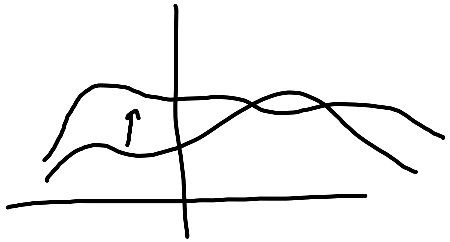
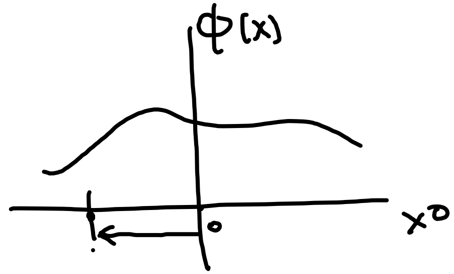
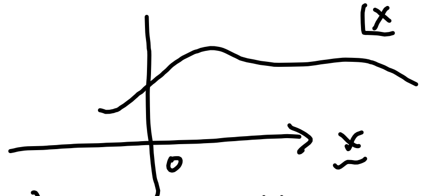
Chap 3. Dirac fields

$\phi(x)$







space-time trasformation ; $\quad x^\mu \rightarrow \Lambda^\mu{}_\nu x^\nu = x'^\mu$

$$g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

(ex) 3-dir boost

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

$SO(3,1)$

$x = \Lambda x_0$

$$\phi(x) \longrightarrow \phi'(x) = \phi(x_0) = \phi(\Lambda^{-1}x) \neq \phi(x)$$

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1} x)$$

$$\partial_\mu \phi \to \partial_\mu \phi' = \partial_\mu \phi(\Lambda^{-1} x) = (\Lambda^{-1})_\mu{}^\nu \, \partial_\nu \phi(\Lambda^{-1} x)$$

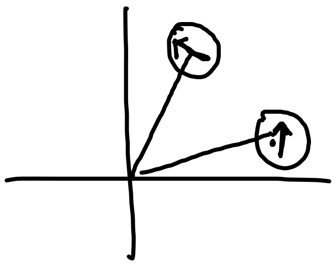
$(\partial_\mu \phi)^2$ invariat

$$\underline{(\Lambda^{-1})_\mu{}^\nu \, (\Lambda^{-1})_\alpha{}^\beta \, g^{\mu\alpha} = g^{\nu\beta}}$$

non-scalar field

(ex) $\vec{V}(\vec{r}) \longrightarrow \hat{R}\cdot\vec{V}(R^{-1}\vec{r})$ 3-vector field

$\vec{r} \to \vec{r}\,' = \hat{R}\cdot\vec{r}$



4-dim ; $V^{\mu}(x) \longrightarrow \Lambda^{\mu}{}_{\nu} V^{\nu}(\Lambda^{-1}x)$ → 4-vector field

tensor field

$V^{\mu\alpha\cdots}(x) \longrightarrow \Lambda^{\mu}{}_{\mu'}\,\Lambda^{\alpha}{}_{\alpha'}\cdots V^{\mu'\alpha'\cdots}(\Lambda^{-1}x)$

Spinor

$$\Phi_a(x) \longmapsto M_{ab}(\Lambda)\, \Phi_b(\Lambda^{-1}x)$$

$a = \mu$

SO(3,1) ; $\Lambda\Lambda' = \Lambda''$

$M(\Lambda)\, M(\Lambda') = M(\Lambda'')$

→ Representation

- singlet
- vector = fundamental rep. $\square$
- tensor ; $V^{\mu\alpha} = V_S^{\mu,\alpha} + \underbrace{V_A^{\mu\alpha}}$

$V_S^{\mu\alpha} = \dfrac{V^{\mu\alpha} + V^{\alpha\mu}}{2}$, $\mu \leftrightarrow \alpha$: $+1$

$V_A^{\mu\alpha} = \underbrace{\dfrac{V^{\mu\alpha} - V^{\alpha\mu}}{2}}$, $\mu \leftrightarrow \alpha$: $-$

- * $\boxed{\text{spinor}}$

Group

Algebra (small elements of group) SO(3,1)

$$\text{Group} = e^{i\theta_a T^a}$$

algebra $\theta_a \ll 1$

→ commutation

$$\underbrace{[T^a, T^b] = i f^{abc} T^c}_{\text{generator}}$$

$$SU(2) = SO(3) \rightarrow \underset{\substack{\downarrow \\ \text{3차원 Rotation}}}{R} = e^{i\theta_a J^a}$$

A.M.

su(2)

$$\boxed{[J^a, J^b] = i\epsilon^{abc}J^c}$$

$$\vec{J} = \vec{r} \times \vec{p} = \vec{r} \times (-i\vec{\nabla})$$

$$J^{(ij)} = -i\left(x^i \vec{\nabla}^j - x^j \vec{\nabla}^i\right)$$

$$J^i = \epsilon^{ijk} J^{(jk)} \rightarrow [J^i, J^j] = i\epsilon^{ijk}J^k$$

SO(3,1)

$$J^{\mu\nu} = i\left(x^\mu \underset{\substack{\downarrow \\ \frac{\partial}{\partial x^\nu}}}{\partial^\nu} - x^\nu \partial^\mu\right)$$

$$\left(\varepsilon^{\mu\nu\alpha\beta} J_{\alpha\beta} = \tilde{J}^{\mu\nu}\right)$$

SO(3,1)

$$[J^{\mu\nu}, J^{\alpha\beta}] = i^2 [x^\mu \partial^\nu, x^\alpha \partial^\beta] + \cdots = i\left(g^{\nu\alpha} J^{\mu\beta} - g^{\nu\beta} J^{\mu\alpha} - g^{\mu\alpha} J^{\nu\beta} + g^{\mu\beta} J^{\nu\alpha}\right)$$

$$x^\mu \overbrace{[\partial^\nu, x^\alpha]}^{=} \partial^\beta = g^{\nu\alpha} x^\mu \partial^\beta$$

vector rep.

$$\left(\underbrace{J^{\underline{\mu\nu}}}\right)_{\alpha\beta} = i\left(\delta^{\mu}_{\alpha}\,\delta^{\nu}_{\beta} - \delta^{\nu}_{\alpha}\,\delta^{\mu}_{\beta}\right)$$

$$J^{\mu\nu} = i \quad \begin{array}{cc} & \nu \\ \mu & \begin{pmatrix} 0 & & 1 & \\ & \ddots & & \\ & -1 & \ddots & \\ & & & \ddots \end{pmatrix} \end{array}$$

$$(\Lambda)_{\alpha}^{\ \beta} = \left(e^{-i\,\underbrace{\omega_{\mu\nu}}\,J^{\mu\nu}}\right)_{\alpha}^{\ \beta} \overset{\omega_{\mu\nu}\ll 1}{\approx} \left(\mathbb{1} - i\,\omega_{\mu\nu}J^{\mu\nu}\right)_{\alpha}^{\ \beta}$$

$$\| $$

$$\delta_{\alpha}^{\beta} - i\,\omega_{\mu\nu}(J^{\mu\nu})_{\alpha}^{\ \beta}$$

params

$$\text{angles, velocities} = \omega_{\mu\nu} = -\omega_{\nu\mu}$$

$$\underline{3} \qquad \underline{3}$$

$$\beta = \frac{v}{c} = v \ll 1 \qquad \gamma = \frac{1}{\sqrt{1-v^2}} \approx 1$$

(ex) z-dir. boost

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \cong \begin{pmatrix} 1 & 0 & 0 & -v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v & 0 & 0 & 1 \end{pmatrix} = \mathbb{1} - \underbrace{i\,\omega_{\mu\nu}J^{\mu\nu}}_{\omega_{03}\underbrace{J^{03}}}$$

$$v\begin{pmatrix} 0 & & -1 \\ 0 & 0 & \\ 1 & & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Spinor

Dirac Matrices

$\gamma^\mu \leftarrow n\times n$ matrix $\quad \mu = 0, \cdots, d-1$

d-dim space-time

Clifford algebra $\longrightarrow$ $\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \mathbb{1}_n$

$$\{\gamma^\mu,\gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix} \quad \overbrace{-1 \cdots -1}^{d-1}$$

γ^μ : Hermitian or antiHermitian matrices

$$\underbrace{\gamma^{\mu\dagger} = \gamma^\mu}, \quad \underbrace{\gamma^{\mu\dagger} = -\gamma^\mu}$$

$2\ n\times n = 2n^2$ indep. elements $= 2n^2 - n^2$

$= \boxed{n^2}$ 개

n^2개 조건

1	$d = {}_dC_1$	$\frac{d(d-1)}{2} = {}_dC_2$	${}_dC_3$		${}_dC_d = 1$
$\mathbb{1}$,	γ^μ,	$\underbrace{\gamma^{[\mu}\gamma^{\nu]}}$,	$\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$,	$\cdots$	$\gamma^{[\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_d]}$

$$\gamma^{[\mu}\gamma^{\nu]} = \frac{[\gamma^\mu,\gamma^\nu] + \cancel{\{\gamma^\mu,\gamma^\nu\}}}{2}$$

$$\gamma^{[\mu}\gamma^\nu\gamma^{\rho]} \ne \mathbb{1}$$

$$\gamma^{[\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_d]} = \gamma^0\gamma^1\cdots\gamma^{d-1}$$

$$= 1 + {}_dC_1 + {}_dC_2 + \cdots + {}_dC_d = (1+1)^d = 2^d = n^2 \Rightarrow \boxed{n = 2^{\frac{d}{2}}}$$

$d=4 \qquad n = 2^2 = 4$

$d=6 \qquad n=8$

$\vdots$

$d=10 \qquad n = 2^5$

$[J^{\mu\nu}, J^{\alpha\beta}] = i \; g^{\mu\alpha} J^{\nu\beta} \; \cdots$

$S^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$

3dim $\to$ $n=2$

$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1} \longrightarrow \{\gamma^i, \gamma^j\} = -2\,\mathbb{1}_2$

σ^i Pauli matrices $\qquad \{\sigma^i, \sigma^j\} = 2\delta_{ij}\mathbb{1}_2$

$\therefore \quad \underline{\gamma^i = i\sigma^i}$

$S^{ij} = \frac{i}{4} i^2 \underbrace{[\sigma^i, \sigma^j]}_{= 2i\epsilon^{ijk}\sigma^k} = \frac{1}{2}\epsilon^{ijk}\sigma^k$

4-dim $\rightarrow n=4$

$$\gamma^\mu : \quad \gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \leftarrow 2\times 2 \text{ Id.} \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \mathbb{1}.$$

(ex) $\gamma^{0\,2} = \mathbb{1}$

$\gamma^{i\,2} = -\mathbb{1}$

$$\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^{i2} & 0 \\ 0 & -\sigma^{i2} \end{pmatrix} = -\mathbb{1}$$

$\mu \neq \nu \quad \{\gamma^\mu, \gamma^\nu\} = 0$

$\gamma^{0\dagger} = \gamma^0$: Hermitian

$\vec{\gamma}^\dagger = -\vec{\gamma}$: anti "

$$S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \leftarrow \text{boost.}$$

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\epsilon^{ijk}\underbrace{\begin{pmatrix} \sigma^k & \\ & \sigma^k \end{pmatrix}}_{\Sigma^k} \leftarrow \text{rotation}$$

Dirac Eq.

