양자장론

M. Peskin → 현상론

1. Scalar field } free (no interaction)
2. Dirac "
3. Interaction (ϕ^4, QED) → Feynman diagrams
4. Renormalization
5*. " Group. (Ken Wilson)

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0. Why QFT?

(ex) electron의 magnetic moment.

$$\vec{\mu} = \underset{\substack{\uparrow \\ \text{g-factor}}}{g} \frac{e}{2m_e} \vec{S}$$

✗ ⟵ | ✓

	고전역학	양자역학	Rel. QM. (Dirac Eq)	QFT	Exp.
g	1	1. ($\vec{S} = \pm\frac{\hbar}{2}$)	2	very accurate	$2.\underbrace{00023\cdots}_{\text{Anomalous M.M.}}$

Relativity (Special) [$c \equiv 1$]

$$\underset{\substack{\uparrow \\ 0,1,2,3}}{x^{\mu}} = (t, \vec{x}), \quad p^{\mu} = (E, \vec{p})$$

$$\downarrow$$

metric tensor

$$p^2 = p \cdot p = p_{\mu} p^{\mu} = (\underbrace{g_{\mu\nu} p^{\nu}}_{p_{\mu}}) p^{\mu} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\hookrightarrow m^2$

$$p^2 = E^2 - \vec{p}^{\,2} = m^2$$

Dirac

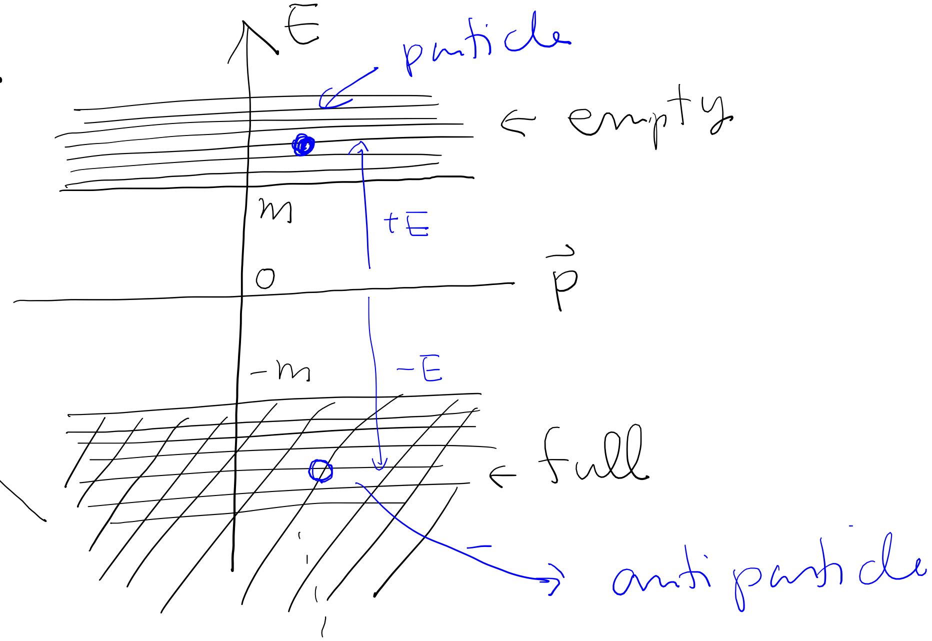
$$E = \pm\sqrt{\vec{p}^{\,2}+m^2}$$

$$|E| \geq m$$

fermion (Dirac field)

$|0\rangle =$

fundamental vacuum



$|\text{excited}\rangle$

$$E-(-E) = 2\sqrt{\vec{p}^{\,2}+m^2}$$

$$\text{Energy} \;\underset{\text{pair creati.}}{\overset{\text{pair annihilati}}{\rightleftarrows}}\; e^- + e^+$$

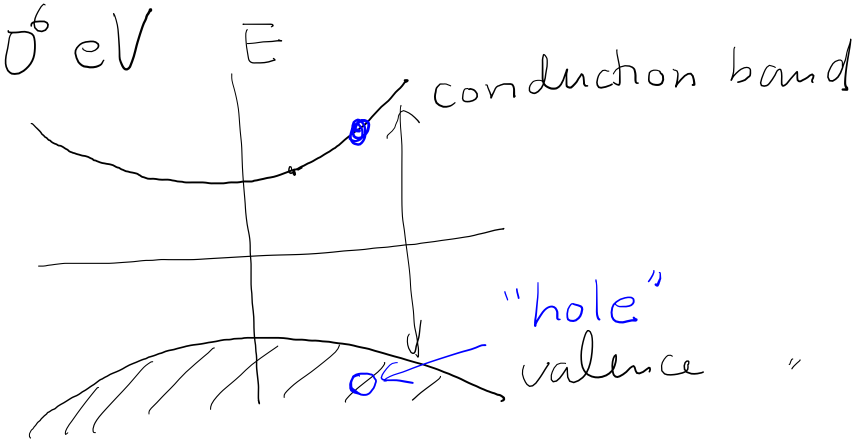
no particle two particle

Condensed matter (non Relativistic)

Quantum Field : phonon, excitor, plasmon, skyrmion, Majorana

$$E \ll m$$

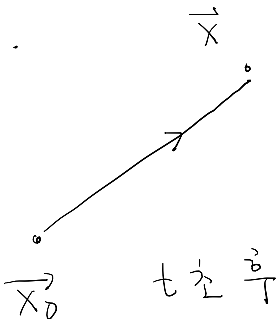
$$\sim eV \ll 0.5 \times 10^6 eV$$



pseudo vac.

Chap 2. Klein-Gordon (Scalar) field.

