Chap 7. Regularization

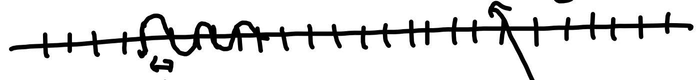
Continuum

$\sum \longrightarrow \int d^3\vec{k}$

$\lambda \to$ wavelength

discrete

$\Lambda \sim \frac{1}{\varepsilon}$



ε

$\int^{|\vec{k}| = \Lambda \to \infty}$

cutoff

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dimensional regularization

LSZ reduction formula

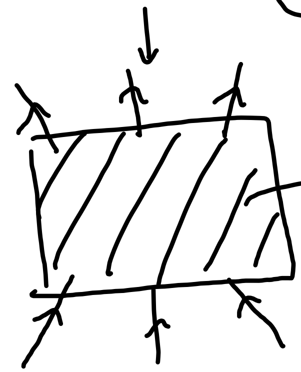
Correlation function

$$\langle \Omega | T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle \quad \Longleftarrow\!\!\!\!\Longrightarrow$$

Scattering matrix

$${}_{out}\langle \vec{p}_1, \cdots | \underbrace{\vec{k}_1, \cdots}_{\text{on-shell}} \rangle_{in} \equiv \langle\{\vec{p}_i\}|S|\{\vec{k}_j\}\rangle$$

on-shell $p_i^2 = E_i^2 - \vec{p}_i^{\,2} = m_i^2$



F. D.

(connected amputated)

$$\int\cdots\int \langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\,\phi(y_1)\cdots\phi(y_m)\}|\Omega\rangle \prod_{i=1}^{n} e^{i p_i\cdot x_i}\prod_{j=1}^{m} e^{-ik_j\cdot y_j}\, d^4x_i\, d^4y_j$$

$$\sim \prod_i\left(\frac{i\sqrt{Z}}{p_i^2-m_i^2+i\varepsilon}\right)\prod_j\left(\frac{i\sqrt{Z}}{k_j^2-m_j^2+i\varepsilon}\right)\langle\{\vec{p}_i\}|S|\{\vec{k}_j\}\rangle$$

on-shell.

$p_i^2 = m_i^2$

$k_j^2 = m_j^2$

Any state $\rightarrow$ $\mathbb{1} = \sum_n |n\rangle\langle n|$

Q.M. $\longrightarrow$ $\mathbb{1}_{\text{1-particle}} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}|\vec{p}\rangle\langle\vec{p}|$.

QFT

$$\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p(\lambda)}|\lambda_{\vec{p}}\rangle\langle\lambda_{\vec{p}}|$$

$\vec{p}$ = total momentum of multi-particle states

(ex) 1-particle ; $p = (p^0, \vec{p})$ $\quad p^0 = \sqrt{\vec{p}^2 + m^2} = E_{\vec{p}}$

2- " : $p = p_1 + p_2 = (\underbrace{p_1^0 + p_2^0}_{E_p}, \underbrace{\vec{p}_1 + \vec{p}_2}_{\vec{p}})$ $\quad E_p = \underbrace{\sqrt{\vec{p}_1^{\,2} + m_1^2} + \sqrt{\vec{p}_2^{\,2} + m_2^2}}$

$= E_{\vec{p}}(\lambda)$ $\leftarrow \vec{p}_1 - \vec{p}_2$ $\quad = \sqrt{(\vec{p}_1+\vec{p}_2)^2 + m_\lambda^2}$

$$\langle\Omega|\phi(x)\overset{\mathbb{1}}{\downarrow}\phi(y)|\Omega\rangle = \underbrace{\langle\Omega|\phi(x)|\Omega\rangle}_{=0}\langle\Omega|\phi(y)|\Omega\rangle$$

$\langle\lambda_0|\phi(0)|\Omega\rangle e^{+ip\cdot y}$

$$+ \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p(\lambda)} \underbrace{\langle\Omega|\phi(x)|\lambda_{\vec{p}}\rangle}\underbrace{\langle\lambda_{\vec{p}}|\phi(y)|\Omega\rangle}$$

$e^{-iE_p x^0 + i\vec{p}\cdot\vec{x}}$

$\downarrow$

$$\langle\Omega|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|\lambda_{\vec{p}}\rangle = \langle\Omega|\phi(0)|\lambda_0\rangle \cdot e^{-ip\cdot x}$$

$\langle\Omega|\overbrace{U\phi(0)U^{-1}}^{\phi(0)}|\lambda_{\vec{p}}\rangle$ $\quad |\lambda_{\vec{0}}\rangle$

$$= \boxed{\langle\Omega|\phi(0)|\lambda_{\vec{p}}\rangle}(e^{-ip\cdot x})\Big|_{p^0 = E_{\vec{p}}(\lambda)}$$

$$= \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_{\vec p}(\lambda)}\, e^{-ip\cdot(x-y)}\Big| \qquad |\langle\Omega|\phi(0)|\lambda_0\rangle|^2$$

$$p^0 = E_{\vec p}(\lambda) = \sqrt{\vec p^2 + m_\lambda^2}$$

$$D_F(x-y, m_\lambda^2) = \int dm^2\, \delta(M^2 - m_\lambda^2)\, D_F(x-y, M^2)$$