$$(\underset{\sim}{\partial}^2 + m^2)\phi = 0$$

$$\sqrt{\partial^2 + m^2} \overset{?}{=}$$

$$[\underset{\sim}{\gamma^\mu}, \underset{\sim}{S^{\rho\sigma}}] = \underset{\sim}{(J^{\rho\sigma})^\mu{}_\nu}\ \underset{\sim}{\gamma^\nu} \leftarrow \{\gamma^\alpha, \gamma^\beta\} = 2g^{\alpha\beta}\mathbb{1}.$$

$$S^{\rho\sigma} = \frac{i}{4}[\gamma^\rho, \gamma^\sigma]$$

$$(J^{\rho\sigma})^\mu{}_\nu = i(\delta^\rho_\nu g^{\sigma\mu} - \delta^\sigma_\nu g^{\rho\mu}) \Leftrightarrow (J^{\rho\sigma})_{\alpha\beta} = i(\delta^\rho_\alpha\delta^\sigma_\beta - \delta^\rho_\beta\delta^\sigma_\alpha)$$

γ^ν: "Lorents vector"

$$\overbrace{e^{\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}}}^{} = \Lambda_{\frac{1}{2}}^{-1}$$

$$\left(1 + \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right)\gamma^\mu\left(1 - \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right) \rightarrow e^{-\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}} \equiv \Lambda_{\frac{1}{2}} \text{ (4x4 matrix)}$$

$$= \gamma^\mu + \frac{i}{2}\sum_{\rho\sigma}\omega_{\rho\sigma}(S^{\rho\sigma}\gamma^\mu - \gamma^\mu S^{\rho\sigma})$$

$$-[\gamma^\mu, S^{\rho\sigma}] = -(J^{\rho\sigma})^\mu{}_\nu\gamma^\nu$$

$$= \left(\delta^\mu{}_\nu - \frac{i}{2}\omega_{\rho\sigma}(J^{\rho\sigma})^\mu{}_\nu\right)\gamma^\nu \rightarrow \left(e^{-\frac{i}{2}\omega_{\rho\sigma}\underline{\underline{J^{\rho\sigma}}}}\right)^\mu{}_\nu$$

general ω ;

$$\underline{\Lambda_{\frac{1}{2}}^{-1} \; \underline{\underline{\gamma^\mu}} \; \Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu \, \gamma^\nu}$$

Λ = Lorentz matrix

(ex) z-dir boost

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

$$e^{-\frac{i}{2} \omega_{\rho\sigma} J^{\rho\sigma}}$$

$J^{\rho\sigma}$ → vector

$$(J^{\rho\sigma})^\mu{}_\nu = i\left(\delta^\rho_\nu g^{\sigma\mu} - \delta^\sigma_\nu g^{\rho\mu}\right)$$

$\omega_{\rho\sigma}$: (3-rotation angles, 3-boost velocities)

$$e^{-\frac{i}{2} \omega_{\rho\sigma} \underbrace{S^{\rho\sigma}}_{\text{spinor}}}$$

Spinor field $\quad \underline{\psi(x) \longmapsto \psi'(x) = \Lambda_{\frac{1}{2}} \psi(\Lambda^{-1} x)}$

(cf) $\phi(x) \longmapsto \phi'(x) = 1 \cdot \phi(\Lambda^{-1} x)$

Dirac eq.

$$\boxed{\Big(i\,\overset{4\times4}{\gamma^\mu}\,\partial_\mu - m\overset{4\times4}{\mathbb{1}}\Big)\,\overset{\begin{pmatrix}\cdot\\\cdot\\\cdot\\\cdot\end{pmatrix}}{\psi}(x) = 0}$$

$$(i\gamma^\mu\partial_\mu + m)(i\gamma^\nu\partial_\nu - m) = -\underbrace{\partial_\mu\partial_\nu}_{\mu\leftrightarrow\nu:\ \text{sym}}\,\gamma^\mu\gamma^\nu - m^2\mathbb{1}$$

$$\gamma^\mu\gamma^\nu = \frac{1}{\cancel{2}}\Big(\cancel{\underbrace{[\gamma^\mu,\gamma^\nu]}_{\text{antisym}}} + \underbrace{\overbrace{\{\gamma^\mu,\gamma^\nu\}}^{\cancel{2}g^{\mu\nu}\mathbb{1}}}_{\text{sym}}\Big)$$

$$= -\underbrace{(\partial^2 + m^2)\,\psi = 0}$$

$$\partial^2 = \left(\frac{\partial}{\partial t}\right)^2 - \nabla^2$$

$$\psi(x) \longmapsto \Lambda_{\frac{1}{2}}\,\psi(\underbrace{\Lambda^{-1}x}_{x'})$$

$$x = \Lambda x' \rightarrow \frac{\partial}{\partial x^\mu} = \Lambda^{-1}{}_\mu{}^\nu\,\frac{\partial}{\partial x'^\nu}, \qquad \frac{\partial}{\partial x^\mu} = \partial_\mu$$

$$\longrightarrow \Big(i\gamma^\mu(\Lambda^{-1})^\nu{}_\mu\,\partial'_\nu - m\mathbb{1}\Big)\Lambda_{\frac{1}{2}}\psi(x')$$

$$= \Lambda_{\frac{1}{2}}\Lambda^{-1}_{\frac{1}{2}}\Big(i\,\underline{\underline{\gamma^\mu}}(\Lambda^{-1})^\nu{}_\mu\,\partial'_\nu - m\underline{\underline{\mathbb{1}}}\Big)\Lambda_{\frac{1}{2}}\psi(x')$$

$$= \Lambda_{\frac{1}{2}} \Big(\underbrace{\Lambda^{-1}_{\frac{1}{2}} \gamma^\mu \Lambda_{\frac{1}{2}}}_{= \Lambda^\mu{}_\alpha \gamma^\alpha} i (\Lambda^{-1})^\nu{}_\mu \, \partial'_\nu - m \underline{\underline{\mathbb{1}}} \Big) \psi(x')$$

$$\underbrace{i \, \underbrace{(\Lambda^{-1})^\nu{}_\mu \Lambda^\mu{}_\alpha}_{\delta^\nu{}_\alpha} \gamma^\alpha \partial'_\nu}_{\gamma^\nu \partial'_\nu}$$

$$= \Lambda_{\frac{1}{2}} \underbrace{\left(i \gamma^\nu \partial'_\nu - m \right) \psi(x')} = 0$$

Lorentz-Invariant Lagrangian.

$$\left((i\gamma^\mu \partial_\mu - m)\psi \right)_x \longrightarrow \underset{\sim}{\Lambda_{\frac{1}{2}}} \left[(\quad) \psi \right]_{x'}$$

$$\psi \to \psi^\dagger$$

$$\frac{\psi^\dagger (i\gamma^\mu \partial_\mu - m)\psi}{\psi^\dagger \to \psi^\dagger \Lambda^\dagger_{\frac{1}{2}}}$$

$$\Lambda^\dagger_{\frac{1}{2}} \Lambda_{\frac{1}{2}} \neq \mathbb{1}$$

$$\Lambda^\dagger_{\frac{1}{2}} \neq \Lambda^{-1}_{\frac{1}{2}}$$

$$(\psi \to \Lambda_{\frac{1}{2}} \psi)$$

$$\Lambda_{\frac{1}{2}} = e^{-\frac{i}{2} \underbrace{\omega_{\rho\sigma} S^{\rho\sigma}}_{\omega_{0i} S^{0i} + \omega_{ij} S^{ij}}}$$

$$S^{\rho\sigma} = \frac{i}{2} [\gamma^\rho, \gamma^\sigma]$$

$$S^{\rho\sigma\dagger} = -\frac{i}{2} [\gamma^{\sigma\dagger}, \gamma^{\rho\dagger}] \neq S^{\rho\sigma}$$

$$S^{0i\dagger} = +\frac{i}{2} [\gamma^i, \gamma^0] = -S^{0i}$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \gamma^{0\dagger}$$

$$\underset{1,2,3}{\gamma^i} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad \gamma^{i\dagger} = \begin{pmatrix} 0 & -\sigma^{i\dagger} \\ \sigma^{i\dagger} & 0 \end{pmatrix} = -\gamma^i$$

$$\{\gamma^0, \gamma^i\} = 0$$

$$[\gamma^0, S^{ij}] = 0 \, ,$$

$$\overline{\psi} = \psi^\dagger \gamma^0$$

$$\gamma^0 \underbrace{\gamma^0 \gamma^i} = \gamma^i$$

$$-)\ \gamma^0 \gamma^i \gamma^0 = -\gamma^i$$

$$\underbrace{\gamma^0 S^{0i}}_{\frac{i}{2}[\gamma^0, \gamma^i]} = i\, \underbrace{\underbrace{\gamma^i \gamma^0}_{\frac{\gamma^i \gamma^0 - \gamma^0 \gamma^i}{2}} \gamma^0}_{-S^{0i}} = -\underbrace{S^{0i} \gamma^0}$$

$$\Lambda_{\frac{1}{2}} = e^{-\frac{i}{2}\left[\omega_{0i} S^{0i} + \omega_{ij} S^{ij}\right]}$$

$$\Lambda^{\dagger}_{\frac{1}{2}} = e^{\frac{i}{2}\left[\underline{-\omega_{0i}}\ \underline{S^{0i}} + \underline{\omega_{ij} S^{ij}}\right]}$$

$$\Lambda^{\dagger}_{\frac{1}{2}} \gamma^0 = \gamma^0 \underbrace{e^{\frac{i}{2}\left[\omega_{0i} S^{0i} + \omega_{ij} S^{ij}\right]}}_{\Lambda^{-1}_{\frac{1}{2}}}$$

$$\therefore \boxed{\Lambda^{\dagger}_{\frac{1}{2}} \gamma^0 = \gamma^0 \Lambda^{-1}_{\frac{1}{2}}}$$

$$\psi \longrightarrow \Lambda_{\frac{1}{2}} \psi \dashrightarrow (i\gamma^\mu \partial_\mu - m)\psi\big|_x \longmapsto \underset{\sim}{\Lambda_{\frac{1}{2}}} (i\gamma^\mu \partial_\mu - m)\psi\big|_{\substack{x' \\ \shortparallel \\ \Lambda^{-1}x}}$$

$$\psi^\dagger \rightarrow \psi^\dagger \Lambda_{\frac{1}{2}}^\dagger$$

$$\bar{\psi} = \psi^\dagger \gamma^0 \longrightarrow \psi^\dagger \Lambda_{\frac{1}{2}}^\dagger \gamma^0 = \underline{\psi^\dagger \gamma^0} \Lambda_{\frac{1}{2}}^{-1} = \bar{\psi} \underset{\sim}{\Lambda_{\frac{1}{2}}^{-1}}$$

$$\therefore \left[\bar{\psi} (i\gamma^\mu \partial_\mu - \underset{\sim}{m}) \psi \right]_x \longmapsto \left[\qquad \qquad \right]_{x'}$$

Lorentz scalar.