Q.M.



$$U(t) = \langle \vec{x} | \underbrace{e^{-iHt}}_{\text{time-evolution}} | x_0 \rangle$$

If $H = \dfrac{\hat{\vec{p}}^{\,2}}{2m}$

$$U(t) = \langle \vec{x} | \underset{\uparrow \atop \int \frac{d^3\vec{p}\,'}{(2\pi)^3} |\vec{p}\,'\rangle\langle\vec{p}\,'|}{\mathbb{1}} \; e^{-\frac{it}{2m}\hat{\vec{p}}^{\,2}} \; \underset{\uparrow \atop \int \frac{d^3\vec{p}}{(2\pi)^3} |\vec{p}\rangle\langle\vec{p}|}{\mathbb{1}} \; | \vec{x}_0 \rangle$$

$$= \underbrace{\int}\ \underbrace{\int} \langle \vec{x} | \vec{p}\,' \rangle \underbrace{\langle \vec{p}\,' | e^{-\frac{it}{2m}\hat{\vec{p}}^{\,2}} | \vec{p} \rangle}_{e^{-\frac{it\vec{p}^{\,2}}{2m}} \underbrace{\langle \vec{p}\,' | \vec{p} \rangle}_{(2\pi)^3 \delta^{(3)}(\vec{p}\,' - \vec{p})}} \langle \vec{p} | \vec{x}_0 \rangle = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{-\frac{it\vec{p}^{\,2}}{2m}} \underbrace{\underbrace{\langle \vec{x} | \vec{p} \rangle}_{e^{i\vec{x}\cdot\vec{p}}} \underbrace{\langle \vec{p} | \vec{x}_0 \rangle}_{e^{-i\vec{x}_0\cdot\vec{p}}}}_{e^{+i\vec{p}\cdot(\vec{x}-\vec{x}_0)}}$$

$$\boxed{2\pi\, d\cos\theta\; p^2 dp} \rightarrow d^3\vec{p}, \qquad \vec{p}^{\,2} = p^2$$

$$\vec{p}\cdot(\vec{x}-\vec{x}_0) = \underset{\overset{|||}{p}}{|\vec{p}|}\ \underbrace{|\vec{x}-\vec{x}_0|}_{x} \cos\theta$$

$$\int_{-1}^{1} d\cos\theta \; e^{ipx\cos\theta} = \frac{1}{ipx}\, e^{ipx\cos\theta}\Big|_{-1}^{1}$$

$$= \frac{2}{px}\,\underbrace{\frac{e^{ipx}-e^{-ipx}}{2i}}_{\sin(px)}$$

$$= 2\pi\int_0^\infty dp\; p^{\cancel{2}}\,\frac{2}{\cancel{p}\,x}\,\underbrace{\sin(px)}_{\frac{e^{ipx}-e^{-ipx}}{2i}}\; e^{-\frac{it}{2m}p^2}$$

$$= \frac{2\pi}{ix}\int_{-\infty}^{\infty} p\; e^{\boxed{ipx-\frac{it}{2m}p^2}}$$

$if(p)$

$\sqrt{p^2+m^2}$

$\to f'(p) = x - \frac{it}{m}p = 0$

$p = \frac{mx}{it}$

$\frac{mx^2}{it} - \frac{it}{2m}\left(\frac{mx}{it}\right)^2 = \frac{mx^2}{2it}$

$dp \approx e^{-i\frac{mx^2}{2t}}$ $\quad x = |\vec{x}-\vec{x}_0|$

$\neq 0$ for any $x, t \gg 1$

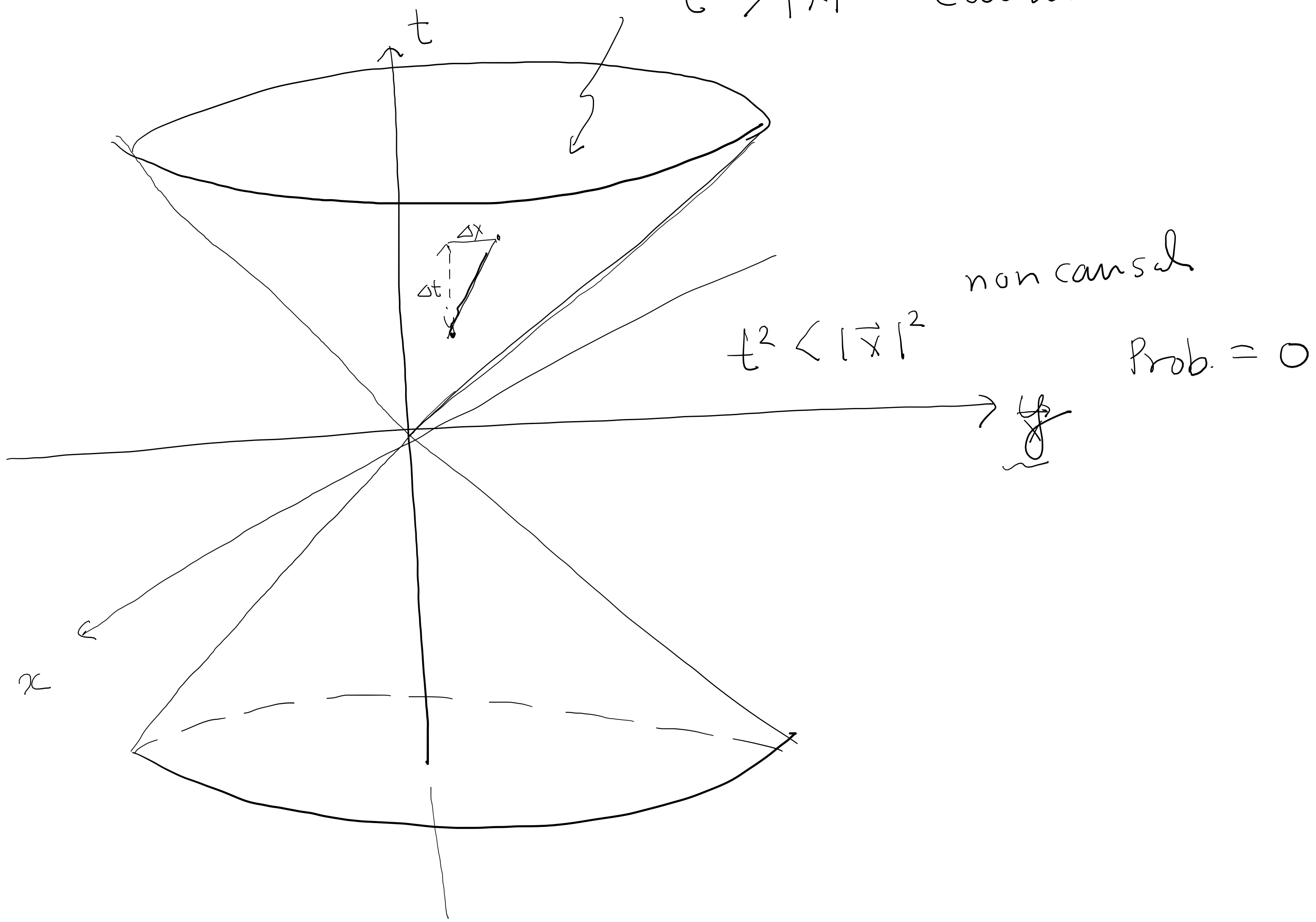
$\frac{1}{\hbar}$

Steepest descent method

$$\int_{-\infty}^{\infty} e^{iNf(x)}\,dx \approx e^{iNf(x_0)} \qquad (f'(x_0)=0) \qquad (N\gg 1)$$