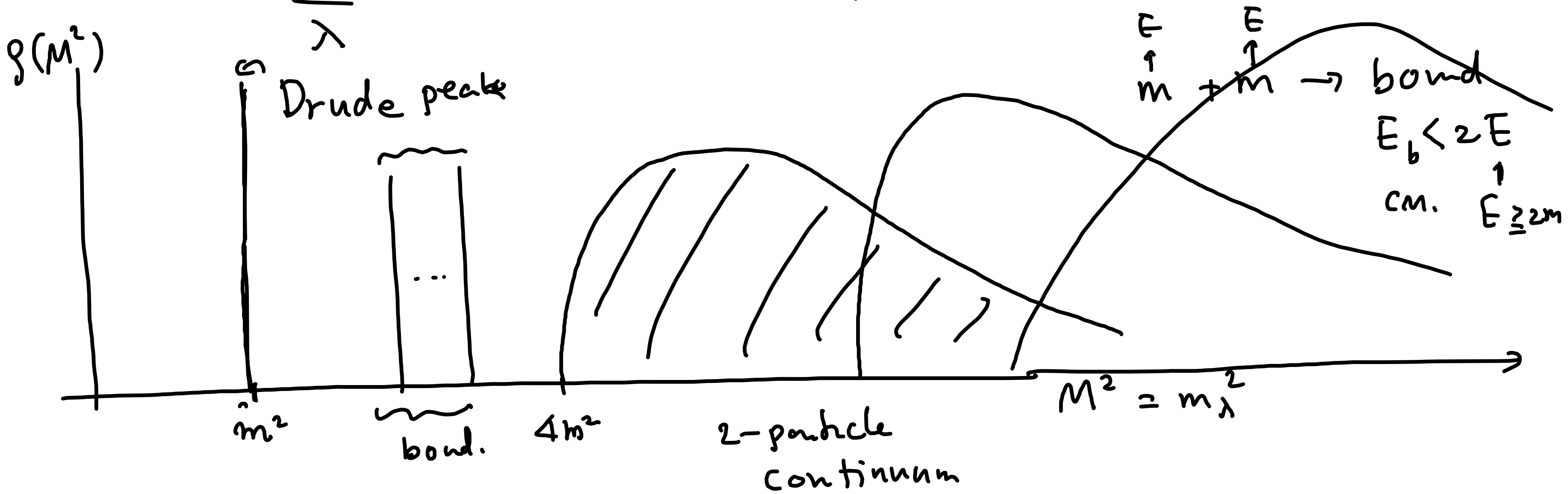
$$= \sum_\lambda \overbrace{\int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2 - m_\lambda^2 + i\varepsilon}\, e^{-ip\cdot(x-y)}} \qquad |\langle\Omega|\phi(0)|\lambda_0\rangle|^2$$

off-shell

spectral density function

$$= \int \frac{dM^2}{2\pi}\,\rho(M^2)\, D_F(x-y, M^2)$$

$$\rho(M^2) \equiv \sum_\lambda (2\pi)\,\delta(M^2 - m_\lambda^2)\,|\langle\Omega|\phi(0)|\underset{\sim}{\lambda_0}\rangle|^2$$

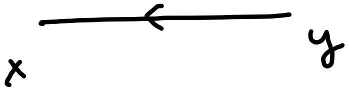


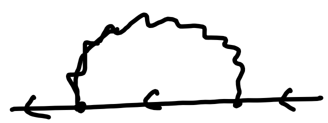
$$\int d^4x\, e^{ip\cdot x}\langle\Omega|\phi(x)\phi(0)|\Omega\rangle = \int \frac{dM^2}{2\pi}\,\rho(M^2)\int D_F(x, M^2)\, d^4x\, e^{ip\cdot x}$$

$2\pi\,\delta(M^2-m^2)\,Z$ $\quad |\langle\Omega|\phi(0)|\lambda_0\rangle|^2$

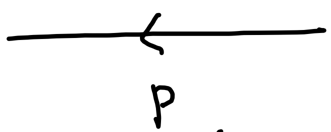
$$= \frac{iZ}{p^2-m^2+i\epsilon} + \int_{4m^2}^{\infty}\frac{dM^2}{2\pi}\,\rho(M^2)\,\frac{i}{p^2-M^2+i\epsilon}$$

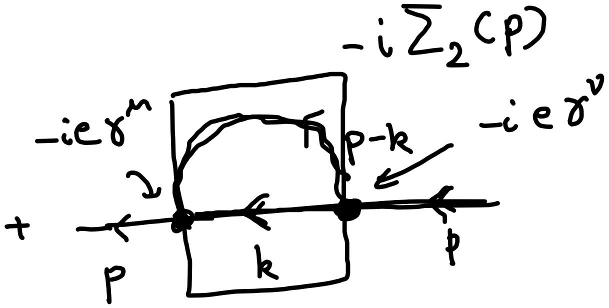
Electron self-energy

$\langle\Omega|T\{\psi(x)\bar\psi(y)\}|\Omega\rangle_{QED} =$ 

$+$  $+ \cdots$

↓ momentum space

$=$ 

$+$ 

$$= \frac{i(\not p+m_0)}{p^2-m_0^2} + \frac{i(\not p+m_0)}{p^2-m_0^2}\left(-i\Sigma_2(p)\right)\frac{i(\not p+m_0)}{p^2-m_0^2}$$

$$-i\Sigma_2(p) \equiv (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i(\not{k}+m_0)}{\underbrace{k^2-m_0^2+i\varepsilon}_{B}} \gamma_\mu \frac{-i}{\underbrace{(p-k)^2-\mu^2+i\varepsilon}_{A}}$$

