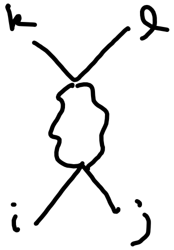
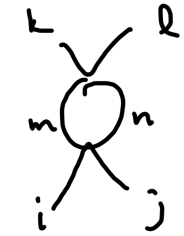
$$(-2i\lambda\,\delta^{il})(-2i\lambda\,\delta^{lj})\frac{-i}{(4\pi)^2}\Gamma\left(2-\frac{d}{2}\right) = 4i\lambda^2\delta^{ij}\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^2}$$

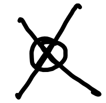
$\times 2$ ($t + u$ chanels)

$$+ \qquad \delta^{ij}\,2i\lambda^2(N+1+3+4)\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^2} = \delta^{ij}\,2i\lambda^2(N+8)\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^2}$$

 $= -2i\,\delta_\lambda\,\delta_{ij} \qquad +$

$$\Rightarrow \quad \underline{\delta_\lambda = \lambda^2(N+8)\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^2}} \quad ✓$$

③  $=$  $+$  $+$ cross chanels

$+$ 

$-2i\delta\lambda(\delta^{ik}\delta^{jl}+\text{cyclic})$

$$= \underbrace{\frac{1}{2}(-2i\lambda\,\delta^{ij})(-2i\lambda\,\delta^{kl})}_{2i\lambda^2\,\delta^{ij}\delta^{kl}\;\cdot\,\frac{-i}{(4\pi)^2}}\frac{-i}{(4\pi)^2}\Gamma\left(2-\frac{d}{2}\right)$$

$$+\frac{1}{2}\left((-2i\lambda)^2(\delta^{ij}\delta^{mn}+\text{cyclic})\cdot(\delta^{kl}\delta^{mn}+\text{cyclic})\right)\frac{-i}{(4\pi)^2}\Gamma\left(2-\frac{d}{2}\right)$$

$$\left(= 2i\lambda^2\left((N+3)\,\underline{\delta^{ij}\delta^{kl}}+2\delta^{ik}\delta^{jl}+2\delta^{il}\delta^{kj}\right)\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2}\right)$$

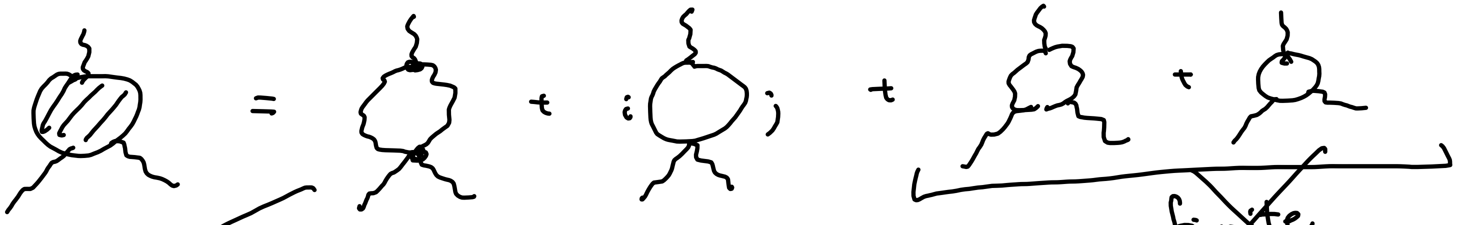
$$+\Big\} = 2i\lambda^2\left((N+4)\,\delta^{ij}\delta^{kl}+2\delta^{ik}\delta^{jl}+2\delta^{il}\delta^{kj}\right)\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2}$$

$+$ cross $\Big)$ " $(i\;j\;k\;l$ cyclic$)$

		$(N+4)$	2
t :	2		
			$(N+4)$
u	2	2	

$$2i\lambda^2\,(N+8)\left(\delta^{ij}\delta^{kl}+\text{cyclic}\right)\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2}$$