$$\langle \vec{x} | e^{-iHt} | \vec{x}_0 \rangle \sim e^{\frac{i m x^2}{2t}}$$

Relativistic



$$\sim \int dp \; e^{\overbrace{-it\sqrt{p^2+m^2} + ipx}} \sim e^{-m\sqrt{x^2-t^2}} \neq 0$$

$$x \gg t$$

$$if(p) = -m\sqrt{x^2-t^2}$$

$$x - t\frac{p}{\sqrt{p^2+m^2}} = 0 \to p = \frac{imx}{\sqrt{x^2-t^2}}$$

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2. QM. $[\hat{x}, \hat{p}] = i\hbar$ Quantization condition. $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$

X QFT. $\hat{\hat{\psi}}(x)$ 2nd "

"field" as dynamical variables.

$$\hat{\phi}(\underbrace{t, \vec{x}}) = \hat{\phi}_{\vec{x}}(t)$$

x^μ; Relativistic Lorentz Transf.

$$A^{\underline{\underline{\mu}}} \longmapsto A'^{\mu} = \underbrace{\Lambda^{\mu}{}_{\nu}} A^{\nu}$$

$$A'^{\mu} \neq A^{\mu}$$

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \ (\nu) \\ \hline 0 & \text{dilation} & \text{boost} & & \\ 1 & & & & \\ 2 & & & \text{Rotation} & \\ 3 \ (\mu) & & & & \end{array}$$

boost in z direction

$$\begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & & \\ 0 & & 1 & \\ -\gamma\beta & & & \gamma \end{pmatrix}$$

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scalar : L.T. scalar. $\boxed{\phi}$

~~μ~~ vector. spinor

$$(A_{\mu}(x), \ \psi_{\alpha}(x))$$

Lagrangian = Lorentz scalar

$$\mathcal{L}(\phi, \underline{\partial_{\mu}\phi})$$

$$\psi_{\alpha}(x) \longrightarrow \psi'_{\alpha}(x') = \sum_{\alpha'} (\Lambda)_{\alpha}{}^{\alpha'} \psi_{\alpha'}(x)$$

$$\mathcal{L}(\phi, \partial_\mu \phi) = V(\phi) \underbrace{f(\partial_\mu \phi)}_{1 + \cancel{\partial_\mu \phi} + (\partial_\mu \phi)^2 + }$$

$$= V(\phi)(\partial_\mu \phi)^2 + W(\phi)$$

$$\phi \to f(\phi)$$

$$(\partial_\mu f(\phi))^2 = (f'(\phi)\, \partial_\mu \phi)^2$$

$$V(\phi) f'(\phi) = 1$$

$$\boxed{\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}(\partial_\mu \phi)^2 + Z(\phi)}$$

$$S = \int \mathcal{L}\, d^4 x \quad : \quad \text{Lorentz scalar}$$

$$d^4 x = dt\, dx\, dy\, dz \longrightarrow \frac{dt}{\cancel{\gamma}}\, dx\, dy\, \cancel{\gamma}\, dz$$

S = Lorentz scalar + " = Poincare inv.

Poincare group = Lorentz group + translation

$SO(3,1)$

$$\left\{\begin{array}{l} t \to t + a_0 \\ \vec{x} \to \vec{x} + \vec{a} \end{array}\right.$$

$$x^\mu \to x^\mu + a^\mu$$

$$\mathcal{L} \xrightarrow{\text{L.T.}} \mathcal{L}$$

$$\xrightarrow{\text{translatn.}} \mathcal{L} + a^\mu \partial_\mu \mathcal{L}$$

$$S \longrightarrow S + \int d^4x \, \underline{a^\mu} \, \partial_\mu \mathcal{L}$$

$$\underbrace{a^\mu \int d^4x \, \partial_\mu \mathcal{L}}_{0}$$

Classical Field Theory

Least action principle $\quad \delta S = 0$

$$\delta S = \int d^4x \left[\underbrace{\mathcal{L}(\phi + \delta\phi, \partial_\mu\phi + \delta\partial_\mu\phi) - \mathcal{L}(\phi, \partial_\mu\phi)}_{\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta\partial_\mu\phi}_{\underbrace{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right)}_{\text{total derivative}} - \delta\phi \cdot \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}}} \right]$$

$$= \int d^4x \underbrace{\delta\phi}_{\text{any}} \left[\underbrace{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}}_{\overset{!}{=}\,0} \right] = 0$$

E-L Eq.

Hamiltonian

$$\left(H(p,q) = p\dot{q} - L \right) \quad \rightarrow \quad \mathcal{L}(\phi, \partial_\mu \phi) = \mathcal{L}(\phi, \underbrace{\partial_0 \phi}_{\dot{\phi}_{\vec{x}}}, \vec{\nabla}\phi)$$

$$p = \frac{\partial L}{\partial \dot{q}}$$

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2}$$

$$(\partial_0 \phi)^2 - (\vec{\nabla}\phi)^2$$

$$\Pi_{\vec{x}}(t) = \Pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$$

$$\therefore \mathcal{H} = \pi \dot{\phi} - \mathcal{L} = \pi^2 - \left(\frac{1}{2}\left(\pi^2 - (\vec{\nabla}\phi)^2\right) - \frac{1}{2} m^2 \phi^2 \right)$$

$$= \frac{1}{2}\left(\pi^2 + (\vec{\nabla}\phi)^2\right) + \frac{1}{2} m^2 \phi^2$$

$$H = \int \mathcal{H} \, d^3x$$

$$(\partial^2 + m^2)\phi = 0$$

$$\downarrow$$

$$(\partial_0^2 - \vec{\nabla}^2 + m^2)\phi = 0$$

Nöther theorem

$$(\alpha \ll 1)$$

$$\phi \longrightarrow \phi'(x) = \phi(x) + \alpha \underline{\underline{\Delta\phi}} \quad \text{; transf}$$

$$\mathcal{L} \longrightarrow \mathcal{L} + \underline{\alpha\, \partial_\mu J^\mu} \quad \text{upto total drivative}$$