Trick "Feynman parametrization"

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[Ax+B(1-x)]^2} = \int_0^1 dx \frac{1}{((A-B)x+B)^2}$$

$$= -\frac{1}{(A-B)x+B} \cdot \frac{1}{A-B}\Big|_0^1$$

$$= -\frac{1}{A-B} \cdot \left(\frac{1}{A} - \frac{1}{B}\right) = \frac{1}{AB} \checkmark$$

$$\Rightarrow \frac{1}{\prod_{i=1}^{n} A_i} = \int_0^1 \cdots \int \frac{1}{\left(\sum_{i=1}^{n} A_i x_i\right)^{n+1}} \delta\left(\sum_i x_i = 1\right) dx_1 \cdots dx_n$$

$$Ax + B(1-x) = \left((p-k)^2 - \mu^2 + i\varepsilon\right)x + \left(k^2 - m_0^2 + i\varepsilon\right)(1-x)$$

$$= \underline{k^2} + p^2 x - \underline{2p\cdot k\, x} - \mu^2 x - m_0^2(1-x) + i\varepsilon$$

$$= (\underbrace{k - xp}_{l})^2 \underbrace{- x^2 p^2 + p^2 x - \mu^2 x - m_0^2(1-x) + i\varepsilon}_{\overset{\text{\tiny +p^2x(1-x)}}{}\;-\Delta^2}$$

$d^4k \to d^4l$

$$= l^2 - \Delta^2 + i\varepsilon$$

$$-i\Sigma_2(p) = -e^2\int_0^1 dx \int \frac{d^4l}{(2\pi)^4}\,\frac{\gamma^\mu(\overbrace{x\not p + \not l}^{\not k}\,+ m_0)\gamma_\mu}{l^2 - \Delta^2 + i\varepsilon}$$

$\not l \to 0$

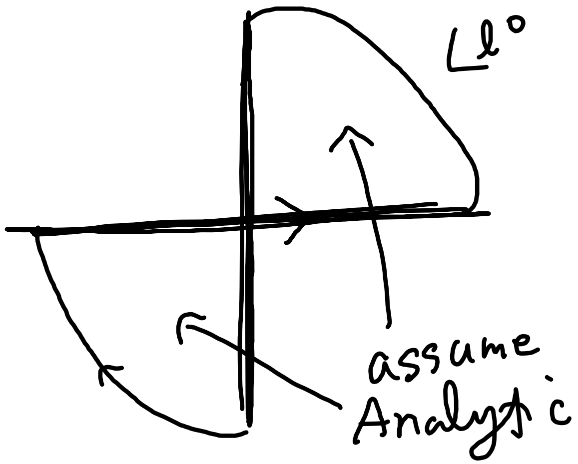
$$\to \underbrace{x\overbrace{\gamma^\mu \not p \gamma_\mu}^{p_\nu \gamma^\mu \overbrace{\gamma^\nu \gamma_\mu}^{-2\gamma^\nu}}}_{-2x\not p} + m_0 \overbrace{\gamma^\mu\gamma_\mu}^{4\mathbb{1}}$$

$$\int \frac{d^4l\ \ \not l}{l^2 - \Delta^2} = 0$$

$\not l$ ← odd $(l \to -l)$

$$= -e^2\int_0^1 dx\,\left(-2x\not p + 4m_0\mathbb{1}\right)\int\frac{d^4l}{(2\pi)^4}\,\frac{1}{l^2 - \Delta^2 + i\varepsilon}$$

$l^2 = l_0^2 - (l_1^2 + l_2^2 + l_3^2) \longrightarrow$ Wick rotation.



$$\int_{-\infty}^{\infty} dl_0 = \int_{-i\infty}^{i\infty} dl_0 = i\int_{-\infty}^{\infty} dl_E \qquad l_0 = il_4$$

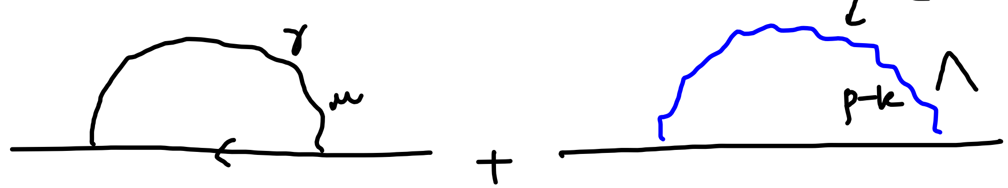
$$l^2 = -(l_1^2 + \cdots + l_4^2) \qquad l_E = (l_1, l_2, l_3, l_4)$$

$$\int d^4 l \,\frac{1}{(l^2-\Delta^2)^2} = -i\int \underbrace{d^4 l_E}_{dl_E\, l_E^3\, d\Omega_3} \frac{1}{(l_E^2+\Delta^2)^2} = -i\int d\Omega_3 \underbrace{\int_0^{\infty} \frac{l_E^3\, dl_E}{(l_E^2+\Delta^2)^2}}_{\infty}$$