//

④ 

~~finite~~

\+ cross channel

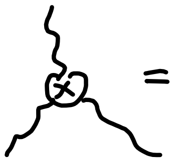
$$\frac{1}{2}\left(-6i\lambda v\right)\left(-6i\lambda\right) i^3 \frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^2} = 18\, i \lambda^2 v \frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^2} + \text{finite}$$

$$\frac{1}{2}\left(-2i\lambda \delta^{ij}\right)\left(-2i\lambda v \delta^{ij}\right) \quad " \quad = 2i\lambda^2 v (N-1) \quad "$$

$(N-1)$

$$3 \times 2i\lambda^2 v\,(N+8)\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^2} + \cdots$$

↑ s, t, u chanels

\+  $= -6iv\delta_\lambda$

$$\therefore \quad \delta_\lambda = \lambda^2 (N+8) \frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^2} + \cdots \quad \checkmark$$

$\Rightarrow$ δ_μ, δ_λ, δ_Z renormalize all divergences

11.3. Effective action

Stat Mech.

with ext. field $H(x)$

$$Z(H) \equiv e^{-\beta F(H)} = \int \mathcal{D}s \; e^{-\beta \int dx \, (\mathcal{H}[s] - H s)} \qquad \beta = \frac{1}{k_B T}$$

$$F = -\frac{1}{\beta} \ln Z$$

$$-\frac{\partial F}{\partial H}\bigg|_{\beta} = \frac{1}{\cancel{\beta}} \frac{1}{Z} \int \mathcal{D}s \; \cancel{\beta} \int s\,dx \; e^{-\beta(\quad)} = \int \langle s \rangle dx \equiv M$$

$$G(M) = F(H) + MH$$

$$\frac{\partial G}{\partial M} = \cancel{\underbrace{\frac{\partial F}{\partial H}}_{-M} \frac{\partial H}{\partial M}} + \cancel{M \frac{\partial H}{\partial M}} + H = H$$

QFT

$$Z[J] = e^{-iE[J]} = \int \mathcal{D}\phi \; e^{i \int d^4x \, (\mathcal{L} + J\phi)}$$

$$\underset{=}{\frac{\delta E[J]}{\delta J(x)}} = i \frac{\delta \ln Z}{\delta J} = \frac{i}{Z} \int \mathcal{D}\phi \; i \underset{\sim}{\phi} \; e^{i \int d^4x \cdots} = \underbrace{-\langle \Omega | \phi | \Omega \rangle}_{\phi_{cl}}$$

$$\Gamma[\phi_{cl}] \equiv -E[J] - \int d^4y \, J(y)\, \phi_{cl}(y)$$

$$\frac{\delta \Gamma}{\delta \phi_{cl}(x)} = -\underbrace{\frac{\delta E}{\delta \phi_{cl}}} - \int d^4y \left(\cancel{\frac{\delta J(y)}{\delta \phi_{cl}(x)} \phi_{cl}(y)} + \underline{J(y)\, \delta^4(x-y)} \right) = -J(x)$$

$$= \int d^4y \, \cancel{\frac{\delta J(y)}{\delta \phi_{cl}(x)}} \, \underbrace{\frac{\delta E[J]}{\delta J(y)}}_{= \phi_{cl}(y)}$$

	Stat. Mech	QFT
dim	$\vec{x}$	$x = (t, \vec{x})$
order	$S(\vec{x})$, M	$\phi(x)$, ϕ_{cl}
external	H	$J(x)$
	$\mathcal{H}$	$\mathcal{L}$
p.F	$Z(H)$	$Z[J]$
free energy	F, G	$E[J]$, $\Gamma[\phi_{cl}]$

if $|\Omega\rangle$ is invariant under translation, + Lorentz transf.

$$\phi_{cl} = \langle\Omega|\underbrace{\phi(x)}|\Omega\rangle = \text{const.}$$

$$e^{i\underset{\sim}{P}\cdot x}\,\phi(0)\,e^{-iP\cdot x}$$

$$\Gamma[\phi_{cl}] = -\underset{\sim}{V}\cdot\underline{T}\;\underbrace{V_{eff}(\phi_{cl})}_{\text{effective potential}}$$