$S \to S$: "invariant" under above transf $=$ symmetry

$$\underline{\delta\mathcal{L}} = \mathcal{L}(\phi', \partial_\mu\phi') - \mathcal{L}(\phi, \partial_\mu\phi)$$

$$\begin{array}{l} \shortparallel \\ \alpha\, \partial_\mu J^\mu \end{array} = \frac{\partial\mathcal{L}}{\partial\phi}(\alpha\Delta\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\alpha\Delta\phi)$$

$$= \alpha\, \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + (\alpha\Delta\phi)\underbrace{\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]}_{=\,0}$$

$$\alpha\, \partial_\mu\Big(\underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - J^\mu}_{\equiv\, j^\mu \text{ : conserved current.}}\Big) = 0$$

$$Q = \int j^0 \, d^3x \quad : \text{ conserved charge}$$

(ex) complex scalar field $\qquad \phi = \phi_1 + i\phi_2$

$$\mathcal{L} = \overbrace{|\partial_\mu \phi|^2}^{\partial_\mu\phi\,\partial^\mu\phi^*} - m^2 |\phi|^2$$

α : 실수. (x-indep)

$$\phi \to e^{i\alpha}\phi \underset{\alpha \ll 1}{\approx} \phi + \underbrace{i\alpha\phi}_{\alpha\Delta\phi}, \qquad \phi^* \to e^{-i\alpha}\phi^* = \phi^* \underbrace{- i\alpha\phi^*}_{+\alpha\Delta\phi^*}$$

$$\Delta\phi = i\phi, \qquad \Delta\phi^* = -i\phi^*$$

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\Delta\phi^* - \cancel{J^\mu}$$

$$= (\partial^\mu\phi^*)\, i\phi + (\partial^\mu\phi)(-i\phi^*) = i\left[(\partial^\mu\phi^*)\phi - (\partial^\mu\phi)\phi^*\right]$$

(ex) space-time sym.

$$A \cdot B = A^\mu B_\mu = A_\mu B^\mu = g^{\mu\nu} A_\nu B_\mu$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$x^\mu \longrightarrow x^\mu + a^\mu \qquad (a^\mu \ll 1)$$

$$\phi \longrightarrow \phi(x+a) = \phi(x) + \underbrace{a_\nu \partial^\nu \phi}_{\Delta^\nu \phi}$$

$$\mathcal{L} \longrightarrow \mathcal{L} + a^\nu \partial_\nu \mathcal{L} \to a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L}) = \underbrace{a_\nu g^{\nu\beta}}_{a^\beta} \partial_\mu (\delta^\mu_\beta \mathcal{L})$$

$$J^\mu_{(\nu)} = \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta^\nu \phi}_{\partial^\mu \phi \, \partial^\nu \phi} - \underbrace{g^{\nu\beta} \delta^\mu_\beta \mathcal{L}}_{g^{\mu\nu} \mathcal{L}}$$

$(\nu) \uparrow a^\nu$

$$\boxed{T^{\mu\nu} = \partial^\mu \phi \, \partial^\nu \phi - g^{\mu\nu} \mathcal{L}}$$

: { stress-energy tensor, energy-momentum tensor }

$$\boxed{T^\mu{}_\nu = g_{\nu\alpha} T^{\mu\alpha}} = \partial^\mu \phi \underbrace{g_{\nu\alpha} \partial^\alpha \phi}_{\partial_\nu \phi} - \underbrace{g_{\nu\alpha} g^{\mu\alpha}}_{\delta^\mu_\nu} \mathcal{L}$$

$$= \partial^\mu \phi \, \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}$$

$$T^{00} = \underset{\substack{\shortparallel \\ \pi\dot{\phi}}}{\dot{\phi}^2} - \mathcal{L} = \mathcal{H}$$

$$H = \int T^{00}\, d^3x$$

$$\vec{P} = -\int T^{0i}\, d^3x = -\int \underbrace{\dot{\phi}}\, \vec{\nabla}^i \phi \; d^3x = -\int \pi\, \vec{\nabla}\phi \; d^3x \qquad i = 1, 2, 3$$

QM $(\hbar = 1)$ $[q_i, p_j] = i\delta_{ij}$, $[q_i, q_j] = 0$, $[p_i, p_j] = 0$

Canonical quantization

$$[\underset{\substack{\shortparallel \\ \phi(t,\vec{x})}}{\phi_{\vec{x}}}, \underset{\substack{\shortparallel \\ \pi(t,\vec{y})}}{\pi_{\vec{y}}}] = i\,\delta^{(3)}(\vec{x}-\vec{y}), \qquad [\phi(\vec{x}), \phi(\vec{y})] = 0$$

$$[\pi(\vec{x}), \pi(\vec{y})] = 0$$

$$\phi(\vec{x},t) \overset{F.T.}{=} \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \phi(\vec{p},t)$$

$$0 = (\partial_\mu^2 + m^2)\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} (\partial^2 + m^2)\left[e^{i\vec{p}\cdot\vec{x}}\phi(\vec{p},t)\right]$$

$$\partial^2 + m^2 = \partial_0^2 - \vec{\nabla}^2 + m^2$$

$$= \int \frac{d^3p}{(2\pi)^3} \underbrace{(\partial_0^2 + \overbrace{\vec{p}^2 + m^2}^{\omega_{\vec{p}}^2})\phi(\vec{p},t)}_{=0} e^{i\vec{p}\cdot\vec{x}}$$

$$\left(\frac{d^2}{dt^2} + \omega_{\vec{p}}^2\right)\phi(\vec{p},t) = 0$$

$$\longrightarrow \phi = \frac{1}{\sqrt{2\omega}}(a + a^\dagger)$$

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2\phi^2$$

"ϕ" $\to$ "x"

$$p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger)$$

$$H = \omega\left(a^\dagger a + \frac{1}{2}\right)$$

$$\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a^{\dagger}_{-\vec{p}}\, e^{+i\vec{p}\cdot\vec{x}} \right)$$

$$\vec{p} \to -\vec{p} \qquad \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}}\, a^{\dagger}_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}$$

$$x' = -x \to dx = -dx'$$

$$\left(\int_{-\infty}^{\infty} dx = \int_{\infty}^{-\infty} (-dx') = \int_{-\infty}^{\infty} dx' \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} + a^{\dagger}_{-\vec{p}} \right) e^{i\vec{p}\cdot\vec{x}}$$