Regularization

① Pauli-Villars

$$\frac{+i}{(p-k)^2-\Lambda^2+i\varepsilon}$$



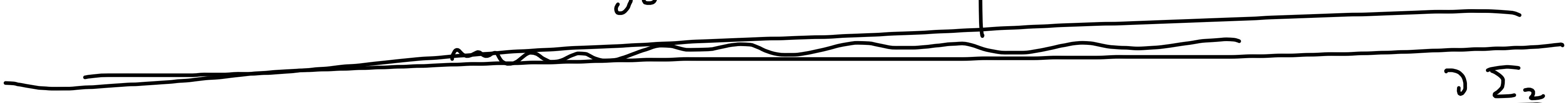
가상의 heavy photon

$$\to \frac{l_E^2(\cdots)}{(l_E^2-\Delta_\mu^2)^2(l_E^2-\Delta_\Lambda^2)^2}$$

$$\frac{i}{(4\pi)^2}\log\frac{\Delta_\mu^2}{\Delta_\Lambda^2} = \frac{i}{(4\pi)^2}\int_0^\infty dl_E^2\left[\frac{l_E^2}{(l_E^2+\Delta_\mu^2)^2} - (\mu\to\Lambda)\right] \int \underbrace{d^4 l}_{l_E^3\, dl_E \,\propto\, l_E^2\, dl_E^2}\left(\frac{1}{(l^2-\Delta_\mu^2)^2} - \frac{1}{(l^2-\Delta_\Lambda^2)^2}\right)$$

$$-i\Sigma_2(p) = \frac{e^2}{8\pi^2}\int_0^1 dx\,\left(2m_0\mathbb{1} - x\not p\right)\log\left(\frac{x\Lambda^2}{(1-x)m_0^2 + x\mu^2 - x(1-x)p^2}\right)$$

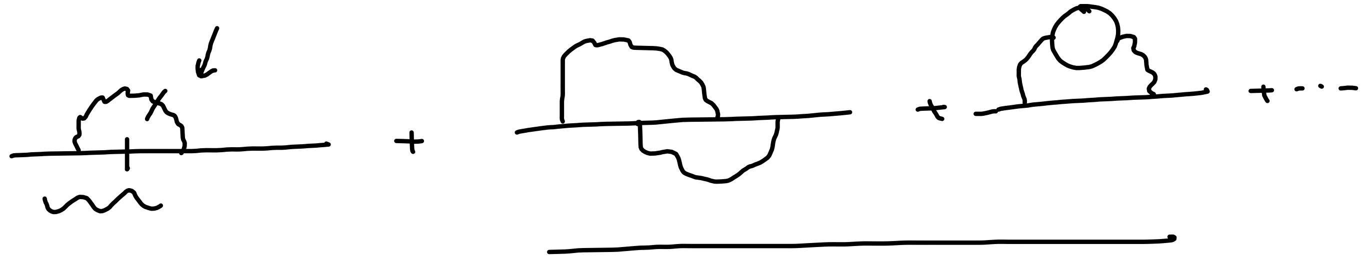
Δ_μ^2 $\quad \not p \not p$



$$\frac{\partial \Sigma_2}{\partial \not p}$$

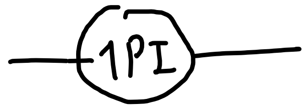
$$-i\Sigma_{all}(p) = \sum\left(-i\Sigma\;(p)\right)$$

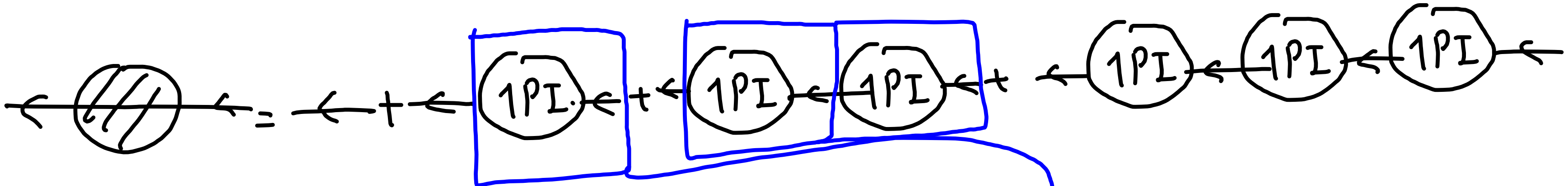
$-i\Sigma$ $=$



$+\cdots$

$=$





$$= \frac{i(\not p + m_0)}{p^2 - m_0^2}\left\{1 + (-i\Sigma)\frac{i(\not p + m_0)}{p^2-m_0^2} + \left((-i\Sigma)\frac{i(\not p+m_0)}{p^2-m_0^2}\right)^2 + \cdots\right\} = \frac{i}{\not p - m_0}\cdot\frac{1}{1 - \frac{\Sigma}{\not p - m_0}}$$

$$= \frac{i}{\not{p} - m_0 - \underset{\substack{\uparrow \\ 4\times 4}}{\Sigma(\not{p})}} \longrightarrow \left.\not{p} - m_0 - \Sigma(\not{p})\right|_{\not{p}=m} = 0$$