Weyl Spinor

4d $\quad \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ ← 2-component. no mix under L.T. → massless (gapless)

Dirac — 4d

Weyl → 2 compo

Majorana ← 4 "

$\frac{i}{4}[\gamma^i, \gamma^j] \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$

$$\Lambda_{\frac{1}{2}} = e^{-\frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}}$$

$$\omega_{\alpha\beta}S^{\alpha\beta} = \omega_{0i}S^{0i} + \omega_{i0}S^{i0} + \omega_{ij}S^{ij}$$

$S^{ij} = 2i\,\varepsilon^{ijk}\sigma^k$

$$S^{ij} = \frac{i}{4}\left[\underbrace{\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}}_{\begin{matrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{matrix}} - \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\right] = -\frac{i}{4}\begin{pmatrix} \{\sigma^i, \sigma^j\} & 0 \\ 0 & \{\sigma^i, \sigma^j\} \end{pmatrix}$$

$$= \frac{1}{2}\varepsilon^{ijk}\underbrace{\Sigma^k}_{= \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}}$$

$\omega_{ij}\varepsilon^{ijk} \equiv 2\theta^k$

$2\theta^k\Sigma^k = 2\vec{\theta}\cdot\vec{\Sigma}$

$$\therefore \quad \omega_{ij}S^{ij} = \underline{\vec{\theta}\cdot\vec{\Sigma}}$$

$$\omega_{01} \underline{S^{01}} + \omega_{10} S^{10} \underset{=}{=} 2\, \omega_{01} S^{01}$$

$$\frac{i}{4} \cdot \quad \underline{\gamma^0 \gamma^1} - \underline{\gamma^1 \gamma^0}$$

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} & \sigma^1 \\ -\sigma^1 & \end{pmatrix} = \begin{pmatrix} -\sigma^1 & \\ & \sigma^1 \end{pmatrix} - \begin{pmatrix} \sigma^1 & \\ & -\sigma^1 \end{pmatrix} = \begin{pmatrix} -2\sigma^1 & 0 \\ 0 & 2\sigma^1 \end{pmatrix}$$

$$\omega_{0i} = \beta_i = v_i \quad (c=1)$$

$$\omega_{0i} S^{0i} = \frac{i}{4}\, 2\, \beta_i \begin{pmatrix} -2\sigma^i & 0 \\ 0 & 2\sigma^i \end{pmatrix} = i \begin{pmatrix} -\vec{\beta}\cdot\vec{\sigma} & 0 \\ 0 & \vec{\beta}\cdot\vec{\sigma} \end{pmatrix}$$

$$\therefore \quad \Lambda_{\frac{1}{2}} = e^{-\frac{i}{2}\left[(-i)\begin{pmatrix} \vec{\beta}\cdot\vec{\sigma} & 0 \\ 0 & -\vec{\beta}\cdot\vec{\sigma} \end{pmatrix} + \begin{pmatrix} \vec{\theta}\cdot\vec{\sigma} & 0 \\ 0 & \vec{\theta}\cdot\vec{\sigma} \end{pmatrix}\right]}$$

$$= e^{-\left(\begin{array}{c|c} \vec{\beta}\cdot\frac{\vec{\sigma}}{2} + i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} & 0 \\ \hline 0 & -\vec{\beta}\cdot\frac{\vec{\sigma}}{2} + i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} \end{array}\right)} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

$$\mathcal{L} \supset -m\, \bar{\psi}\psi = -m\, \psi^\dagger \underline{\gamma^0}\, \psi$$

$$= -m\, (\psi_L^\dagger, \psi_R^\dagger)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = -m\left(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R\right)$$

if $\underline{m=0} \rightarrow \psi_L, \psi_R$ indep.

L.T. $\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} \Rightarrow \begin{pmatrix} \left[e^{-(\vec{\beta}\cdot\frac{\vec{\sigma}}{2}+i\vec{\theta}\cdot\frac{\vec{\sigma}}{2})}\right]_{2\times 2}\psi_L \\ \hline \left[e^{-(-\vec{\beta}\cdot\frac{\vec{\sigma}}{2}+i\vec{\theta}\cdot\frac{\vec{\sigma}}{2})}\right]\psi_R \end{pmatrix}$

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x^μ, $\frac{\partial}{\partial x^\mu} \equiv \partial_\mu$

$$(i\gamma^\mu\partial_\mu - m)\psi = \begin{pmatrix} -m & i\underbrace{(\mathbb{1}\partial_0 + \vec{\sigma}\cdot\vec{\nabla})}_{\sigma\cdot\partial} \\ i\underbrace{(\mathbb{1}\partial_0 - \vec{\sigma}\cdot\vec{\nabla})}_{\bar{\sigma}\cdot\partial} & -m \end{pmatrix}\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$$

$\mathbb{1}\partial^0 + \vec{\gamma}\cdot\vec{\nabla}$
$\quad$ ''

$$\gamma^\mu\partial_\mu \equiv \underbrace{g_{\mu\nu}}_{\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}}\gamma^\mu\partial^\nu = \gamma^0\partial^0 - \gamma^i\underbrace{\partial^i}_{''}$$

$\frac{\partial}{\partial x_i} = -\frac{\partial}{\partial x^i}$

$\therefore \quad \underline{\underline{m=0}} \longrightarrow$ decouple ψ_L from ψ_R

$$(\mathbb{1}\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\psi_L = 0 \longrightarrow i\bar{\sigma}\cdot\partial\psi_L = 0$$

$$(\mathbb{1}\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\psi_R = 0 \longrightarrow i\sigma\cdot\partial\psi_R = 0$$

$$(\mathbb{1}, \vec{\sigma}) = \sigma^{\overset{0,1,2,3}{\downarrow}\mu} \quad (\sigma^0 \equiv \mathbb{1})$$

$$(\mathbb{1}, -\vec{\sigma}) = \bar{\sigma}^\mu$$

3.3. free particle solutions of Dirac eq.

$$\psi(x) = \underbrace{u(p)}_{\text{spinor}} \underbrace{e^{-ip\cdot x}}_{\text{function}}, \qquad m^2 = p^2$$

spinor $\begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$

Feynman notation $\not{p} \equiv \gamma^\mu p_\mu$

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \quad \rightarrow \quad \underline{(\overbrace{\gamma^\mu p_\mu} - m\mathbb{1})\, u(p)} = 0$$

$u(p)$: ?

① rest frame ; $\quad p_\mu = (m, \vec{0}) \quad (p^2 = m^2)$

$$\not{p} = m\gamma^0 \quad \rightarrow \quad m\,(\underbrace{\gamma^0 - \mathbb{1}_4}_{\begin{pmatrix} -\mathbb{1}_2 & \mathbb{1}_2 \\ \mathbb{1}_2 & -\mathbb{1}_2 \end{pmatrix}})\, u(p) = 0$$

2차원 vector $\begin{pmatrix} \xi \\ \hdashline \xi \end{pmatrix}$

$(m, \vec{0})$ normalization

$$\therefore\ u(p_0) = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}, \qquad \xi^\dagger \xi = 1 \ \rightarrow\ \xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \underline{\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$\Sigma_3 = \begin{pmatrix} \sigma^3 & \\ & \sigma^3 \end{pmatrix} \qquad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \sigma^3 & \\ & \sigma^3 \end{pmatrix} = \begin{pmatrix} \sigma^3 & \\ & \sigma^3 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$\sigma^3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$\sigma^3 \xi^1 = \xi^1,\ \sigma^3 \xi^2 = -\xi^2 \rightarrow$ z-compo spin $\quad 1 \rightarrow \uparrow \quad 2 \rightarrow \downarrow$

$$u(p_0)_\uparrow = \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \quad u(p_0)_\downarrow = \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

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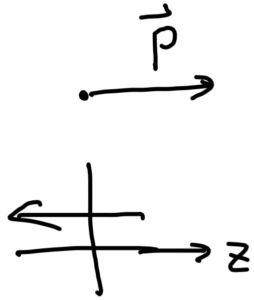
② $\vec{p} \neq 0$



$$\Lambda = \begin{pmatrix}\gamma & 0 & 0 & \beta\gamma\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \beta\gamma & 0 & 0 & \gamma\end{pmatrix} = e^{-\frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}}$$

$$A^3 = \begin{pmatrix} & & & 1\\ & & & \\ & & & \\ 1 & & & \end{pmatrix}\begin{pmatrix}1 & & & \\ & & & \\ & & & \\ & & & 1\end{pmatrix} = \begin{pmatrix} & & & 1\\ & & & \\ & & & \\ 1 & & & \end{pmatrix} = A$$

$$A^2 = \begin{pmatrix} & & & 1\\ & & & \\ & & & \\ 1 & & & \end{pmatrix}\begin{pmatrix} & & & 1\\ & & & \\ & & & \\ 1 & & & \end{pmatrix} = \begin{pmatrix}1 & & & \\ & & & \\ & & & \\ & & & 1\end{pmatrix}$$

$$= e^{\eta\underbrace{\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}}_{A}} = \mathbb{1} + \eta A + \frac{1}{2!}\eta^2 A^2 + \cdots$$

$$= \mathbb{1} + \underbrace{\left(\sum_{n=\text{even}\neq 0}\frac{1}{n!}\eta^n + \mathbb{1} - \mathbb{1}\right)}_{\cosh\eta} A^2 + \underbrace{\left(\sum_{\text{odd}}\frac{\eta^n}{n!}\right)}_{\sinh\eta} A$$

$$= \underbrace{(\mathbb{1} - A^2)}_{\begin{pmatrix}0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0\end{pmatrix}} + \cosh\eta\, A^2 + \sinh\eta\, A$$

$$= \begin{pmatrix}\cosh\eta & 0 & 0 & \sinh\eta\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \sinh\eta & 0 & 0 & \cosh\eta\end{pmatrix}$$

$$\sinh\eta = \gamma\beta, \quad \cosh\eta = \gamma$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$1 = \cosh^2\eta - \sinh^2\eta = \gamma^2 - \gamma^2\beta^2 = 1$$

$$\boxed{\tanh\eta = \beta} \rightarrow \eta = \tanh^{-1}\beta$$

cosh η

$$\Lambda = \begin{pmatrix} c & & & s \\ & 1 & & \\ & & 1 & \\ s & & & c \end{pmatrix} \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = P = \begin{pmatrix} mc \\ 0 \\ 0 \\ ms \end{pmatrix}$$

$$\begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = p_0$$

sinh η

$$\boxed{-\frac{i}{\cancel{2}}\cdot\cancel{2}\,\omega_{03}\,\underbrace{J^{03}}_{i(\delta^0_\nu g^{3\mu} - \delta^3_\nu g^{0\mu})}} = \eta A \implies \eta = \omega_{03}$$

$$i(\delta^0_\nu g^{3\mu} - \delta^3_\nu g^{0\mu}) = i \begin{array}{c|c} & A \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} & & & 1 \\ & & & \\ & & & \\ 1 & & & \end{pmatrix} \end{array}$$

$$\Lambda_{\frac{1}{2}} = e^{-\frac{i}{2}\left[\omega_{03} S^{03} + \omega_{30} S^{30}\right]} = e^{-i \underbrace{\omega_{03}}_{\eta} \underbrace{S^{03}}}$$

$$-\frac{i}{2}\begin{pmatrix}\sigma^3 & 0 \\ 0 & -\sigma^3\end{pmatrix} = \frac{i}{4}\underbrace{[\gamma^0, \gamma^3]}_{\begin{pmatrix}-2\sigma^3 & 0 \\ 0 & 2\sigma^3\end{pmatrix}} \qquad \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} & \sigma^3 \\ -\sigma^3 & \end{pmatrix}$$

$$= e^{-\frac{1}{2}\eta\begin{pmatrix}\sigma^3 & 0 \\ 0 & -\sigma^3\end{pmatrix}} = \begin{pmatrix} e^{-\frac{\eta}{2}\sigma^3} & 0 \\ 0 & e^{\frac{\eta}{2}\sigma^3}\end{pmatrix}$$