$$a^{\dagger}_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}$$

$$\pi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - \underbrace{a^{\dagger}_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}}_{a^{\dagger}_{-\vec{p}}\, e^{i\vec{p}\cdot\vec{x}}} \right)$$

$$\left(a_{\vec{p}} - a^{\dagger}_{-\vec{p}} \right) e^{i\vec{p}\cdot\vec{x}}$$

$$[\phi(\vec{x}), \pi(\vec{x}')] = i\,\delta^{(3)}(\vec{x}-\vec{x}') \quad \checkmark$$

$$= \left[\int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p + a^\dagger_{-p}) e^{ip\cdot x},\ \int \frac{d^3p'}{(2\pi)^3} (-i)\sqrt{\frac{\omega_{p'}}{2}} (a_{p'} - a^\dagger_{-p'}) e^{i\vec{p}'\cdot\vec{x}'}\right]$$

$$[a_{\vec{p}}, a^\dagger_{\vec{p}'}] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}'), \quad [a, a] = [a^\dagger, a^\dagger] = 0$$

$$= \int \frac{d^3p}{(\ )} \int \frac{d^3p'}{\cancel{(\ )}} \frac{1}{\sqrt{2\omega_p}} (-i) \sqrt{\frac{\omega_{p'}}{2}} \left(-\underbrace{[a_p, a^\dagger_{-p'}]}_{(2\pi)^3\delta^{(3)}(\vec{p}+\vec{p}')} + \underbrace{[a^\dagger_{-p}, a_{p'}]}_{-(2\pi)^3\delta^{(3)}(\vec{p}'+\vec{p})} \right) e^{i\vec{p}\cdot\vec{x} + i\vec{p}'\cdot\vec{x}'}$$

$$-2\,\cancel{(2\pi)^3}\,\delta^{(3)}(\vec{p}+\vec{p}')$$

$$\omega_{-\vec{p}} = \omega_{\vec{p}}$$

$$= \int \frac{d^3p}{(2\pi)^3} \underbrace{\frac{(-2)(-i)}{2}}_{i} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} = i\,\delta^{(3)}(\vec{x}-\vec{x}')$$

$$H = \int d^3x\, \mathcal{H} = \int d^3x \left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right) = \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \left(a^\dagger_{\vec{p}} a_{\vec{p}} + \frac{1}{2} \underbrace{[a_{\vec{p}}, a^\dagger_{\vec{p}}]}_{(2\pi)^3(\infty)}\right)$$

$$\vec{P} = -\int \pi \, \vec{\nabla}\phi \, d^3\vec{x} = \int \frac{d^3p}{(2\pi)^3} \, \vec{p} \, a^\dagger_{\vec{p}} a_{\vec{p}}$$

$$\underbrace{A(\vec{p})}_{\sqrt{2E_{\vec{p}}}} \, a^\dagger_{\vec{p}} |0\rangle = |\vec{p}\rangle$$

$$\langle \vec{p} | \vec{q} \rangle = A(\vec{p}) A(\vec{q}) \langle 0 | a_{\vec{q}} \, a^\dagger_{\vec{p}} | 0 \rangle = (2\pi)^3 \underbrace{A(\vec{p}) A(\vec{q})}_{A^2(\vec{p})} \delta^{(3)}(\vec{p} - \vec{q})$$

$$a_{\vec{q}} \, a^\dagger_{\vec{p}} = \underbrace{[a_{\vec{q}}, a^\dagger_{\vec{p}}]}_{(2\pi)^3 \delta^{(3)}(\vec{q} - \vec{p})} + a^\dagger_{\vec{p}} a_{\vec{q}}$$

$$A^2(\vec{p}) \to 2E_{\vec{p}} \qquad A = \sqrt{2E_{\vec{p}}}$$

$$a_{\vec{p}} |0\rangle = 0 \qquad \left(\langle 0 | a^\dagger_{\vec{p}} = 0 \right)$$

Lorentz invariant.

$$\delta^{(3)}(\vec{p} - \vec{q}) = \delta^{(3)}(\vec{p}\,' - \vec{q}\,') \frac{dp_3'}{dp_3} = \delta^{(3)}(\vec{p}\,' - q') \frac{E'}{E}$$

$$\delta(\alpha x) = \frac{\delta(x)}{|\alpha|}$$

$$\begin{pmatrix} E'_{p} \\ \vec{p}\,' \end{pmatrix} = \Lambda \cdot \begin{pmatrix} E_{\vec{p}} \\ \vec{p} \end{pmatrix} \qquad p_3' = \gamma(p_3 + \beta E_{\vec{p}})$$

$$\gamma\left(\frac{\overbrace{E + \beta p_3}^{E'}}{E}\right) = \frac{dp_3'}{dp_3} = \gamma\left(1 + \beta \underbrace{\frac{dE}{dp_3}}_{\frac{p_3}{E}}\right)$$

$$E = \sqrt{\vec{p}^{\,2} + m^2} \qquad \vec{p}^{\,2} = p_3^2 + p_1^2 + p_2^2 + m^2 \qquad \frac{p_3}{\sqrt{}} = \frac{p_3}{E}$$

$$\boxed{E_{\vec{p}} \, \delta^{(3)}(\vec{p} - \vec{q}) = E_{\vec{p}\,'} \, \delta^{(3)}(\vec{p}\,' - \vec{q}\,') = \text{Lorentz inv.}}$$

$$\boxed{|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\, a^{\dagger}_{\vec{p}}\,|0\rangle}$$

$$(\mathbb{1}) = \boxed{\int \underbrace{\frac{d^3\vec{p}}{(2\pi)^3}}_{\cancel{\text{L.I.}}} \frac{1}{2E_{\vec{p}}}}\,|\vec{p}\rangle\langle\vec{p}| = \int \frac{d^4 p}{(2\pi)^4}\, |\vec{p}\rangle\langle\vec{p}|\,(2\pi)\,\delta(p^2-m^2)\Big|_{p_0>0}$$

L.I.