(cf) $\leftarrow = \dfrac{i}{\not{p} - m_0} = \dfrac{i(\not{p}+m_0)}{p^2 - m_0^2}$

$\nwarrow$

$\not{p} \sim m_0$

① mass renormalization.

$$\underbrace{\overset{\substack{0.5\,\text{MeV} \\ \downarrow}}{m} - m_0}_{\delta m} - \underbrace{\Sigma(\not{p}=m)}_{= c e^2 \ln \Lambda^2} = 0$$

$$\underbrace{\delta m}_{+\infty} = \Sigma(\not{p}=m) \leftarrow$$

$\uparrow$

$-c e^2 \ln \Lambda^2 + \cdots$

② wave function renorm.

$$\Sigma(\not{p}) \approx \Sigma(m) + (\not{p}-m)\Sigma'(m) + \cdots$$

$$\frac{i}{\underbrace{\not{p} - m_0 - \Sigma(m)}_{-m} - (\not{p}-m)\Sigma'(m)} = \frac{i}{(\not{p}-m)\underbrace{(1-\Sigma'(m))}}$$

$$\frac{1}{Z_2} = \frac{1}{1+\delta Z_2} = 1 - \delta Z_2$$

$$= \underline{\frac{i Z_2}{\not{p} - m}}$$

1-loop.

$$\alpha = \frac{e^2}{4\pi} \qquad \delta m = \frac{\alpha}{2\pi} m_0 \int_0^1 dx\, (2-x) \ln\left[\frac{x \Lambda^2}{(1-x)^2 m_0^2 + x\mu^2} \right].$$

$$Z_2 = 1 + \delta Z_2$$

$$\delta Z_2 = \Sigma'(m) = \left. \frac{\partial \Sigma(\not p)}{\partial \not p} \right|_{\not p = m} = \frac{\alpha}{2\pi} \int_0^1 dx \left[\cdots\cdots \right]$$

$$\int d^4x \, \langle \Omega | T\{ \phi(x) \phi(0) \} | \Omega \rangle \underset{p^2 \to m^2}{=} \frac{iZ}{p^2 - m^2 + i\varepsilon}$$

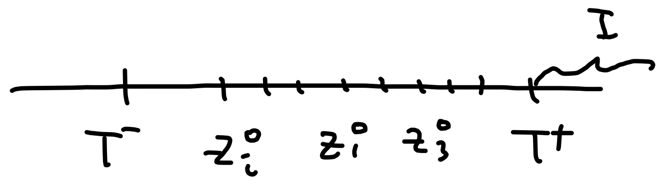
LSZ reduction

$$\underbrace{\int d^4x}\; e^{ip\cdot x} \langle \Omega | T\{ \underbrace{\phi(x)}\; \underbrace{\phi(z_1)\,\phi(z_2)\cdots} \} | \Omega \rangle$$

$$\underbrace{\int d^3\vec{x} \int_{-\infty}^{\infty} dx^0}$$

$$\underline{\underline{I}} \to \langle \Omega | \phi(x) T\{\phi(z_1)\cdots\} | \Omega\rangle \qquad x^0 > z_i^0$$

$$\underbrace{\int_{T^+}^{\infty}} + \int_{T^-}^{T^+} + \int_{-\infty}^{T^-}$$



$$\langle \Omega | \phi(0) | \lambda_0 \rangle \left. \underbrace{e^{-iq\cdot x}} \right|_{q^0 = E_{\vec{q}}(\lambda)}$$

$$= \int d^3x \int_{T^+}^{\infty} dx^0 \; e^{ip^0x^0} e^{-i\vec{p}\cdot\vec{x}} \sum_\lambda \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{2E_{\vec{q}}(\lambda)} \overbrace{\langle \Omega | \phi(x) | \lambda_{\vec{q}} \rangle} \times \langle \lambda_{\vec{q}} | T\{\phi(z_1)\cdots\} | \Omega \rangle$$

$$\overbrace{\underbrace{e^{-i q^0 x^0}}_{q^0 = E_{\vec{q}}(\lambda)} \; \underline{e^{+i\vec{q}\cdot\vec{x}}} \cdot \underbrace{e^{-\epsilon x^0}}_{\epsilon \to 0^+}}$$

$$\int_{T^+}^{\infty} dx^0 \; e^{i x^0 (p^0 - q^0 + i\epsilon)} \qquad q^0 = E_{\vec{q}}(\lambda)$$

$$=$$

$$\int d^3\vec{x} \;\to\; (2\pi)^3 \, \delta^{(3)}(\vec{p} - \vec{q})$$

$$\Rightarrow \sum_\lambda \underbrace{\frac{1}{2E_{\vec p}(\lambda)} \frac{i}{p^0 - E_{\vec p}(\lambda) + i\varepsilon}}_{\boxed{\frac{i}{p^2 - m_\lambda^2 + i\varepsilon}}} e^{i(p^0 - E_{\vec p} + i\varepsilon)T^+} \underbrace{\langle \Omega | \phi(0) | \lambda_0 \rangle}_{\sqrt{Z}} \langle \lambda_{\vec p} | T\{\phi(z_1) \cdots\} | \Omega \rangle$$