$$= \begin{pmatrix} c\,\mathbb{1} - s\,\sigma^3 & 0 \\ 0 & c\,\mathbb{1} + s\,\sigma^3 \end{pmatrix} \qquad c = \cosh\frac{\eta}{2}, \quad s = \sinh\frac{\eta}{2}, \quad \frac{1}{2}\left(e^{\frac{\eta}{2}} \pm e^{-\frac{\eta}{2}}\right)$$

$$e^{\alpha\sigma^3} = \underbrace{\sum_{n_{\text{even}}} \frac{\alpha^n}{n!}}_{\cosh\alpha} \underbrace{(\sigma^3)^n}_{\mathbb{1}} + \underbrace{\sum_{n_{\text{odd}}} \frac{\alpha^n}{n!}}_{\sinh\alpha} \underbrace{(\sigma^3)^n}_{\sigma^3} = \cosh\alpha \cdot \mathbb{1} + \sinh\alpha \cdot \sigma^3$$

$$u(p) = \Lambda_{\frac{1}{2}}\, u(p_0) = \begin{pmatrix} c\,\mathbb{1} - s\,\sigma^3 & 0 \\ 0 & c\,\mathbb{1} + s\,\sigma^3 \end{pmatrix} \sqrt{m}\begin{pmatrix}\xi \\ \xi\end{pmatrix}$$

$$\sqrt{m}\left(c\,\mathbb{1} \mp s\,\sigma^3\right) = \underbrace{\sqrt{m}\,e^{\frac{\eta}{2}}}_{\sqrt{E+p^3}}\left(\frac{\mathbb{1} \mp \sigma^3}{2}\right) + \underbrace{\sqrt{m}\,e^{-\frac{\eta}{2}}}_{\sqrt{E-p^3}}\left(\frac{\mathbb{1} \pm \sigma^3}{2}\right)$$

$$E = m\underbrace{\gamma}_{\cosh\eta} \qquad P_3 = m\underbrace{\gamma\beta}_{\sinh\eta} \quad \rightarrow \sqrt{E \pm P_3} = \sqrt{m}\, e^{\pm\eta/2}$$

$$\sqrt{E+p^3}\,\frac{\mathbb{1} \mp \sigma^3}{2} + \sqrt{E-p^3}\,\frac{\mathbb{1} \pm \sigma^3}{2}$$

$$\begin{pmatrix} \left[\sqrt{E+p^3}\,\frac{1-\sigma^3}{2} + \sqrt{E-p_3}\,\frac{1+\sigma^3}{2}\right]\xi \\ \left[\sqrt{E+p^3}\,\frac{1+\sigma^3}{2} + \sqrt{E-p^3}\,\frac{1-\sigma^3}{2}\right]\xi \end{pmatrix}$$

$$\begin{pmatrix} \left[\sqrt{E+p^3}\,\overbrace{\underbrace{\frac{1-\sigma^3}{2}}_{P_-}}^{\sqrt{\frac{1-\sigma^3}{2}}} + \sqrt{E-p_3}\,\overbrace{\underbrace{\frac{1+\sigma^3}{2}}_{P_+}}^{\sqrt{\frac{1+\sigma^3}{2}}}\right]\xi \\ \left[\sqrt{E+p^3}\,\underbrace{\frac{1+\sigma^3}{2}}_{P_+} + \sqrt{E-p^3}\,\frac{1-\sigma^3}{2}\right]\xi \end{pmatrix}$$

$$\frac{(\mathbb{1}\pm\sigma^3)^2}{4} = \frac{(\mathbb{1}\pm\sigma^3)}{2}$$

$$P_\pm \equiv \frac{1\pm\sigma^3}{2} \qquad P_\pm^2 = P_\pm \qquad P_+P_- = 0$$

$$\left(\sqrt{E+p_3}\;P_- + \underbrace{\sqrt{E-p_3}}\;P_+\right)^2 = \underbrace{(E+p_3)}\,P_- + (E-p_3)\,P_+ \qquad (E,\vec{p}) \quad (\mathbb{1},\vec{\sigma})$$

$$= E\,\underbrace{(P_-+P_+)}_{\mathbb{1}} + p_3\,\underbrace{(P_- - P_+)}_{-\sigma^3} = p\cdot\sigma$$

$$\left(\sqrt{E+p_3}\;P_+ + \underbrace{\sqrt{E-p_3}}\;P_-\right)^2 = (E+p_3)\,P_+ + (E-p_3)\,P_- \qquad (E,\vec{p}) \quad (\mathbb{1},\vec{\sigma})$$

$$= E\,\underbrace{(P_-+P_+)}_{\mathbb{1}} + p_3\,\underbrace{(P_+ - P_-)}_{\sigma^3} = p\cdot\bar{\sigma}$$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma}\;\xi \\ \sqrt{p\cdot\bar{\sigma}}\;\xi \end{pmatrix}$$

$$\sigma^3\xi^1 = \xi^1 \rightarrow P_-\xi^1 = 0 \quad P_+\xi^1 = \xi^1$$

$$\xi^1 = \begin{pmatrix}1\\0\end{pmatrix}$$

$$E = m\gamma \simeq p_3 = m\gamma\beta$$

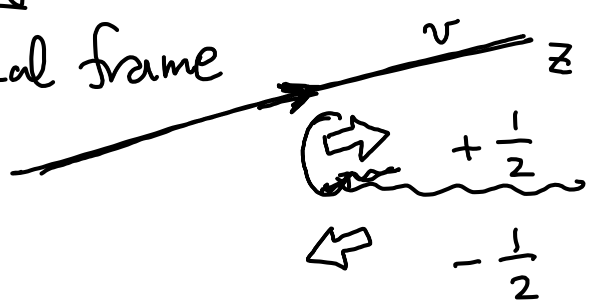
$$\xi = \xi^1: \quad \begin{pmatrix} \sqrt{E-p_3}\;\xi^1 \\ \sqrt{E+p_3}\;\xi^1 \end{pmatrix} \xrightarrow{\text{big boost}} \begin{matrix}\gamma \gg 1\\ (\beta\simeq 1)\end{matrix} \begin{pmatrix} 0 \\ \sqrt{2E}\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix}$$

$$\xi = \xi^2 \qquad P_-\xi^2 = 1,\; P_+\xi^2 = 0$$

$$\begin{pmatrix} \sqrt{E+p_3}\;\xi^2 \\ \sqrt{E-p_3}\;\xi^2 \end{pmatrix} \xrightarrow{\text{big boost}} \begin{pmatrix} \sqrt{2E}\begin{pmatrix}0\\1\end{pmatrix} \\ \begin{pmatrix}0\\0\end{pmatrix} \end{pmatrix}$$

helicity :

original frame



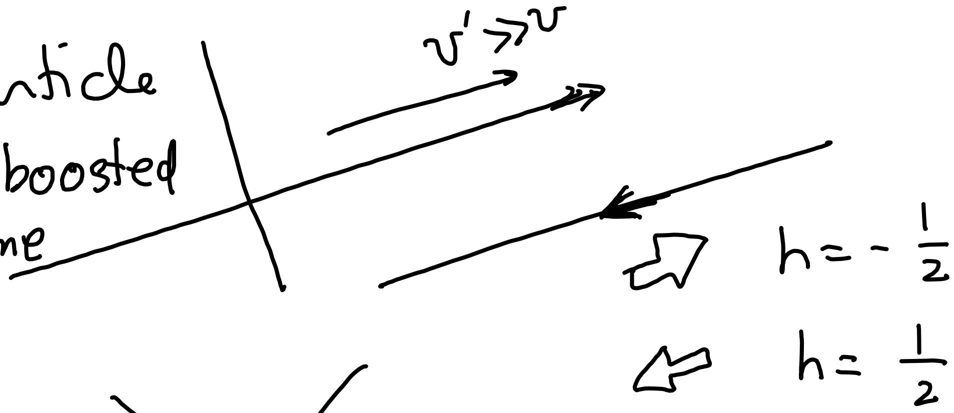
$h \equiv \hat{p} \cdot \vec{S}$

Right-handed.

Left-handed

massive particle

new boosted frame



helicity $\neq$ good ~~quantum #~~

massless particle (neutrino)

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi \\ \sqrt{p\cdot\bar{\sigma}}\ \xi \end{pmatrix} \longrightarrow u^{+}(p) = \left(\xi^{+}\sqrt{p\cdot\sigma},\ \xi^{+}\sqrt{p\cdot\bar{\sigma}}\right)$$

$$u^{+}u = \xi^{+}\, p\cdot\sigma\, \xi + \xi^{+}\, p\cdot\bar{\sigma}\, \xi$$

$$= \xi^{+}\big(\underset{(1,\vec{\sigma})}{p\cdot\sigma} + \underset{(1,-\vec{\sigma})}{p\cdot\bar{\sigma}}\big)\xi = 2\underset{E_{\vec{p}}}{\underline{p^{0}}}\ \xi^{+}\xi$$

$$\bar{u}u = \left(\underset{\sim}{\xi^\dagger}\sqrt{p\cdot\sigma},\ \xi^\dagger\sqrt{p\cdot\bar\sigma}\right)\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\ \xi \\ \sqrt{p\cdot\bar\sigma}\ \xi\end{pmatrix}$$

$$= \xi^\dagger\Big(\underbrace{\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma}}_{\displaystyle \sqrt{(p^0-\vec p\cdot\vec\sigma)(p^0+\vec p\cdot\vec\sigma)}} + \underbrace{\sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma}}_{m}\Big)\xi = \underset{\sim}{2m}\,\xi^\dagger\xi$$

$$\sqrt{(p^0-\vec p\cdot\vec\sigma)(p^0+\vec p\cdot\vec\sigma)} = \sqrt{p_0^2\mathbb{1}-(\vec p\cdot\vec\sigma)^2} = \sqrt{\underset{p_0^2-\vec p^2=m^2}{\underset{\sim}{p^2}}\mathbb{1}} = m$$

$$(\vec p\cdot\vec\sigma)^2 = (p^i\sigma^i)^2 = p^i\sigma^i p^j\sigma^j = \underset{\sim}{p^i p^j}\,\underbrace{\sigma^i\sigma^j}_{\frac12\left(\cancel{[\sigma^i,\sigma^j]}+\{\sigma^i,\sigma^j\}\right)} = p^i p^j\delta^{ij}\mathbb{1} = \vec p^{\,2}\mathbb{1}$$

$$\{\sigma^i,\sigma^j\} = 2\delta^{ij}\mathbb{1}$$

$$\boxed{u^s(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\ \xi^s \\ \sqrt{p\cdot\bar\sigma}\ \xi^s\end{pmatrix}}$$

$s = 1,\ 2$

$\uparrow\ \ \uparrow$

$\uparrow\ \ \downarrow$,

$$\bar u^r u^s = 2m\,\overbrace{\underset{\sim}{\xi^{r\dagger}\xi^s}}^{\delta^{rs}}$$

$$(1,0)\begin{pmatrix}1\\0\end{pmatrix} = 1$$

$$(0,1)\begin{pmatrix}1\\0\end{pmatrix} = 0$$

$$\boxed{v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \eta^s \\ -\sqrt{p\cdot\bar\sigma}\ \eta^s \end{pmatrix}}$$

$s = 1, 2$

$$\begin{array}{ccc} & s=1 & s=2 \\ \eta^s = & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{array}$$

$(\cdots\cdots)\begin{pmatrix} \vdots \end{pmatrix}$

$\shortparallel$

number

$$\bar v^r(p)\, v^s(p) = -2m \underbrace{\eta^{r\dagger}\eta^s}_{\delta^{rs}} = -2m\,\delta^{rs}$$

$$v^{r\dagger}(p)\, v^s(p) = 2E_{\vec p}\,\delta^{rs}$$

$$\bar u^r v^s = \bar v^r u^s = 0 \qquad (u^{r\dagger} v^s \neq 0)$$

$\uparrow$ cH?