11.4. Effective action

$$\mathcal{L} = \mathcal{L}_1 + \underbrace{\delta\mathcal{L}}_{\text{counter terms}} \qquad \delta_\lambda\phi^4 + \cdots$$

source term $J = J_1 + \delta J$

cl. eq.

$$\frac{\delta\mathcal{L}}{\delta\phi} + J = 0 \quad\longrightarrow\quad \left.\frac{\delta\mathcal{L}_1}{\delta\phi} + J_1\right|_{\phi=\phi_{cl}} = 0$$

$$e^{-iE[J]} = \int \mathcal{D}\phi \; e^{i\int(\mathcal{L}_1 + J_1\phi)} \; \underbrace{e^{i\int(\delta\mathcal{L} + \delta J\,\phi)}}$$

$$\underline{\underline{\phi_{cl}}} \quad \phi = \phi_{cl} + \eta \qquad \mathcal{D}\phi \to \mathcal{D}\eta$$

$$\int d^4x\,(\mathcal{L}_1 + J_1\phi) = \int d^4x\,\Big(\underbrace{\mathcal{L}_1[\phi_{cl}]} + J_1\phi_{cl} + \eta(x)\overbrace{\left(\frac{\delta\mathcal{L}_1}{\delta\phi}\Big|_{\phi_{cl}} + J_1\right)}^{0}\Big)$$

$$\uparrow \; \mathcal{L}_1(\phi_{cl} + \eta)$$

$$+ \frac{1}{2}\int \eta(x)\,\eta(y)\,\frac{\delta^2\mathcal{L}_1}{\delta\phi(x)\,\delta\phi(y)}\Big|_{\phi_{cl}} d^4x\,d^4y$$

$$+ \cdots \eta^3 + \cdots$$

η- indp.

$$e^{-iE[J]} = e^{i\int d^4x\,(\mathcal{L}_1[\phi_{cl}] + J_1\phi_{cl})}$$

$$\times \int \mathcal{D}\eta \; \underbrace{e^{\frac{1}{2}i\int d^4x\,d^4y\,\eta(x)\eta(y)\left(\frac{\delta^2\mathcal{L}_1}{\delta\phi(x)\delta\phi(y)}\right)_{\phi_{cl}}} \left(1 + [\eta^3] + \cdots\right)}$$

$$\underbrace{\left(\det\left[-\frac{\delta^2\mathcal{L}_1}{\delta\phi(x)\,\delta\phi(y)}\right]\right)^{-\frac{1}{2}}} = e^{-\frac{1}{2}\underbrace{\log\det[\;]}_{\text{tr"}\log}}$$

$$\delta \mathcal{L}[\phi = \phi_{cl} + \eta] + \delta J(\phi_{cl} + \eta)$$
$$- \delta \mathcal{L}[\phi_{cl}] + \delta \mathcal{L}[\phi_{cl}]$$

$$= \delta \mathcal{L}[\phi_{cl}] + \delta J \phi_{cl} + \underbrace{\delta \mathcal{L}[\phi_{cl} + \eta] - \delta \mathcal{L}[\phi_{cl}] + \delta J \cdot \eta}_{\mathcal{O}(\eta)}$$

$$\Rightarrow -i E[J] = \left[i \int d^4x \, \mathcal{L}_1[\phi_{cl}] + \underset{\sim\sim}{J_1} \phi_{cl} \right] - \frac{1}{2} \log \det \left[- \frac{\delta^2 \mathcal{L}_1}{\delta\phi \delta\phi} \right]$$

$$+ \; i \int d^4x \left(\underbrace{\delta \mathcal{L}[\phi_{cl}]}_{C.T.} + \underbrace{\delta J \phi_{cl}}_{J \phi_{cl}} \right) + \left(\text{pert. diagrams for } \eta\text{-field.} \right)$$

$$\Gamma[\phi_{cl}] = \int d^4x \, \mathcal{L}_1[\phi_{cl}] + \frac{i}{2} \log \det [\quad] + \int d^4x \, \delta \mathcal{L}[\phi_{cl}]$$
$$+ \cdots$$

apply to linear σ-model

x-indep.