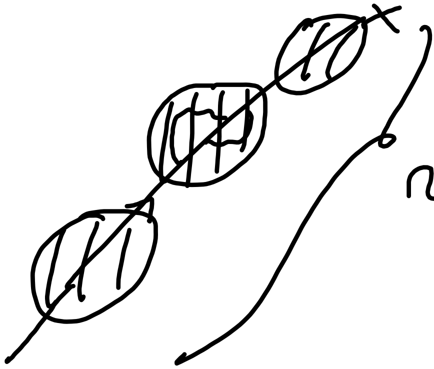
$p^2 = m_\lambda^2$

$$\frac{i}{p^{0\,2} - (\underbrace{\vec p^{\,2} + m_\lambda^2}_{E_{\vec p}(\lambda)}) + i\varepsilon} = \frac{i}{(p^0 - E_p)(p_0 + E_p)} \approx \frac{i}{2E_p(p^0 - E_p)}$$

$$= \sum_\lambda \frac{i\sqrt{Z}}{p^2 - m_\lambda^2 + i\varepsilon} \langle \lambda_{\vec p} | T\{\phi(z_1) \cdots\} | \Omega \rangle$$

$$= \frac{i\sqrt{Z}}{p^2 - m^2 + i\varepsilon} \langle \vec p | T\{\phi(z_1) \cdots\} | \Omega \rangle + \text{continuum}$$

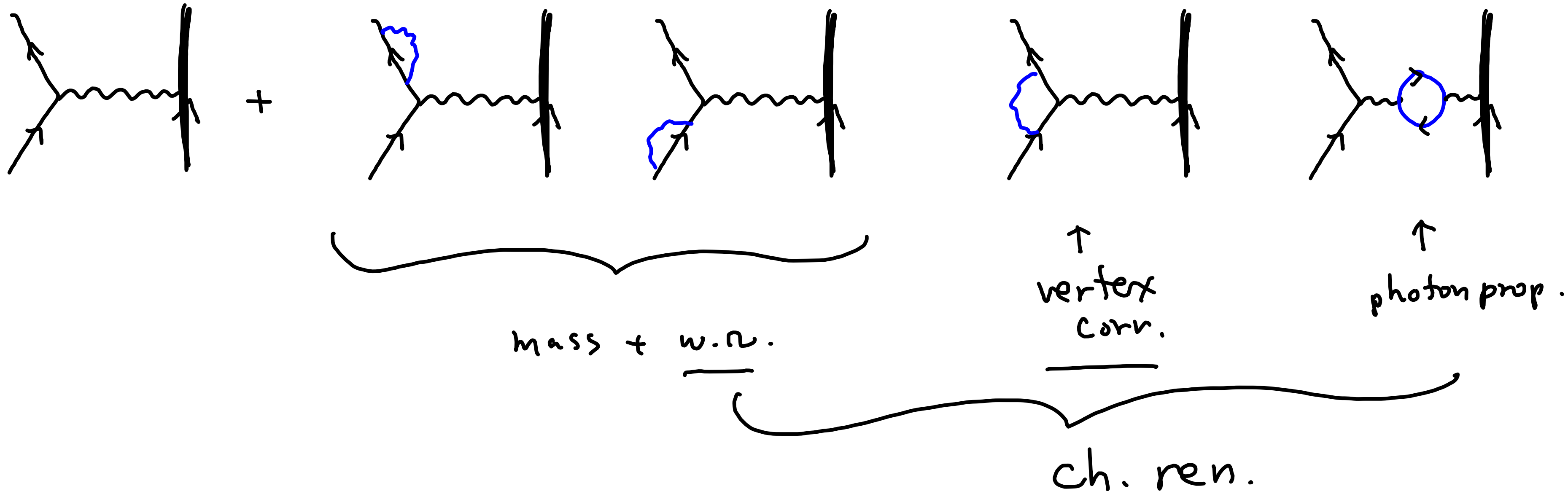
$$\underbrace{\prod_i \frac{i\sqrt{Z}}{p_i^2 - m^2 + i\varepsilon} \prod_j \frac{i\sqrt{Z}}{k_j^2 - m^2 + i\varepsilon}}\ {}_{\text{out}}\underbrace{\langle \vec p_1, \vec p_2 \cdots | \vec k_1, \cdots k_m \rangle_{\text{in}}}_{\text{S-matrix}} = \int \langle \Omega | T\{\phi(x_1) \cdots \phi(y_1) \cdots\} | \Omega \rangle e^{ip\cdot x} e^{-ik\cdot y}$$

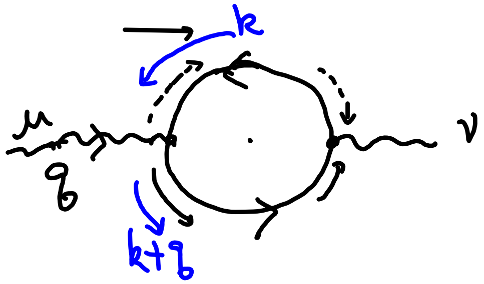


remove $\Rightarrow$ amputated

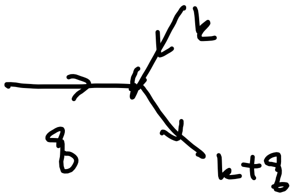
QED : $e_0, m_0 \longrightarrow e, m$

③ charge renorm.