$\delta(p^2-m^2)$

mass shell

on-shell

$$\int \underbrace{\frac{d^4 p}{(2\pi)^4}(2\pi)}\,\delta(\underbrace{p^2}_{p_0^2-\vec{p}^{\,2}}-m^2)\Big|_{p_0>0}$$

$$= \int \frac{d^3 p}{(2\pi)^3}\int dp^0\, \delta\big(p_0^2-\underbrace{(\vec{p}^{\,2}+m^2)}_{E_{\vec{p}}^2}\big)$$

$$\delta\big((p_0-E_{\vec{p}})(p_0+E_{\vec{p}})\big)\Big|_{p_0>0}$$

$$\times$$

$$\parallel$$

$$\frac{\delta(p_0-E_{\vec{p}})}{p_0+E_{\vec{p}}} \longrightarrow \frac{1}{2E_{\vec{p}}}$$

$$\phi(\vec{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (\cancel{a_p} + a^+_{+p}) e^{-i\vec{p}\cdot\vec{x}} |0\rangle$$

$$a^+_p|0\rangle = \frac{1}{\sqrt{2E_p}} |\vec{p}\rangle$$

$$\omega_{\vec{p}} = \sqrt{\vec{p}^2+m^2} = E_{\vec{p}}$$

$(\hbar=1)$

$$\boxed{\phi(\vec{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-i\vec{p}\cdot\vec{x}} |\vec{p}\rangle}$$

$$\langle 0|\phi(\vec{x})|\vec{p}\rangle = \frac{1}{\cancel{2E_{\vec{p}}}} e^{ip\cdot\vec{x}} \cdot \cancel{2E_{\vec{p}}} = e^{i\vec{p}\cdot\vec{x}}$$

2.4. time (Schrodinger picture → Heisenberg picture)

$\mathcal{O}_i$, $\underset{\text{w.f.}}{\psi(t)}$ → $\mathcal{O}_i(t)$

$\downarrow$

$\mathcal{O}(\vec{x}, t)$

$$e^{iHt}\phi(\vec{x})\,e^{-iHt} = \phi(\vec{x}, t)$$

H-eq ; $i\dfrac{\partial \mathcal{O}}{\partial t} = [\mathcal{O}, H]$

$$i\frac{\partial \phi(\vec{x},t)}{\partial t} = \left[\phi(\vec{x},t), \int d^3x' \left\{ \frac{1}{2}\pi(\vec{x}',t)^2 + \frac{1}{2}\left(\vec{\nabla}\phi(x',t)\right)^2 + \frac{1}{2}m^2\phi(x',t)^2 \right\}\right]$$

$$= \int d^3x' \frac{1}{\cancel{2}} \underbrace{[\phi(\vec{x},t), \pi(\vec{x}',t)^2]}_{\pi[\phi,\pi] + [\phi,\pi]\pi}$$

$[A, BC] = B[A,C] + [A,B]C$

$[\phi,\pi] = i\,\delta^{(3)}(\vec{x}-\vec{x}')$

$= \cancel{2}\,i\,\delta^{(3)}(\vec{x}-\vec{x}')$

$$= i\,\pi(\vec{x}, t) \quad ✓$$

$$\to \quad \pi = \frac{\partial\phi}{\partial t}$$

$$\pi = \dot{\phi}$$

$$i\frac{\partial \pi(\vec{x},t)}{\partial t} = \left[\pi(\vec{x},t), \int d^3x' \left\{ \frac{1}{2}\cancel{\pi(\vec{x}',t)^2} + \frac{1}{2}\left(\vec{\nabla}_{x'}\phi(x',t)\right)^2 + \frac{1}{2}m^2\phi(x',t)^2 \right\}\right]$$

$$= \int d^3x' \left\{ \underbrace{\vec{\nabla}_{x'}\phi\ \cancel{\tfrac{1}{2}\cdot 2}\ \left[\pi(\vec{x},t), \vec{\nabla}_{x'}\phi\right]} + m^2\phi\ \underbrace{[\pi,\phi]}_{-i\delta^{(3)}} \right\}$$

$$\vec{\nabla}_{x'}\left([\pi,\phi]\vec{\nabla}_{x'}\phi\right) - \underbrace{[\pi,\phi]}_{-i\delta^{(3)}}\nabla^2_{x'}\phi$$

$$= \int d^3x' (-i)\left(-\nabla^2_{x'} + m^2\right)\phi(x',t)\,\delta^{(3)}(\vec{x}-\vec{x}')$$

$$= (-i)\left(-\vec{\nabla}^2_{\vec{x}} + m^2\right)\phi(\vec{x},t)$$

$$i\frac{\partial^2\phi}{\partial t^2} = (-i)\left(-\nabla^2 + m^2\right)\phi \rightarrow \frac{\partial^2\phi}{\partial t^2} = \left(\nabla^2 - m^2\right)\phi$$

missing part

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$\phi(x) = e^{iHt}\,\phi(\vec{x})\,e^{-iHt} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \Big(\underbrace{e^{iHt} a_{\vec{p}} e^{-iHt}}_{a_{\vec{p}} e^{-iE_{\vec{p}}t}} e^{i\vec{p}\cdot\vec{x}} + \underbrace{e^{iHt} a^\dagger_{\vec{p}} e^{-iHt}}_{a^\dagger_{\vec{p}} e^{iE_{\vec{p}}t}} e^{-i\vec{p}\cdot\vec{x}} \Big)$$

$$\left(a_{\vec{p}}\, e^{-ip\cdot x} + a^\dagger_{\vec{p}}\, e^{ip\cdot x} \right)_{p^0 \equiv E_{\vec{p}}}$$

$$\boxed{\phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}\, e^{ip\cdot x}\, |\vec{p}\rangle}$$

$$p = (E_{\vec{p}}, \vec{p})$$

$$E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$$

$$p^2 = m^2$$

Campbell-Baker-Hausdorff (CBH) formula

$$e^{\mu}\lambda e^{-\mu} = \lambda + [\lambda,\mu] + \cdots$$

$f(1)$

(pf) $\hat{f}(t) \equiv e^{t\mu}\lambda e^{-t\mu} \longmapsto \frac{df}{dt} = \mu e^{t\mu}\lambda e^{-t\mu} + e^{t\mu}\lambda e^{-t\mu}(-\mu)$

$$= e^{t\mu}[\mu,\lambda]e^{-t\mu}$$

$$f(t) = \sum_{n=0}^{\infty}\frac{1}{n!}t^n f^{(n)}(0)$$

$$f'(0) = [\mu,\lambda]$$

$$f''(0) = [\mu,[\mu,\lambda]]$$

$$f^{(3)}(0) = [\mu,[\mu,[\mu,\lambda]]] \quad \cdots$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(0)\,t^n$$

$$f(1) = \mu + [\mu,\lambda] + \frac{1}{2!}[\mu,[\mu,\lambda]] + \frac{1}{3!}[\mu,[\mu,[\mu,\lambda]]] + \cdots$$

CBH.