$$\mathcal{L} = \mathcal{L}[\phi^i_{cl} + \eta^i] = \frac{1}{2} \mu^2 \phi^{i\,2}_{cl} - \frac{\lambda}{4} (\phi^{i\,2}_{cl})^2 + (\mu^2 - \lambda \phi^{i\,2}_{cl}) \phi^i_{cl} \eta^i$$
$$+ \frac{1}{2} (\partial_\mu \eta^i)^2 + \frac{1}{2} \mu^2 \eta^{i\,2} - \frac{\lambda}{2} \left[\phi^{i\,2}_{cl} \eta^{i\,2} + 2 \phi^i_{cl} \eta^i \right]$$
$$+ \mathcal{O}(\eta^3, \eta^4)$$

$$\frac{\delta^2 \mathcal{L}}{\delta\phi^i \delta\phi^j} = \frac{\delta^2 \mathcal{L}}{\delta\eta^i \delta\eta^j} = -\partial^2 \delta^{ij} + \mu^2 \delta^{ij} - \lambda \Big[\underbrace{\vec{\phi}_{cl}^{\,2}}_{\phi_{cl}^2} \delta^{ij} + \underbrace{2\phi_{cl}^i \phi_{cl}^j}_{2\phi_{cl}^2 \delta_{iN}\delta_{jN}} \Big]$$

$\to$ SSB $\quad \vec{\phi}_{cl} = (\vec{0}, \phi_{cl})$

$$m_i^2 = \begin{cases} \lambda\phi_{cl}^2 - \mu^2 & i \neq N \\ 3\lambda\phi_{cl}^2 - \mu^2 & i = N \end{cases}$$

$$\det(\quad) = \det\left(\partial^2 + \underbrace{\lambda\phi_{cl}^2 - \mu^2}\right)^{N-1} \times \det\left(\partial^2 + \underbrace{3\lambda\phi_{cl}^2 - \mu^2}\right)$$

$$\log\det(\partial^2 + m^2) = \mathrm{Tr}\log(\partial^2 + m^2) = \sum_k \log(-k^2 + m^2)$$

$$= \sum_k \phi_k^* \log(\quad) \phi_k = (VT)\int \frac{d^4k}{(2\pi)^4} \log(-k^2 + m^2)$$

$$\phi_k = e^{ik\cdot x}$$

$$\int \frac{d^4k}{(2\pi)^4} \log(-k^2 + m^2) = i\int \frac{d^4k_E}{(2\pi)^4} \log(k_E^2 + m^2) = i\left(-\frac{\partial}{\partial\alpha}\right) \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{(k_E^2 + m^2)^\alpha}\Bigg|_{\alpha=0}$$

$$\frac{1}{(k_E^2+m^2)^\alpha} = e^{-\alpha \ln(k_E^2 + m^2)}$$

$$= -i\frac{\partial}{\partial\alpha}\left(\frac{1}{(4\pi)^{d/2}} \frac{\Gamma(\alpha - \frac{d}{2})}{\Gamma(\alpha)} \frac{1}{(m^2)^{\alpha - \frac{d}{2}}}\right)\Bigg|_{\alpha=0}$$

$$= -i\frac{\Gamma(-\frac{d}{2})}{(4\pi)^{d/2}} (m^2)^{d/2}$$

$$V_{eff}[\phi_{cl}] = -\frac{1}{VT}\Gamma[\phi_{cl}]$$

$$= -\frac{1}{2}\mu^2\phi_{cl}^2 + \frac{\lambda}{4}\phi_{cl}^4$$

$$-\frac{1}{2}\frac{\Gamma(-\frac{d}{2})}{(4\pi)^{d/2}}\left\{(N-1)\left[\lambda\phi_{cl}^2-\mu^2\right]^{d/2} + 1\cdot\left[3\lambda\phi_{cl}^2-\mu^2\right]^{d/2}\right\}$$