Spin sums

$$\sum_{s=1,2} u^s(p)\,\bar u^s(p) = 4\times 4 \text{ matrix} = \sum_{s=1,2} \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^s \\ \sqrt{p\cdot\bar\sigma}\ \xi^s \end{pmatrix} \underbrace{(\xi^{s\dagger}\sqrt{p\cdot\sigma},\ \xi^{s\dagger}\sqrt{p\cdot\bar\sigma})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{(\xi_s^\dagger\sqrt{p\cdot\bar\sigma},\ \xi_s^\dagger\sqrt{p\cdot\sigma})}$$

$$\underbrace{\begin{pmatrix} u \end{pmatrix}(u^\dagger)\begin{pmatrix} \gamma^0 \end{pmatrix}}_{(\quad)}$$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma}\sum_s \xi^s\xi_s^\dagger\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\sigma}\sum_s\xi^s\xi^{s\dagger}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar\sigma}\sum_s\xi^s\xi_s^\dagger\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\bar\sigma}\sum_s\xi^s\xi_s^{s\dagger}\sqrt{p\cdot\sigma} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma}\,\mathbb{1}\,\sqrt{p\cdot\bar{\bar{\sigma}}} \;,\; \sqrt{p\cdot\sigma} & \sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\bar\sigma} \;,\; \sqrt{p\cdot\bar\sigma} \quad \sqrt{p\cdot\sigma} \end{pmatrix}$$

$$\sum_s \xi^s \xi^{s\dagger} = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}$$
$$= \mathbb{1}_2$$

$$= \begin{pmatrix} \overbrace{\sqrt{(p\cdot\sigma)(p\cdot\bar\sigma)}}^{=m\mathbb{1}} & p\cdot\sigma \\ p\cdot\bar\sigma & \overbrace{\sqrt{(p\cdot\bar\sigma)(p\cdot\sigma)}}^{=m\mathbb{1}} \end{pmatrix} = \begin{pmatrix} m\mathbb{1} & p\cdot\sigma \\ p\cdot\bar\sigma & m\mathbb{1} \end{pmatrix}$$

$$= p_\mu \underbrace{\begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}}_{\gamma^\mu} + m\mathbb{1}$$

$$\boxed{\sum_{s=\uparrow,\downarrow} u^s(p)\bar{u}^s(p) = \not{p} + m}$$

$$\boxed{\sum_{s=\uparrow,\downarrow} v^s(p)\bar{v}^s(p) = \not{p} - m}$$

$\mathbb{1}, \quad \gamma^\mu, \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu], \quad \gamma^\mu\gamma^5, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$\{\gamma^\mu, \gamma^\nu\} = 0 \quad \mu \neq \nu$

$$\gamma^{5\dagger} = +i\,\gamma^3\gamma^2\gamma^1\gamma^0 = \gamma^5$$

$$= \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

ψ 를 이용해 쓸 수 있는

- scalar ; $\quad \bar{\psi}\psi$

- 4-vector ; $\quad j^\mu = \bar{\psi}\gamma^\mu\psi$

$$\rightarrow \psi^\dagger \underbrace{\Lambda^\dagger_{\frac{1}{2}}\gamma^0}_{\gamma^0\Lambda^{-1}_{\frac{1}{2}}} \gamma^\mu \Lambda_{\frac{1}{2}}\psi = \Lambda^\mu{}_\nu \bar{\psi}\gamma^\nu\psi \quad ✓$$

$$\gamma^0 \Lambda^{-1}_{\frac{1}{2}} \gamma^\mu \Lambda_{\frac{1}{2}} : \quad \Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu \gamma^\nu$$

$\frac{T}{P}$

Parity $\left\{ \begin{array}{l} t \rightarrow t \\ \vec{x} \rightarrow -\vec{x} \end{array} \right\}$

- "pseudo"-4-vector

$$\rightarrow \bar{\psi}\gamma^\mu\gamma^5\psi = j^{\mu 5}$$

- pseudo-scalar $\quad \rightarrow \bar{\psi}\gamma^5\psi$

Chiral 4-vector $\underset{\sim}{j^\mu_L} = \bar\Psi \gamma^\mu \frac{1-\gamma^5}{2} \psi$

$\{\gamma^5, \gamma^\mu\} = 0$

$\underset{\sim}{j^\mu_R} = \bar\Psi \gamma^\mu \frac{1+\gamma^5}{2} \psi$

$\psi \to e^{i\alpha} \psi$ ✓ $\psi^\dagger \to \psi^\dagger \underset{\sim}{e^{-i\alpha}}$

global U(1)

$\psi \to e^{i\alpha\gamma^5} \psi$

$j^\mu = \bar\Psi \gamma^\mu \psi \longrightarrow \psi^\dagger \underset{\sim}{e^{-i\alpha\gamma^5}} \underset{\sim}{\gamma^0 \gamma^\mu} \underset{\sim}{e^{i\alpha\gamma^5}} \psi$

$\to \psi^\dagger \gamma^0 \gamma^\mu \psi = \bar\Psi \gamma^\mu \bar\Psi$ ✓

$j^{\mu 5} = \bar\Psi \gamma^\mu \gamma^5 \psi \longrightarrow \psi^\dagger \underline{e^{-i\alpha\gamma^5}} \gamma^0 \gamma^\mu \underline{\gamma^5 e^{i\alpha\gamma^5}} \psi$

$= \psi^\dagger \gamma^0 \gamma^\mu \gamma^5 \psi = \bar\Psi \gamma^\mu \gamma^5 \psi$

global $U_A(1)$ $\longrightarrow$ quantum field theory not sym. (Anomaly)

3.5. Quantization of Dirac field.

Dirac Eq. $(i\not{\partial} - m)\psi = 0$

scalar field

$$\phi = \phi_1 + i\phi_2$$
$$\bar{\phi} = \phi_1 - i\phi_2$$

$$\boxed{\mathcal{L}(\psi, \underline{\bar{\psi}}) = \bar{\psi}(i\not{\partial} - m)\psi}$$

$$(\cdots\cdots)$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0 = \frac{\partial \mathcal{L}}{\partial \bar{\psi}}$$

$$\bar{\psi}\left(i\gamma^0\dot{\psi} + (i\vec{\gamma}\cdot\vec{\nabla} - m)\psi\right)$$
$$= i\psi^\dagger\dot{\psi} + \bar{\psi}(i\vec{\gamma}\cdot\vec{\nabla} - m)\psi$$

conjugate momentum:

$$\frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \bar{\psi}\, i\gamma^0 = \underline{i}\,\psi^\dagger = p_\psi$$

$$\mathcal{H} = \cancel{\underbrace{i\psi^\dagger}_{p_\psi}\dot{\psi}} - \mathcal{L} = \bar{\psi}\left(-i\vec{\gamma}\cdot\vec{\nabla} + m\right)\psi$$

$$\not{\partial} = \gamma^\mu\partial_\mu = \gamma^0\partial_0 + \vec{\gamma}^i\cdot\vec{\partial}_i$$

$$\frac{\partial}{\partial x^i} = \vec{\nabla}$$

$$H = \int d^3x\, \mathcal{H} = \int d^3x\, \psi^\dagger\gamma^0\left(-i\vec{\gamma}\cdot\vec{\nabla} + m\right)\psi$$

Quantization condition

$$\left[i\,\psi_a^\dagger(\vec{x}) ,\ \psi_b(\vec{y}) \right] = i\hbar\,\delta_{ab}\,\delta^{(3)}(\vec{x}-\vec{y}) \qquad (\times)$$

for Majorana

$$\underbrace{\left[\psi_a(\vec{x}) ,\ \psi_a(\vec{y}) \right]}_{\vec{x} \leftrightarrow \vec{y} \text{ antisym.}} = \underbrace{\hbar \qquad \delta^{(3)}(\vec{x}-\vec{y})}_{\text{sym}} \longrightarrow \times$$

$$\boxed{\left\{ \psi_a^\dagger(\vec{x}) ,\ \psi_b(\vec{y}) \right\} = \hbar\,\delta_{ab}\,\delta^{(3)}(\vec{x}-\vec{y})}$$

$$\hat{\psi}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s=1,2} \left(\hat{a}^s_{\vec{p}}\, u^s(\vec{p})\, e^{i\vec{p}\cdot\vec{x}} + \hat{b}^s_{\vec{p}}\, v^s(\vec{p})\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^s \\ \sqrt{p\cdot\bar{\sigma}}\ \xi^s \end{pmatrix}$$

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\ \eta^s \end{pmatrix}$$

$$\hat{\psi}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s=1,2} \left(\hat{a}^s_{\vec{p}}\, \underbrace{u^s(\vec{p})} \underline{e^{i\vec{p}\cdot\vec{x}}} + \hat{b}^s_{\vec{p}}\, \underbrace{v^s(\vec{p})} \underline{e^{-i\vec{p}\cdot\vec{x}}} \right)$$

$$\hat{\psi}^\dagger(\vec{x}) = \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}'}}} \sum_{s'=1,2} \left(\hat{a}^{s'\dagger}_{\vec{p}'}\, \underbrace{u^{s'\dagger}(\vec{p}')}\, e^{-i\vec{p}'\cdot\vec{x}} + \hat{b}^{s'}_{\vec{p}'}\, \underbrace{v^{s'\dagger}(\vec{p}')}\, e^{+i\vec{p}'\cdot\vec{x}} \right)$$

$$H = \int d^3x\, \underbrace{\psi^\dagger}_{a^\dagger + b^\dagger} \gamma^0 \left(-i\,\vec{\gamma}\cdot\underbrace{\vec{\nabla}} + m\right) \underbrace{\psi}_{a+b}$$

$a^\dagger a$:

$$\underbrace{\int d^3x} \int \frac{d^3p}{(2\pi)^3} \underbrace{\int \frac{d^3p'}{(2\pi)^3}} \frac{1}{\sqrt{2E_{\vec{p}}\, 2E_{\vec{p}'}}} \sum_{s,s'} u^{s'\dagger}(\vec{p}')\, \gamma^0 \left(\vec{\gamma}\cdot\vec{p} + m\right) u^s(\vec{p})$$

$$a^{s'\dagger}_{\vec{p}'}\, a^s_{\vec{p}}\, \underbrace{e^{i(\vec{p}-\vec{p}')\cdot\vec{x}}}_{(2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}}\, a^{s'\dagger}_{\vec{p}}\, a^s_{\vec{p}} \sum_{s,s'} \underbrace{u^{s'\dagger}(\vec{p})}_{\left(\xi^{s'\dagger}\sqrt{p\cdot\sigma},\ \xi^{s'\dagger}\sqrt{p\cdot\bar{\sigma}}\right)} \gamma^0 \left(\vec{\gamma}\cdot\vec{p} + m\right) \underbrace{u^s(\vec{p})}_{\begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^s \\ \sqrt{p\cdot\bar{\sigma}}\ \xi^s \end{pmatrix}}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\sum_{s s'}}{2E_{\vec{p}}} a^{s'\dagger}_{\vec{p}} a^{s}_{\vec{p}} \quad \underbrace{\left(\xi^{s'\dagger}\sqrt{p\cdot\sigma},\ \xi^{s'\dagger}\sqrt{p\cdot\bar\sigma}\right)}\ \underbrace{\begin{pmatrix} -\vec{\sigma}\cdot\vec{p} & m \\ m & \vec{\sigma}\cdot\vec{p} \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^{s} \\ \sqrt{p\cdot\bar\sigma}\ \xi^{s} \end{pmatrix}}$$