$$e^{iHt} a_{\vec p}\, e^{-iHt} = a_{\vec p} + it \overbrace{[H, a_{\vec p}]}^{-E_{\vec p} a_{\vec p}} + \frac{1}{2!}(it)^2 \overbrace{[H,[H,a_{\vec p}]]}^{(-E_{\vec p})^2 a_{\vec p}} + \cdots$$

$$\left[H = \int \frac{d^3\vec q}{(2\pi)^3} E_{\vec q}\, a^\dagger_{\vec q} a_{\vec q} \quad \rightarrow \quad [H, a_{\vec p}] = \int \frac{d^3\vec q}{\cancel{(2\pi)^3}} E_{\vec q} \underbrace{[a^\dagger_{\vec q} a_{\vec q}, a_{\vec p}]}_{[a^\dagger_{\vec q}, a_{\vec p}]\, a_{\vec q}} = E_{\vec p} a_{\vec p} \right]$$

$$[a^\dagger_{\vec q}, a_{\vec p}] = -\cancel{(2\pi)^3}\, \delta^{(3)}(\vec q - \vec p)$$

$$= a_{\vec p} \Big(1 + \underbrace{(-itE_{\vec p}) + \frac{1}{2!}(-itE_{\vec p})^2 + \cdots}_{itE_{\vec p}} \Big)$$

$$e^{iHt} a^\dagger_{\vec p}\, e^{-iHt} = a^\dagger_{\vec p} + it \overbrace{[H, a^\dagger_{\vec p}]}^{E_{\vec p} a^\dagger_{\vec p}} + \frac{1}{2!}(it)^2 \overbrace{[H,[H,a^\dagger_{\vec p}]]}^{(E_{\vec p})^2 a^\dagger_{\vec p}} + \cdots$$

$$\left[H = \int \frac{d^3\vec q}{(2\pi)^3} E_{\vec q}\, a^\dagger_{\vec q} a_{\vec q} \quad \rightarrow \quad [H, a^\dagger_{\vec p}] = \int \frac{d^3\vec q}{\cancel{(2\pi)^3}} E_{\vec q} \underbrace{[a^\dagger_{\vec q} a_{\vec q}, a^\dagger_{\vec p}]}_{a^\dagger_{\vec q} [a_{\vec q}, a^\dagger_{\vec p}]} = E_{\vec p} a^\dagger_{\vec p} \right]$$

$$[a_{\vec q}, a^\dagger_{\vec p}] = \cancel{(2\pi)^3}\, \delta^{(3)}(\vec q - \vec p)$$

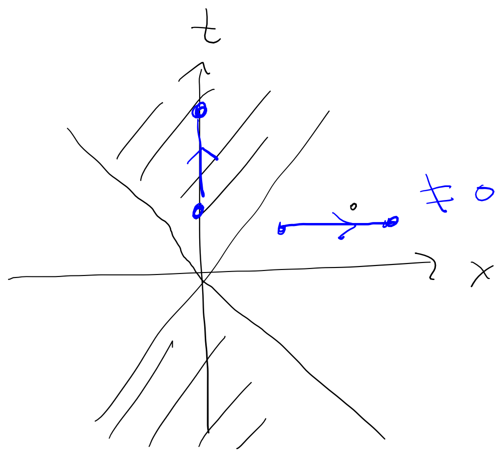
$$= a^\dagger_{\vec p} \Big(1 + \underbrace{it(E_{\vec p}) + \frac{1}{2!}(itE_{\vec p})^2 + \cdots}_{e^{itE_{\vec p}}} \Big)$$

Causality

$\langle 0|\phi(x)\,\phi(y)|0\rangle$

$\parallel$

$D(x-y)$



$\neq 0$ ~~causality~~.

$$\underbrace{\phi(y)|0\rangle} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec p}}e^{ip\cdot y}|\vec p\rangle \qquad 2\pi\, p^2\, dp\, d\cos\theta$$

$$\times\quad \Big(\underbrace{\phi(x)|0\rangle}\Big)^\dagger = \int \frac{d^3p'}{(2\pi)^3}\frac{1}{2E_{\vec p'}}e^{-ip'\cdot x}\langle \vec p'| \qquad = \underbrace{\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip\cdot(x-y)}}$$

$(2\pi)^3\, 2E_{\vec p}\,\delta^{(3)}(\vec p-\vec p')$

$$= \frac{4\pi}{(2\pi)^3}\int_0^\infty dp\,\frac{p^2}{2\sqrt{p^2+m^2}}\,e^{-i\underbrace{\sqrt{p^2+m^2}}_{E}\,t} \qquad (\vec x-\vec y=0,\ x^0-y^0=t)$$

$$dE = \frac{p\,dp}{\sqrt{}} = \frac{p}{E}\,dp$$

$$= \frac{1}{4\pi^2}\int_m^\infty dE\,\frac{\cancel{E}}{\cancel{p}}\,\frac{p^{\cancel{2}}\overbrace{}^{\sqrt{E^2-m^2}}}{\cancel{E}}\,e^{-iEt} = \frac{1}{4\pi^2}\int_m^\infty\sqrt{E^2-m^2}\,e^{-iEt}\,dE$$

$$t\gg 1 \qquad e^{\underbrace{\ln(\sqrt{E^2-m^2})-iEt}} \qquad \sim e^{-imt}$$

$\frac{\partial}{\partial E}(\ )=0 \xrightarrow{t\to\infty} E=m+O(\frac{1}{t})$

//

$x^0 = y^0, \quad |\vec{x} - \vec{y}| = r \gg 1$ $\quad p \cdot (x-y) = -\vec{p} \cdot \underbrace{(\vec{x} - \vec{y})}_{\vec{r}}$

$$\cdot D(x-y) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i \underbrace{\vec{p} \cdot \vec{r}}}$$

$e^{i p r \cos\theta}$

$d \cos\theta$

$$= \int_0^\infty dp \, - - \quad \frac{1}{ipr} \left(\underbrace{e^{ipr}} \underbrace{- e^{-ipr}}_{\int_{-\infty}^{0}} \right)$$

$$= \int_{-\infty}^{\infty} dp \left(\frac{-i}{2(2\pi)^2 r} \right) \frac{p \, e^{ipr}}{\sqrt{p^2 + m^2}} \sim e^{-mr} \neq 0$$