$$(-\tfrac{d}{2})\Gamma(-\tfrac{d}{2})$$

$$=\Gamma(1-\tfrac{d}{2})\times(1-\tfrac{d}{2})$$

$$+\frac{1}{2}\underbrace{\delta_\mu}\phi_{cl}^2 + \frac{1}{4}\underbrace{\delta_\lambda}\phi_{cl}^4$$

$$\Gamma(\tfrac{d}{2}) = \frac{\cancel{\Gamma(2-\frac{d}{2})}}{(-\frac{d}{2})(1-\frac{d}{2})}$$

$$\delta_\mu \leftarrow -\lambda\mu^2(N+2)\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2}$$

$$\delta_\lambda \rightarrow \lambda^2(N+8)\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2}$$

Minimal subtraction

$$4-d=\epsilon$$

$$\delta \otimes = \frac{-1}{16\pi^2}\left(\underbrace{\frac{2}{\varepsilon}} - \gamma - \ln\frac{M^2}{4\pi}\right)$$

finite

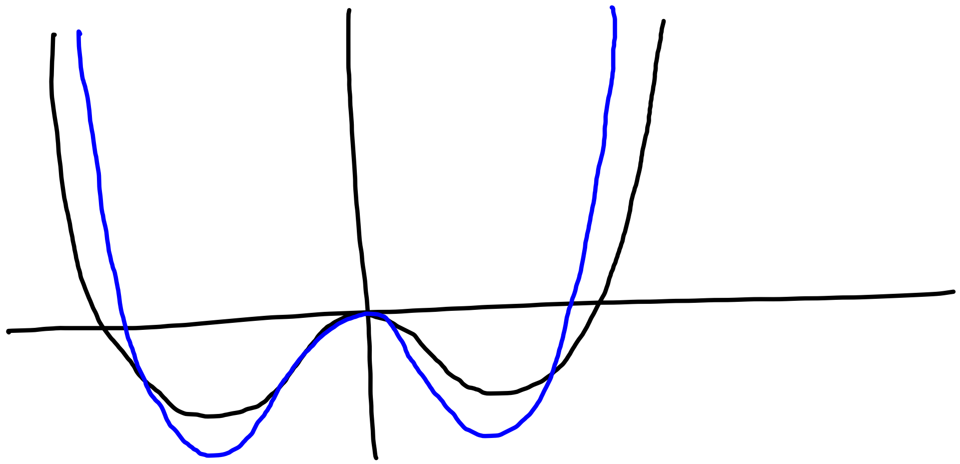
$$\frac{\Gamma(\overbrace{2-\frac{d}{2}}^{\frac{\epsilon}{2}})}{(4\pi)^2\,(4\pi)^{-2+d/2}(m^2)^{\underbrace{2-\frac{d}{2}}_{\frac{\epsilon}{2}}}} = \underbrace{\frac{\Gamma(\frac{\varepsilon}{2})}{(4\pi)^2}}_{\frac{1}{(4\pi)^2}\left(\underbrace{\frac{2}{\varepsilon}}-\gamma+\dots O(\varepsilon)\right)}\;\underbrace{e^{-\frac{\varepsilon}{2}\ln\frac{m^2}{4\pi}}}_{1-\underbrace{\varepsilon\ln\frac{m^2}{4\pi}}+\dots O(\varepsilon^2)}$$

$$= \frac{1}{(4\pi)^2}\left(\underbrace{\frac{2}{\varepsilon}} - \gamma - \ln\frac{m^2}{4\pi} + \dots\right)$$

$$= \frac{1}{(4\pi)^2}\log\frac{m^2}{\underbrace{M^2}}$$

$$V_{eff} = \underbrace{-\frac{1}{2}\mu^2\phi_{cl}^2 + \frac{\lambda}{4}\phi_{cl}^4}$$