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & m \end{pmatrix} = \begin{pmatrix} -\vec{\sigma}\cdot\vec{p} & m \\ m & \vec{\sigma}\cdot\vec{p} \end{pmatrix} \qquad \begin{pmatrix} -\vec{\sigma}\cdot\vec{p}\sqrt{p\cdot\sigma}\,\xi^{s} + m\sqrt{p\cdot\bar\sigma}\,\xi^{s} \\ m\sqrt{p\cdot\sigma}\,\xi^{s} + (\vec{\sigma}\cdot\vec{p})\sqrt{p\cdot\bar\sigma}\,\xi^{s} \end{pmatrix}$$

$$-(\vec{\sigma}\cdot\vec{p})(\cancel{p^0\mathbb{1}} - \vec{\sigma}\cdot\vec{p})$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\sum_{s s'}}{2E_{\vec{p}}} a^{s'\dagger}_{\vec{p}} a^{s}_{\vec{p}} \Bigg(\xi^{s'\dagger}\sqrt{p\cdot\sigma}\left(-\vec{\sigma}\cdot\vec{p}\sqrt{p\cdot\sigma}\right)\xi^{s} + m\,\xi^{s'\dagger}\underbrace{\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma}}_{m\cdot\mathbb{1}}\xi^{s}$$

$$+ \xi^{s'\dagger}\underbrace{\sqrt{p\cdot\bar\sigma}\left(\vec{\sigma}\cdot\vec{p}\sqrt{p\cdot\bar\sigma}\right)}_{\vec{\sigma}\cdot\vec{p}\,(\cancel{p^0\mathbb{1}}+\vec{\sigma}\cdot\vec{p})}\xi^{s} + m\,\xi^{s'\dagger}\underbrace{\sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma}}_{m\mathbb{1}}\xi^{s} \Bigg)$$

$$\sigma = (\mathbb{1}, \vec{\sigma}),\quad \bar\sigma = (\mathbb{1}, -\vec{\sigma})$$

$$\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} = \sqrt{(p^0\mathbb{1} - \vec{\sigma}\cdot\vec{p})(p^0\mathbb{1} + \vec{\sigma}\cdot\vec{p})} = \sqrt{p^{0^2}\mathbb{1} - \vec{p}^{\,2}\mathbb{1}} = m\mathbb{1}$$

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = (\vec{A}\cdot\vec{B})\mathbb{1}$$

$$(\vec{\sigma}\cdot\vec{p})^2 = \vec{p}^{\,2}$$

$$\sqrt{p\cdot\sigma}\,(\vec{p}\cdot\vec{\sigma}) = \sqrt{p^0\mathbb{1} - (\vec{\sigma}\cdot\vec{p})}\cdot\vec{\sigma}\cdot\vec{p}\sqrt{p\cdot\sigma} = \vec{\sigma}\cdot\vec{p}\sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} = (\vec{\sigma}\cdot\vec{p})(p\cdot\sigma)$$

$$\times\sqrt{p\cdot\sigma}$$

$$E_{\vec{p}} = \sqrt{\vec{p}^{\,2}+m^2}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\sum_{s s'}}{\cancel{2}\cancel{E_{\vec{p}}}} a^{s'\dagger}_{\vec{p}} a^{s}_{\vec{p}} \underbrace{\xi^{s'\dagger}\xi^{s}}_{\delta_{ss'}} \underbrace{(2\vec{p}^{\,2} + 2m^2)}_{\cancel{2}E_{\vec{p}}^{\cancel{2}}} = \boxed{\int \frac{d^3p}{(2\pi)^3} \sum_{s} a^{s\dagger}_{\vec{p}} a^{s}_{\vec{p}} E_{\vec{p}}}$$

$b^\dagger b$:

$$\underbrace{\int d^3x}\int\frac{d^3p}{(2\pi)^3}\underbrace{\int\frac{d^3p'}{(2\pi)^3}}\frac{1}{\sqrt{2E_{\vec p}2E_{\vec p'}}}\sum_{s,s'}v^{s'\dagger}(\vec p')\,\gamma^0\left(-\vec\gamma\cdot\vec p+m\right)v^s(\vec p)\;b^{s'\dagger}_{\vec p'}\,b^s_{\vec p}\;\underbrace{e^{-i(\vec p-\vec p')\cdot\vec x}}_{(2\pi)^3\delta^{(3)}(\vec p-\vec p')}$$

$$=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec p}}\,b^{s'\dagger}_{\vec p}\,b^s_{\vec p}\sum_{s,s'}\underbrace{v^{s'\dagger}(\vec p)}_{\left(\eta^{s'\dagger}\sqrt{p\cdot\sigma},\;-\eta^{s'\dagger}\sqrt{p\cdot\bar\sigma}\right)}\gamma^0\left(-\vec\gamma\cdot\vec p+m\right)\underbrace{v^s(\vec p)}_{\begin{pmatrix}\sqrt{p\cdot\sigma}\,\eta^s\\-\sqrt{p\cdot\bar\sigma}\,\eta^s\end{pmatrix}}$$

H.W.

$$-\quad a^\dagger b+b^\dagger a\quad\Rightarrow\quad 0$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\sum_{ss'}}{2E_{\vec p}} b^{s'\dagger}_{\vec p} b^s_{\vec p} \quad \left(\eta^{s'\dagger}\sqrt{p\cdot\sigma}, -\eta^{s'\dagger}\sqrt{p\cdot\bar\sigma}\right) \underbrace{\begin{pmatrix} +\vec\sigma\cdot\vec p & m \\ m & -\vec\sigma\cdot\vec p \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar\sigma}\,\eta^s \end{pmatrix}}$$

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} m & \vec\sigma\cdot\vec p \\ -\vec\sigma\cdot\vec p & m \end{pmatrix} = \begin{pmatrix} -\vec\sigma\cdot\vec p & m \\ m & \vec\sigma\cdot\vec p \end{pmatrix} \qquad \begin{pmatrix} +\vec\sigma\cdot\vec p\sqrt{p\cdot\sigma}\,\eta^s - m\sqrt{p\cdot\bar\sigma}\,\eta^s \\ m\sqrt{p\cdot\sigma}\,\eta^s + (\vec\sigma\cdot\vec p)\sqrt{p\cdot\bar\sigma}\,\eta^s \end{pmatrix}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\sum_{ss'}}{2E_{\vec p}} b^{s'\dagger}_{\vec p} b^s_{\vec p} \Big(\eta^{s'\dagger} \overbrace{\sqrt{p\cdot\sigma}\left(-\vec\sigma\cdot\vec p\sqrt{p\cdot\sigma}\right)}^{+(\vec\sigma\cdot\vec p)(\cancel{p\cdot}\mathbb{1} - \vec\sigma\cdot\vec p)} \eta^s - m\,\eta^{s'\dagger} \underbrace{\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma}}_{m\cdot\mathbb{1}} \eta^s$$

$$+ \eta^{s'\dagger} \underbrace{\sqrt{p\cdot\bar\sigma}\left(-\vec\sigma\cdot\vec p\sqrt{p\cdot\bar\sigma}\right)}_{-\vec\sigma\cdot\vec p(\cancel{p\cdot}\mathbb{1} + \vec\sigma\cdot\vec p)} \eta^s - m\,\eta^{s'\dagger} \underbrace{\sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma}}_{m\mathbb{1}} \eta^s \Big)$$

$$\eta^{s'\dagger}\eta_s(-2\vec p^{\,2} - 2m^2)$$

$$= \frac{d^3p}{(2\pi)^3} \frac{\sum_{ss'}}{\cancel{2E_{\vec p}}} b^{s'\dagger}_{\vec p} b^s_{\vec p} \underbrace{\eta^{s'\dagger}\eta^s}_{\delta_{ss'}} \underbrace{(-2\vec p^{\,2} - 2m^2)}_{-\cancel{2}E^{\cancel{2}}_{\vec p}} = -\int \frac{d^3p}{(2\pi)^3} \sum_s b^{s\dagger}_{\vec p} b^s_{\vec p} E_{\vec p}$$

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_{\vec p} \left(a^{s\dagger}_{\vec p}\, a^s_{\vec p} - \underbrace{b^{s\dagger}_{\vec p}\, b^s_{\vec p}}_{\to\, -b^s_{\vec p}\, b^{s\dagger}_{\vec p} \,=\, -\tilde b^{s\dagger}_{\vec p}\, \tilde b^s_{\vec p}} \right)$$

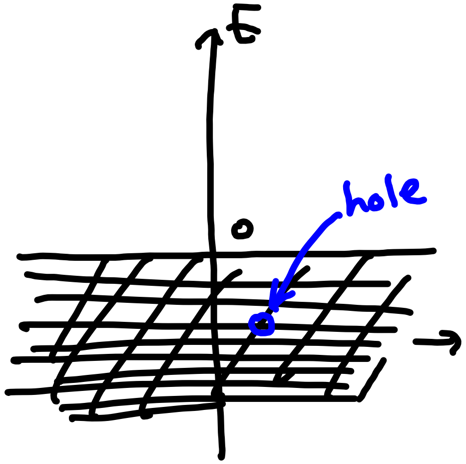
ordinary vacuum

$$\underline{a^s_{\vec p}|0'\rangle = b^s_{\vec p}|0'\rangle} \quad \to \quad H\, b^\dagger_{\vec p}|0\rangle = -E_{\vec p}\, b^\dagger_{\vec p}|0\rangle$$

$b^\dagger_{\vec p}$ → creation operator of negative energy state

Dirac vacuum

$|0\rangle$ = all negative energy states are filled.



$$b^\dagger b^\dagger \cdots |0'\rangle = |0\rangle$$

$$\to \; b^\dagger|0\rangle = 0 \;, \quad b|0\rangle$$

b ↑ hole creation operator

$$\therefore \; b^s_{\vec p} = \tilde b^{s\dagger}_{\vec p} \;, \quad b^{s\dagger}_{\vec p} = \tilde b^s_{\vec p}$$

$$\psi(\vec{x}) = \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}}}\sum_{s=1,2}\left(a^s_{\vec{p}}\, \underset{\sim\sim}{u^s(\vec{p})}\, \underline{e^{i\vec{p}\cdot\vec{x}}} + \overset{*}{b}{}^{s\dagger}_{\vec{p}}\, \underset{\sim\sim}{v^s(\vec{p})}\, \underline{e^{-i\vec{p}\cdot\vec{x}}} \right)$$