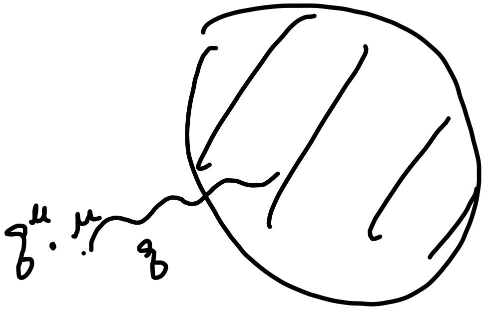




$$= (-ie)^2(-1)\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\left[\gamma^\mu\frac{i}{\cancel{k}-m}\gamma^\nu\frac{i}{\cancel{k}+\cancel{q}-m}\right]$$



$$\equiv i\Pi_2^{\mu\nu}(q) = i\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\underline{\underline{\Pi_2(q)}}$$



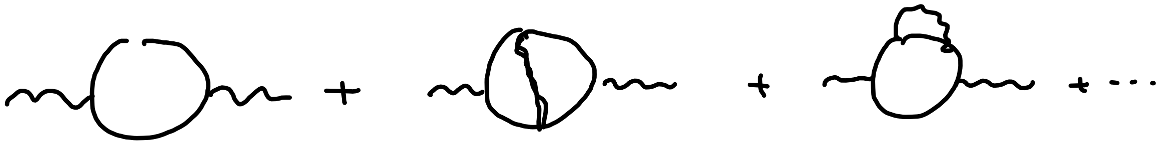
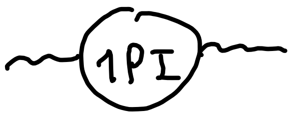
$= 0$

Ward-Takahashi Identity

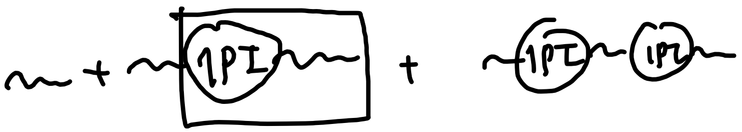
$$q_\mu \Pi_2^{\mu\nu} = q_\nu \Pi_2^{\mu\nu} = 0$$

$$q^2 q^\nu + \alpha\, q^2 q^\nu = 0$$

$$\alpha = -1$$

 $+ \cdots =$  $= i\Pi^{\mu\nu}$

$$= i\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\Pi(q)$$

 $+ \cdots =$

$$\frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\left(i\Pi^{\rho\sigma}\right)\frac{-ig_{\sigma\nu}}{q^2} + \cdots$$

$$= \frac{-i}{q^2\left(1-\Pi(q^2)\right)}\left(g_{\mu\nu} - \cancel{\frac{q_\mu q_\nu}{q^2}}\right) + \cancel{\frac{-i}{q^2}\left(\frac{q_\mu q_\nu}{q^2}\right)}$$

$$\because \quad q_\mu\left(\quad\right) \sim q^2 = 0$$

$$= \frac{-i\, g_{\mu\nu}}{q^2\left(1-\Pi(q^2)\right)} = \frac{-i\, g_{\mu\nu}}{q^2\left(1-\Pi(0)+\cdots\right)} = \frac{-i\, g_{\mu\nu} Z_3}{q^2}$$

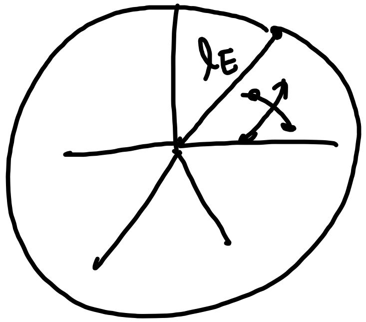
$$Z_3 = \frac{1}{1-\Pi(0)}$$

Dimensional Reg.

$$4 \to d$$

$$\int \frac{d^4 l_E}{(l_E^2+\Delta^2)^2} \sim \int \frac{\cancel{l_E^3}\, d l_E}{\cancel{l_E^4}\; l_E} \sim \text{logarithmic Div.}$$

$$\int \frac{d^d l_E}{(2\pi)^d}\cdots = \int \frac{d\Omega_d}{(2\pi)^d} \int_0^\infty l_E^{d-1}\, d l_E \cdots$$



$$(\sqrt{\pi})^d = \left(\int_{-\infty}^{\infty} dx\, e^{-x^2}\right)^d = \int\cdots\int_{-\infty}^{\infty} dx_1 \cdots dx_d\; e^{-\underbrace{(x_1^2+x_2^2+\cdots+x_d^2)}_{=\,x^2}} = \int d\Omega_d \underbrace{\int_0^\infty x^{d-1} dx\; e^{-x^2}}$$