$$\delta V \left\{ + \frac{1}{4}\frac{1}{(4\pi)^2} \left\{ (N-1)(\lambda\phi_{cl}^2 - \mu^2)^2 \left(\log \frac{\lambda\phi_{cl}^2 - \mu^2}{M^2} - \frac{3}{2} \right) + (3\lambda\phi_{cl}^2 - \mu^2)^2 \left(\log \frac{3\lambda\phi_{cl}^2 - \mu^2}{M^2} - \frac{3}{2} \right) \right\} \right.$$



$$H = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j \qquad \uparrow \ \uparrow \ \nearrow \ \searrow \ \nearrow \qquad O(3)$$

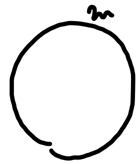
$$\downarrow$$

$$\vec{\uparrow} \ \vec{\uparrow} \ \uparrow \ \uparrow \qquad T < T_c$$

SSB $\rightarrow$ Goldstone boson : "magnon"

Coleman-Mermin-Wagner theorem.

$d \leq 3 \rightarrow$ no GB, no long range order.

$d \rightarrow$ scalar field  $= \displaystyle\int \frac{d^d p}{(2\pi)^d} \frac{i}{p^2 + m^2} = i\Omega_d \int_0^\infty dp\, p^{d-1} \underbrace{\frac{1}{p^2}}$

$m = 0$ $\qquad \displaystyle\propto \int_0^\infty dp\, p^{d-3}$

$\left.\begin{array}{l} d = 3 \rightarrow 2+1 \text{ dim} \\ d = 2 \rightarrow 1+1 \text{ dim} \end{array}\right\}$

no IR div if $d-3>0$

$$M \to M + \delta M$$

$$\lambda = \lambda_0 + \lambda_0^2 (N+8) \frac{1}{(4\pi)^2} \log \frac{\Lambda^2}{M^2}$$

$\uparrow$ bare

$$\to \quad \lambda(M) = \lambda_0 - \frac{\lambda_0^2 (N+8)}{(4\pi)^2} \log \frac{\Lambda^2}{M^2} \quad \to \quad \lambda = \lambda_0 + \mathcal{O}(\lambda_0^2)$$

$$\lambda(M+\delta M) = \lambda_0 - \;\text{''}\; \log \frac{\Lambda^2 M^2}{(M+\delta M)^2 M^2}$$

$$= \lambda(M) - \frac{\lambda_0^2 (N+8)}{(4\pi)^2} \ln \underbrace{\frac{M^2}{(M+\delta M)^2}}$$

$$\delta M^2 = 2M\delta M$$

$$-2 \ln\left(1 + \frac{\delta M}{M}\right) \simeq -2\frac{\delta M}{M} = -\frac{\delta M^2}{M^2}$$

$$\boxed{\lambda(M+\delta M) = \lambda(M) + \frac{\lambda(M)^2 (N+8)}{(4\pi)^2} \frac{\delta M^2}{M^2}}$$

running coupling constant

$$\boxed{\mu^2(M+\delta M) = \mu^2(M) - \frac{\lambda \mu^2}{(4\pi)^2} (N+2) \frac{\delta M^2}{M^2}}$$

11.5. Effective action vs. Generating Function

$$e^{-i\,\underbrace{E[J]}} = \int \mathcal{D}\phi \; e^{i\int d^d x\,(\mathcal{L} + J\phi)} = Z[J]$$

$$E[J] = i \log Z[J]$$

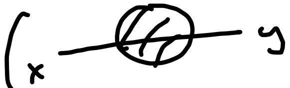
$$i \int \mathcal{D}\phi \; \phi(x) \; e^{i\int(\mathcal{L}+J\phi)} = \frac{\partial Z}{\partial J(x)}$$