$\vec{p} \to -\vec{p}$

$\downarrow$

Heisenberg picture

$$\psi(x) = e^{-iHt}\, \psi(\vec{x})\, e^{iHt}$$

$$\overset{H}{a}{}^s_{\vec{p}} = e^{-ip^0t}\, a^s_p$$

$$b^{s\dagger}_{\vec{p}} = e^{ip^0t}\, b^{s\dagger}_{\vec{p}}$$

$$\boxed{\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1,2}\left(a^s_{\vec{p}}\, u^s(\vec{p})\, e^{-ip\cdot x} + b^{s\dagger}_{\vec{p}}\, v^s(\vec{p})\, e^{ip\cdot x} \right)}$$

$$\boxed{\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1,2}\left(a^{s\dagger}_{\vec{p}}\, \bar{u}^s(\vec{p})\, e^{+ip\cdot x} + b^s_{\vec{p}}\, \bar{v}^s(\vec{p})\, e^{-ip\cdot x} \right)}$$

$$\boxed{\left\{ \psi_a(x) , \psi_b^\dagger(y) \right\}_{x^0=y^0} = \delta_{ab}\, \delta^{(3)}(\vec{x}-\vec{y})}$$

$$\{ \psi , \psi \} = \{ \psi^\dagger, \psi^\dagger \} = 0$$

$$\underline{\left\{ a^r_{\vec{p}'} , a^{s\dagger}_{\vec{p}} \right\} = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')\, \delta_{rs}}$$

$$\left\{ b^r_{\vec{p}'} , b^{s\dagger}_{\vec{p}} \right\} = \qquad \text{''}$$

$|0\rangle$; annihilated by a, b

$$\vec{\mathbb{P}} = \int d^3x \; \psi^\dagger (-i\vec{\nabla}) \psi = \int \frac{d^3p}{(2\pi)^3} \sum_s \vec{p} \left(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right)$$

$|\vec{p}, s\rangle = \sqrt{2E_{\vec{p}}}\, a^{\dagger s}_{\vec{p}} |0\rangle$, $\sqrt{2E_{\vec{p}}}\, b^{s\dagger}_{\vec{p}} |0\rangle$

$\vec{p}$, E_p, s of 1-particle state

$\vec{p}$, E_p, s of 1-antiparticle state

$$\psi(x) \longrightarrow \Lambda_{\frac{1}{2}}\, \psi(\Lambda^{-1}x) = \psi'(x)$$

$$\delta\psi = \Lambda_{\frac{1}{2}}\, \psi(\Lambda^{-1}x) - \psi(x)$$

θ: infinitessimal rotation about z-axis

$$\Lambda_{\frac{1}{2}} = \mathbb{1} - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu} = \mathbb{1} - \frac{i}{2}\theta\,\underbrace{\Sigma^3}_{\begin{pmatrix}\sigma^3 & \\ & \sigma^3\end{pmatrix}}$$

$$\omega_{12} = \theta$$

$$(x, y) \longrightarrow (x+\theta y,\ y-\theta x)$$

$$\delta\psi = \Big(\mathbb{1} - \frac{i}{2}\theta\Sigma^3\Big)\underbrace{\psi(t, z, x+\theta y, y-\theta x)}_{\theta y \frac{\partial\psi}{\partial x} + (-\theta x)\frac{\partial\psi}{\partial y}} - \psi(t, z, x, y)$$

$$= -\theta\Big(x\partial_y - y\partial_x + \frac{i}{2}\Sigma^3\Big)\psi$$

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi = +i\bar{\psi}\gamma^\mu\delta\psi$$

$$Q_3 = J^0 = \int d^3x \ \bar\Psi \ \gamma^0 \left(x\,\partial_y - y\,\partial_x + \frac{i}{2}\Sigma^3 \right) \Psi$$

$$\vec{Q} = \int d^3x \ \bar\Psi \ \gamma^0 \left(\Big[\vec{x} \times \underbrace{(-i\vec\nabla)}_{\vec{p}} \Big] + \frac{1}{2}\vec\Sigma \right) \Psi$$

particle <u>at rest.</u>

$$\cancel{(2\pi)^3}\,\delta(\vec{p}-\vec{p}\,')$$

$$J_z = \underbrace{\int d^3x}\ \int \frac{d^3p\ d^3p'}{\cancel{(2\pi)^{3\cdot 2}}} \frac{1}{\sqrt{2E_{\vec p}\,2E_{\vec p'}}}\ e^{-i\vec{p}\,'\cdot\vec{x}}\ e^{i\vec{p}\cdot\vec{x}}$$

$$\sum_{s,s'} \left(a^{s'\dagger}_{\vec p'}\ u^{s'\dagger}(\vec p\,') + b^{s'}_{-\vec p'}\ v^{s'\dagger}(-\vec p\,') \right) \underbrace{\frac{\Sigma^3}{2}} \left(a^s_{\vec p}\ u^s(\vec p) + b^{s\dagger}_{-\vec p}\ v^s(-\vec p) \right)$$

$$\cdot\ a^{r\dagger}_{\vec 0}|0\rangle$$

$$\supset (\ \ )\times a^{s\dagger}_{\vec p}\ \underbrace{a^s_{\vec p}\ a^{r\dagger}_{\vec 0}|0\rangle}_{(2\pi)^3\,\delta^{(3)}(\vec p-\vec 0)\,\delta_{sr} - \cancel{a^{r\dagger}_{\vec 0}\,a^s_{\vec p}\,|0\rangle}} = \cancel{(2\pi)^3}\underbrace{\delta^{(3)}(\vec p)}\ a^{r\dagger}_{\vec 0}|0\rangle\cdot\frac{1}{2m}$$

$$\underbrace{u^{s'\dagger}(\vec p)\frac{\Sigma^3}{2}\,u^s(\vec p)}_{\propto\ \delta_{ss'}}$$

$$\therefore \quad J_3\, a^{s\dagger}_{\vec{0}} |0\rangle = \frac{1}{\cancel{2m}\,2}\Big(\underbrace{u^{s\dagger}(\vec{0})\,\Sigma^3\, u^s(0)}\Big)\, a^{s\dagger}_{\vec{0}}|0\rangle = \pm\tfrac{1}{2} a^{s\dagger}_0|0\rangle$$

$$\begin{pmatrix} + ; s=1 \\ - ; s=2 \end{pmatrix}$$

$$\Sigma^3 = \left(\begin{array}{c|c} \sigma^3 & 0 \\ \hline 0 & \sigma^3 \end{array}\right)$$

$$\underbrace{\xi^{s\dagger}\sigma^3\xi^s}\cdot \cancel{2m}$$

$$\to (\xi^{s\dagger}\sqrt{p\cdot\sigma}, \xi^{s\dagger}\sqrt{p\cdot\bar\sigma})\overbrace{\left(\begin{array}{c|c} \sigma^3 & \\ \hline & \sigma^3 \end{array}\right)}^{\Sigma^3}\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s\end{pmatrix}$$

$$=\xi^{s\dagger}\Big(\overbrace{\sqrt{p\cdot\sigma}\,\sigma^3\sqrt{p\cdot\sigma}}^{\text{"} m\sigma^3 \text{"}} + \underbrace{\sqrt{p\cdot\bar\sigma}\,\sigma^3\sqrt{p\cdot\bar\sigma}}_{m\sigma^3}\Big)\xi^s = 2m\,\xi^{s\dagger}\sigma^3\xi^s$$

$$J_3\, b^{s\dagger}_0|0\rangle = \mp\tfrac{1}{2} b^{s\dagger}_0|0\rangle$$

Charge current

$$j^\mu = \bar\psi\gamma^\mu\psi \quad \to \quad \text{charge density} \quad j^0 = \psi^\dagger\psi$$

$$\text{charge } Q = \int d^3x\, j^0 = \int d^3x\, \psi^\dagger\psi$$

$$\sum_s \left(a^{s\dagger}_{\vec{p}}\, u^{s\dagger}(\vec{p}) + \underline{b^s_{-\vec{p}}}\, v^{s\dagger}(-\vec{p}) \right) \left(a^s_{\vec{p}}\, u^s(\vec{p}) + \underline{b^{s\dagger}_{-\vec{p}}}\, v^s(-\vec{p}) \right)$$

$$u^{s'\dagger} u^s = \delta^{s's}\, 2E_{\vec{p}}$$

$$\hat{Q} = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(\underline{a^{s\dagger}_{\vec{p}}\, a^s_{\vec{p}}} + \underbrace{b^s_{-\vec{p}}\, b^{s\dagger}_{-\vec{p}}} \right)$$

$$- \underbrace{b^{s\dagger}_{-\vec{p}}\, b^s_{-\vec{p}}} \quad (\underline{+ \infty \text{ const.}})$$

particle # operator

anti-particle # operator.

$$\text{charge(particle)} = -\,\text{charge}(\overline{\text{particle}})$$

	E	$\vec{p}$	s	Q
p	E	$\vec{p}$	s	Q
$\bar{p}$	E	$\vec{p}$	$-s$	$-Q$

Dirac propagator

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s=1,2} \left(a^s_{\vec{p}} u^s(\vec{p}) e^{-ip\cdot x} + b^{s\dagger}_{\vec{p}} v^s(\vec{p}) e^{ip\cdot x} \right)$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s=1,2} \left(a^{s\dagger}_{\vec{p}} \bar{u}^s(\vec{p}) e^{+ip\cdot x} + \underline{b^s_{\vec{p}}} \bar{v}^s(\vec{p}) e^{-ip\cdot x} \right)$$

$$\langle 0 | \psi_a(x) \bar{\psi}_b(y) | 0 \rangle = \int \cdots \sum_{s,s'} u^s \bar{u}^{s'} e^{-ip\cdot x} e^{ip'\cdot y} \underbrace{\langle 0 | a^s_{\vec{p}} a^{s'\dagger}_{\vec{p}'} | 0 \rangle}_{(2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}') \delta_{ss'}}$$

4×4

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \underbrace{\sum_s u^s(p) \bar{u}^s(p)}_{(\not{p} + m)_{ab}} e^{-ip(x-y)}$$

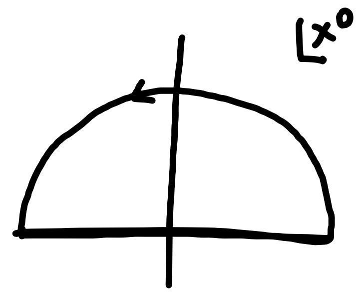
$$= (i \not{\partial}_x + m) \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)}$$

$$\langle 0 | \bar{\psi}(y) \, \psi(x) | 0 \rangle = \int \cdots \sum_{s,s'} v^s \, \bar{v}^{s'} \, e^{ip\cdot x} e^{-ip'\cdot y} \underbrace{\langle 0 | b^s_{\vec p} b^{s'+}_{\vec p'} | 0 \rangle}_{(2\pi)^3 \delta^{(3)}(\vec p - \vec p') \delta_{ss'}}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}} \underbrace{\sum_s v^s(p) \, \bar{v}^s(p)}_{(\not{p} - m)_{ab}} \, e^{+ip(x-y)} \cdots$$

$$= -(i \not\partial_x + m) \underline{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}} e^{+ip\cdot(x-y)}}$$

$$S_R(x-y) = \langle 0 | \{ \psi(x), \bar{\psi}(y) \} | 0 \rangle \, \theta(x^0 - y^0)$$



$$= (i \not\partial_x + m) \, D_R(x-y)$$

$\uparrow$ scalar

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} \cdot (\not{p} + m) \, e^{-ip\cdot(x-y)}$$

$$\frac{i(\not p + m)}{p^2 - m^2} = \frac{i\cancel{(\not p + m)}}{\cancel{(\not p + m)}(\not p - m)} = \frac{i}{\not p - m} = \tilde{S}_R(p)$$

$$(\not p + m)(\not p - m) = \not p^2 - m^2 \mathbb{1} = p^2 - m^2$$

$$\not p \not p = \gamma^\mu p_\mu \gamma^\nu p_\nu$$

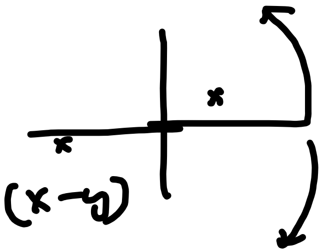
$$= \underbrace{\gamma^\mu \gamma^\nu}_{\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \cancel{\frac{1}{2}[\gamma^\mu, \gamma^\nu]}} p_\mu p_\nu = \not p^2$$

$$\underbrace{\{\gamma^\mu, \gamma^\nu\}}_{2g^{\mu\nu}\mathbb{1}}$$

Feynman.