$$x^2 \equiv t \to 2x\,dx = dt$$

$$dx = \frac{dt}{2\sqrt{t}}$$

$$= \int d\Omega_d \underbrace{\int_0^\infty dt\; \frac{t^{\overbrace{\frac{d-1}{2}-\frac{1}{2}}^{\frac{d}{2}-1}}}{2}\; e^{-t}}_{\frac{\Gamma(\frac{d}{2})}{2}}$$

$$\therefore \int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \quad //$$

$$d=3 \quad 4\pi = \frac{2\pi^{3/2}}{\Gamma(\frac{3}{2})} \quad \checkmark$$

$$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\underbrace{\Gamma\left(\tfrac{1}{2}\right)}_{\sqrt{\pi}}, \quad \tfrac{3}{2} \to \tfrac{1}{2}+1$$

$$d=4; \quad \frac{2\pi^2}{\Gamma(2)} = 2\pi^2 \quad //$$

$$\int \frac{1}{(2\pi)^d} \frac{d^d l}{(l^2+\Delta)^2} = \int d\Omega_d \int_0^\infty \frac{l^{d-2} \overbrace{l\,dl}^{\to \frac{1}{2} d l^2}}{(l^2+\Delta)^2}$$

$$d l^2 = 2 l\, dl$$

$$\boxed{\Delta^{\frac{d}{2}-2}\,\Gamma\left(2-\frac{d}{2}\right)\pi^{d/2}}\ \boxed{\times\frac{1}{(2\pi)^d}} = \underbrace{\int d\Omega d}_{\frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}}\ \underbrace{\frac{1}{2}\int_0^\infty \frac{{l^2}^{\frac{d-2}{2}}\, d l^2}{(l^2+\Delta)^2}}$$

$$=$$

$$\frac{1}{\cancel{2}}\frac{\cancel{2}\pi^{d/2}}{\cancel{\Gamma\left(\frac{d}{2}\right)}} \times \Delta^{\frac{d-2}{2}-1}\frac{\cancel{\Gamma\left(\frac{d}{2}\right)}\Gamma\left(-\frac{d}{2}+2\right)}{\Gamma(2)}$$

$$\underbrace{\frac{d}{2}-1}_{\sim}\qquad -\frac{d}{2}+2-1$$

$$=$$

$$\underbrace{\Delta^{\frac{d-2}{2}-1}\int_0^1 dx\,(1-x)^{\frac{d-2}{2}}x^{-\frac{d-2}{2}}}_{\beta-積分} =$$

$$\frac{\Delta}{l^2+\Delta} \equiv x \quad \to l^2+\Delta = \frac{\Delta}{x}$$

$$l^2 = \frac{\Delta(1-x)}{x} \qquad d l^2 = -\frac{\Delta}{x^2}dx$$

$$\int_1^0 \frac{\left(\Delta\frac{(1-x)}{x}\right)^{\frac{d-2}{2}}\left(-\frac{\Delta}{x^2}\right)dx}{\left(\frac{\Delta}{x}\right)^2}$$

$$\int_0^1 dx\ x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\underbrace{\Delta^{\frac{d}{2}-2}}\ \underbrace{\Gamma\left(2-\frac{d}{2}\right)}\pi^{d/2} \quad \times\frac{1}{(2\pi)^d}$$

$d \to 4 \quad \Gamma(0) = \infty$

$4 - d \equiv \epsilon$

$$\Gamma\left(2-\frac{d}{2}\right) = \Gamma\left(\frac{\epsilon}{2}\right) \sim \frac{2}{\epsilon} - \gamma + O(\epsilon)$$

$\epsilon \to 0$

Euler-Mascheroni

0.5772...

$$\pi^{d/2} = \pi^{2-\frac{\epsilon}{2}}$$

$$\Delta^{-\frac{\epsilon}{2}} \simeq e^{\log\left(\Delta^{-\frac{\epsilon}{2}}\right)} = e^{-\frac{\epsilon}{2}\log\Delta} \simeq 1 - \frac{\epsilon}{2}\log\Delta + \cdots$$

$$= \frac{1}{2^4\pi^2}\underbrace{\left(\frac{2}{\epsilon} - \gamma + \cdots\right)\left(1 - \frac{\epsilon}{2}\log\Delta + \cdots\right)}$$

$$\underline{\underline{\frac{2}{\epsilon}}} - \gamma - \underline{\underline{\log\Delta}} + \cdots$$

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}$$

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{l_E^2}{(l_E^2+\Delta)^n} = \frac{d/2}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1}$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu \qquad d=4$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1} \qquad = \underline{-(d-2)\gamma^\nu}$$

$$g^{\mu\nu} g_{\mu\nu} = \delta^\mu_\mu = d \qquad \Rightarrow \Pi_2 = \int \frac{d^d l_E}{(2\pi)^d} \frac{(1-\frac{d}{2})\, g^{\mu\nu} l_E^2}{(l_E^2+\Delta)^2}$$

$$\Pi_2(q^2) = \frac{-8e^2}{(4\pi)^{d/2}} \int_0^1 dx\, x(1-x) \quad \underbrace{\Gamma(2-\tfrac{d}{2}) \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}}_{\left(\frac{2}{\epsilon} - \log\Delta(x) - \gamma + \log(4\pi) \cdots\right)}$$