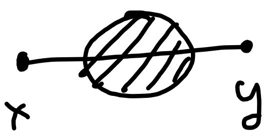
$$\frac{\delta E[J]}{\delta J(x)} = i \, \frac{1}{Z} \, \boxed{\frac{\partial Z}{\partial J(x)}}$$

$$(i)^2 \int \mathcal{D}\phi \; \phi(x)\phi(y) \; e^{i\int(\mathcal{L}+J\phi)}$$

$$\Big|_{J=0} \qquad \frac{-i}{Z}\frac{\partial Z}{\partial J}\Big|_{J\to 0} = i\langle\phi\rangle$$

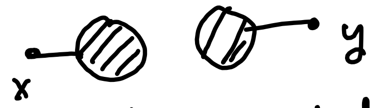
$$\frac{\delta^2 E}{\delta J(y)\,\delta J(x)} = \underbrace{-i \, \frac{1}{Z^2} \frac{\partial Z}{\partial J(y)} \frac{\partial Z}{\partial J(x)}}_{i^2 \langle\phi(y)\rangle\langle\phi(x)\rangle} + \underbrace{i \, \frac{1}{Z} \frac{\partial^2 Z}{\partial J(x)\partial J(y)}}_{i^2 \langle\phi(x)\,\phi(y)\rangle}$$

$$= -i\left(\langle\phi(x)\,\phi(y)\rangle - \langle\phi(x)\rangle\langle\phi(y)\rangle \right) = -i\Big(\quad$$  $$\Big)$$



connected

+



disconnected

$$\therefore \quad \frac{\delta^2 E[J]}{\delta J(x)\,\delta J(y)} = -i \,\langle \phi(x)\,\phi(y)\rangle_{\text{conn.}}$$

$$\therefore \quad \frac{\delta^n E[J]}{\delta J(x_1)\cdots\delta J(x_n)} = i^{n+1}\,\langle \phi(x_1)\cdots\phi(x_n)\rangle_{\text{conn.}}$$

$$\underbrace{\frac{\delta}{\delta J(y)}\,\frac{\delta \Gamma(\phi_{cl})}{\delta \phi_{cl}(x)}}_{\int d^4z\, \frac{\delta \phi_{cl}(z)}{\delta J(y)}\,\frac{\delta^2 \Gamma(\phi_{cl})}{\delta \phi_{cl}(z)\,\delta\phi_{cl}(x)}} = -\frac{\delta J(x)}{\delta J(y)} = -\delta(x-y)$$

$$\left[\phi_{cl}(z) = -\underline{\frac{\delta E}{\delta J(z)}}\right]$$

$$= -\underbrace{\int d^4z\, \frac{\delta^2 E}{\delta J(y)\,\delta J(z)}}_{\left(\frac{\delta^2 E}{\delta J \delta J}\right)_{yz}} \cdot \underbrace{\frac{\delta^2 \Gamma}{\delta \phi_{cl}(z)\,\delta\phi_{cl}(x)}}_{\left(\frac{\delta^2 \Gamma}{\delta \phi_{cl}^2}\right)_{zx}} = -\delta(x-y)$$

$$\left(\frac{\delta^2 E}{\delta J \delta J}\right)_{yz} \cdot \left(\frac{\delta^2 \Gamma}{\delta \phi_{cl}^2}\right)_{zx} = \mathbb{1}$$

$$\Rightarrow \quad \underbrace{\frac{\delta^2 E}{\delta J(x)\,\delta J(y)}}_{-i\,\underbrace{\langle\phi(x)\phi(y)\rangle_{conn}}_{D(x-y)}} = \left(\frac{\delta^2\Gamma}{\delta\phi_{cl}\,\delta\phi_{cl}}\right)^{-1}.$$

$$D(x-y) = \int \frac{d^4p}{(2\pi)^4}\,\tilde{D}(p)\,e^{-ip\cdot(x-y)} \qquad \text{(cf)}\ \tilde{D} = \frac{i}{p^2+m^2}$$

$$\frac{\delta^2\Gamma}{\delta\phi_{cl}(x)\,\delta\phi_{cl}(y)} = i\,D^{-1}(x-y) = i\int\frac{d^4p}{(2\pi)^4}\,\underbrace{\frac{1}{\tilde{D}(p)}}_{\text{(cf)}\ -i(p^2+m^2)}\,e^{-ip\cdot(x-y)}$$