$$S_F(x-y) \equiv \begin{cases} \langle 0 | \psi(x) \bar\psi(y) | 0 \rangle & x^0 > y^0 \\ -\langle 0 | \bar\psi(y) \psi(x) | 0 \rangle & y^0 > x^0 \end{cases}$$

$$= \langle 0 | T(\psi(x) \bar\psi(y)) | 0 \rangle$$



$$= \int \frac{d^4p}{(2\pi)^4} \left[\frac{i(\not p + m)}{p^2 - m^2 + i\epsilon} \right]_{\epsilon > 0} e^{-ip\cdot(x-y)}$$

3.6. Discrete symmetries

$$P^2 = 1 = T^2 = C^2$$

P, T, C

parity — time reversal — Charge conjugation

$\vec{x} \to -\vec{x}$ $\quad$ $t \to -t$ $\quad$ $p \longleftrightarrow \bar{p}$

$x = (t, \vec{x})$

$\tilde{x} = (t, -\vec{x})$

$+\vec{p}' \cdot \vec{x} = -\vec{p}' \cdot (-\vec{x}) = -\vec{p} \cdot \vec{x}$

$$P\psi(t,\vec{x})P = \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}}}_{\vec{p} \to -\vec{p}'} \sum_{s=1,2} \Big(\eta_a a^s_{-\vec{p}} \underbrace{u^s(\vec{p})}_{u(\tilde{p}')} \underbrace{e^{-ip\cdot\tilde{x}}}_{e^{-ip'\cdot\tilde{x}}} + \eta_b^* b^{s\dagger}_{-\vec{p}} v^s(\vec{p}) e^{ip\cdot x} \Big), \quad (t')$$

$-\vec{p}'$ $\quad$ $(E_{\vec{p}}, -\vec{p}')$

$$u^s(\tilde{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} = \gamma^0 u^s(p)$$

$$v^s(\tilde{p}) = -\gamma^0 v(p)$$

$P\psi P \to$

$$P a^s_{\vec{p}} P = \eta_a a^s_{-\vec{p}}, \quad P a^{\dagger s}_{\vec{p}} P = \eta_a^* a^{\dagger s}_{-\vec{p}}$$

$$P b^s_{\vec{p}} P = \eta_b b^s_{-\vec{p}} \qquad \mathcal{L} \supset \bar{\psi}\cdot\psi$$

$$P^2 a^s_{\vec{p}} P^2 = \eta_a^2 a^s_{\vec{p}} \qquad \eta_a^2, \eta_b^2 = \pm 1$$

$$P\,\underbrace{\psi(x)}_{\psi(t,\vec{x})}\,P = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\,\gamma^0 \sum_s \Big(\eta_a\, a^s_p\, u^s(p)\, e^{-ip\cdot\tilde{x}} - \eta_b^*\, b^{s\dagger}_p\, v^s(p)\, e^{ip\cdot\tilde{x}}\Big)$$

$$v^s(\tilde{p}) = -\gamma^0 v^s(p) \qquad\qquad (t,-\vec{x})$$

$$\propto \psi(t,-\vec{x}) = \psi(\tilde{x}) \qquad \boxed{-\eta_b^* = \eta_a}$$

$$\eta_a \eta_b = -|\eta_b|^2 = -1$$

$$\therefore \quad \boxed{P\,\psi(t,\vec{x})\,P = \eta_a\,\gamma^0\,\psi(t,-\vec{x})}$$

$$P\,\bar{\psi}\,\psi\,P = \underbrace{P\,\bar{\psi}\,P}\,\underbrace{P\,\psi\,P}$$

$$P^2 = 1$$

$$= \eta_a\,\eta_a^*\,\bar{\psi}(t,-\vec{x})\,\underbrace{\gamma^0\gamma^0}_{\mathbb{1}}\,\psi(t,-\vec{x})$$

$$= \underbrace{|\eta_a|^2}_{=1}\,\bar{\psi}\psi(t,-\vec{x}) \quad \leftarrow \text{scalar}$$

$$P\, i\bar{\Psi}\underset{\uparrow}{\gamma^5}\Psi\, P = \underbrace{P\,\bar{\Psi}\,P}\gamma^5\underbrace{P\,\Psi\,P}$$

$P^2 = 1$

$$= \eta_a\,\eta_a^*\,\bar{\Psi}(t,-\vec{x})\,\overbrace{\gamma^0\gamma^5\gamma^0}^{-\gamma^5}\,\psi(t,-\vec{x})$$

$$= -i\,\bar{\Psi}\gamma^5\Psi\,(t,-\vec{x}) \qquad \leftarrow \text{pseudo scalar}$$

$$P\,\bar{\Psi}\underset{\uparrow}{\gamma^\mu}\Psi\, P = \underbrace{P\,\bar{\Psi}\,P}\gamma^\mu\underbrace{P\,\Psi\,P} \qquad \leftarrow \text{vector}$$

$P^2 = 1$

$$= \eta_a\,\eta_a^*\,\bar{\Psi}(t,-\vec{x})\,\underline{\gamma^0\gamma^\mu\gamma^0}\,\psi(t,-\vec{x})$$

$$\gamma^0\gamma^\mu\gamma^0 \to \begin{cases} \gamma^\mu & \mu = 0 \\ -\gamma^\mu & \mu = 1,2,3 \end{cases}$$

$$= (-1)^\mu\,\bar{\Psi}\gamma^\mu\Psi\,(t,-\vec{x})$$

$$P\, i\bar{\Psi}\overbrace{}^{\gamma^5\gamma^\mu}\Psi\, P = \underbrace{P\,\bar{\Psi}\,P}\gamma^5\gamma^\mu\underbrace{P\,\Psi\,P} \qquad \leftarrow \text{axial vector}$$

$P^2 = 1$

$$= \eta_a\,\eta_a^*\,\bar{\Psi}(t,-\vec{x})\,\underbrace{\gamma^0\gamma^5\gamma^\mu\gamma^0}\,\psi(t,-\vec{x})$$

$$\begin{cases} -\gamma^5\gamma^\mu & \mu = 0 \\ \gamma^5\gamma^\mu & \mu = 1,2,3 \end{cases}$$

$$= -(-1)^\mu\,\bar{\Psi}\,\Psi\,(t,-\vec{x})$$

Time Reversal.

$$T\,\psi(t,\vec{x})\,T = \underline{\psi(-t,\vec{x})}$$

$$T\,\underbrace{e^{iHt}}_{e^{-iEt}}\,\psi(\vec{x})\,\underline{e^{-iHt}}\;T\quad|0\rangle$$

if $[H,T]=0$

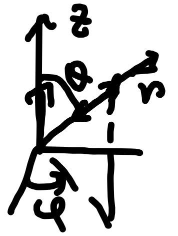
$$\psi(-t,\vec{x})|0\rangle \sim \underbrace{e^{-iHt}}_{e^{-iEt}} \quad\cdots\quad |0\rangle$$

$\therefore$

$$\boxed{T\,(c\text{-number}) = c^{*}\,T}$$

$$T\,e^{iHt} = e^{-iHt}\,T.$$

$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \to \xi^{s}$



$$\xi^{\uparrow} = \begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix} \qquad \xi^{\downarrow} = \begin{pmatrix}e^{-i\phi}\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix} \;\to\; \underline{\xi^{-s} = -i\sigma^{2}(\xi^{s})^{*}}$$

$$\underbrace{(\vec{\sigma}\cdot\vec{n})}_{\sigma^{3}}\,\xi^{s} = \pm\,\xi^{s} \qquad s=1,2$$

$$(\vec{\sigma}\cdot\vec{n})\,\xi^{-s} = \mp\,\xi^{-s} \to \eta^{s}$$

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^{-s} \\ -\sqrt{p\cdot\bar\sigma}\ \xi^{-s} \end{pmatrix}$$

$$\begin{pmatrix} \xi^{-1} \to \downarrow \\ \xi^{-2} \to \uparrow \end{pmatrix}$$

$s=1$

$$a^{-s}_{\vec p} \equiv \left(a^2_{\vec p},\ -a^1_{\vec p} \right)$$

$$b^{-s}_{\vec p} = \left(b^2_{\vec p},\ -b^1_{\vec p} \right)$$

$$u^{-s}(\tilde p) = \begin{pmatrix} \sqrt{\tilde p\cdot\sigma}\ \underline{\xi^{-s}} \\ \sqrt{\tilde p\cdot\bar\sigma}\ \xi^{-s} \end{pmatrix} = \underbrace{-i\left(\begin{array}{c|c} \sigma^2 & 0 \\ \hline 0 & \sigma^2 \end{array}\right)}_{-\gamma^1\gamma^3} \left(u^s(p)\right)^*$$

$$\left(u^s(p)\right)^* \to \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^s \\ \sqrt{p\cdot\bar\sigma}\ \xi^s \end{pmatrix}^*$$

$$\tilde p = (E_{\vec p}, -\vec p)$$

$$\to\ v^{-s}(\tilde p) = -\gamma^1\gamma^3\left(v^s(p)\right)^*$$

$$T a^s_{\vec p} T = a^{-s}_{-\vec p} \qquad T b^s_{\vec p} T = b^{-s}_{-\vec p}$$

$$T\psi(t,\vec{x})T = \underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}}\sum_{s=1,2}\left(a^s_{-\vec{p}}\, u^s(\vec{p})\, \underbrace{e^{-i\tilde{p}\cdot x}}_{\Downarrow} + b^{s\dagger}_{-\vec{p}}\, v^s(\vec{p})\, e^{ip\cdot x}\right)$$

$$= \underline{-\gamma^1\gamma^3}\,\psi(-t,\vec{x})$$

$$\Rightarrow \quad T\ \bar{\psi}\psi\ T = \bar{\psi}\psi(-t,\vec{x})$$

C. C.

$$C\, a^s_{\vec{p}}\, C = b^s_{\vec{p}}$$

$$u^s(p) = -i\gamma^2 v(p)^*$$

$$\rightarrow \quad \underline{C\,\psi(x)\,C = -i\left(\bar{\psi}\gamma^0\gamma^2\right)^T} \quad \text{(H.W.)}$$

$$C \bar{\psi}\psi C = \bar{\psi}\psi .$$

$$\bar{\psi} \not{\partial} \psi$$

	$\bar{\psi}\psi$	$i\bar{\psi}\gamma^5\psi$	$\bar{\psi}\gamma^\mu\psi$	$i\bar{\psi}\gamma^5\gamma^\mu\psi$	∂^μ
P	1	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu$
T	1	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu$
C	1	1	-1	$+1$	1
CPT	1	1	-1	-1	-1

* $\boxed{\text{CPT - theorem}}$ all physical $\mathcal{L}$ is CPT - invariant