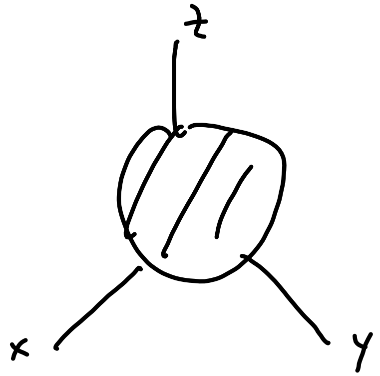
$$\frac{\delta^3 E[J]}{\delta J_x\,\delta J_y\,\delta J_z} = \frac{\delta}{\delta J_z}\left(\frac{\delta^2\Gamma}{\delta\phi_{cl}\,\delta\phi_{cl}}\right)^{-1}. = \frac{\delta\phi_w}{\delta J_z}\,\frac{\delta}{\delta\phi_w}\left(\frac{\delta^2\Gamma}{\delta\phi_x\,\delta\phi_y}\right)^{-1}$$

$$MM^{-1} = \mathbb{1} \ \rightarrow\ \frac{\partial M}{\partial\alpha}M^{-1} + M\frac{\partial M^{-1}}{\partial\alpha} = 0$$

$$\therefore\ \frac{\partial M^{-1}}{\partial\alpha} = -M^{-1}\frac{\partial M}{\partial\alpha}M^{-1}$$

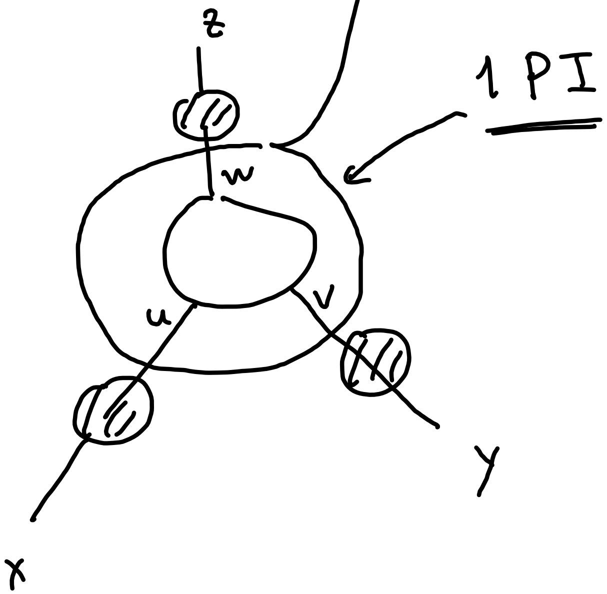
$$\underbrace{\frac{\delta\phi_w}{\delta J_z}}\;\underbrace{\frac{\delta}{\delta\phi_w}\left(\underbrace{\frac{\delta^2\Gamma}{\delta\phi_x\delta\phi_y}}_{M}\right)^{-1}} = \underbrace{\left(\frac{\delta^2\Gamma}{\delta\phi_x\delta\phi_u}\right)^{-1}}_{iD(x-u)}\;\underbrace{\frac{\delta^3\Gamma}{\delta\phi_w\delta\phi_u\delta\phi_v}}\;\underbrace{\left(\frac{\delta^2\Gamma}{\delta\phi_v\delta\phi_y}\right)^{-1}}_{iD(v-y)}$$

$$= -\frac{\delta^2 E}{\delta J_z\delta J_w} = \underline{iD(z-w)}$$



$$\frac{\delta^3 E}{\delta J^3}$$

=



$$\frac{\delta^n\Gamma}{\delta\phi_{cl}(x_1)\ldots\delta\phi_{cl}(x_n)} = -i\left\langle \phi(x_1)\;\ldots\;\phi(x_n)\right\rangle_{1PI}.$$