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-ip\cdot x} + a^\dagger_{\vec{p}}\, e^{ip\cdot x} \right)_{p^0 = E_{\vec{p}}}$$

$$\overset{=}{(a + a^\dagger)_x}$$

Causality

$$\langle 0 | \phi(x)\phi(y) | 0 \rangle = D(x-y)$$

$x^0 = y^0$

$$\langle 0 | [a + \cancel{a^\dagger}]_x [\cancel{a} + a^\dagger]_y | 0 \rangle = \langle 0 | [a]_x [a^\dagger]_y | 0 \rangle$$

$$= \langle 0 | \left[[a]_x, [a^\dagger]_y \right] + \cancel{[a^\dagger]_y [a]_x} | 0 \rangle$$

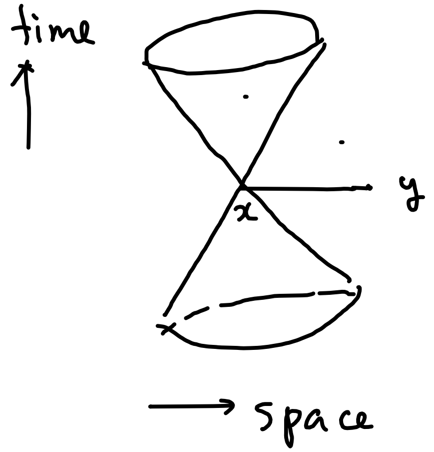
0 0

$$\langle 0 | \left[[a]_x, [a^\dagger]_y \right] | 0 \rangle = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} e^{-ip\cdot x} \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{q}}}} e^{iq\cdot y} \underbrace{[a_{\vec{p}}, a^\dagger_{\vec{q}}]}_{(2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)}$$

$''$

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\, e^{-ip\cdot(x-y)} = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2\sqrt{\vec{p}^2+m^2}}\, e^{i\vec{p}\cdot\vec{r}}$$

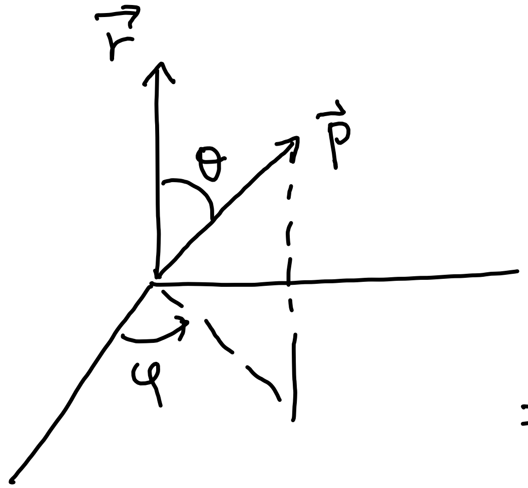


$x^0 = y^0$

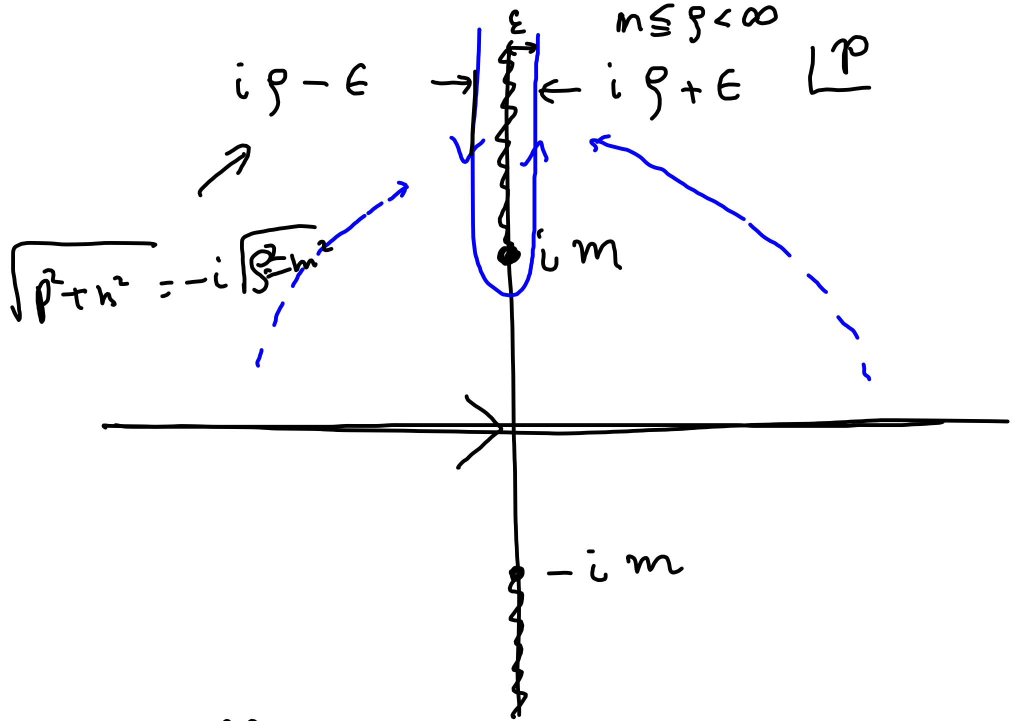
$$p\cdot(x-y) = \underbrace{p^0(x^0-y^0)}_{0} - \vec{p}\cdot(\underbrace{\vec{x}-\vec{y}}_{\equiv\vec{r}}) = -\vec{p}\cdot\vec{r}$$

$$= \int_0^\infty \frac{\cancel{p^2}\, dp}{(2\pi)^{\cancel{3}}}\cdot\cancel{2\pi}\int_{-1}^{1} d\cos\theta\,\frac{1}{2\sqrt{p^2+m^2}}\, e^{ipr\cos\theta}$$

$$\frac{e^{ipr} - e^{-ipr}}{i\cancel{p}r} \to p\to -p$$

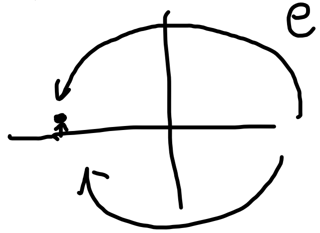


$$= \int_{-\infty}^{\infty} \frac{p\, e^{ipr}\, dp}{(2\pi)^2\, 2ir\sqrt{p^2+m^2}} = \frac{1}{8\pi^2 ir}\int_{-\infty}^{\infty} dp\,\frac{p e^{ipr}}{\sqrt{p^2+m^2}}$$



$$\sqrt{p^2+m^2} = \sqrt{\underbrace{-\rho^2+m^2}_{-(\rho^2-m^2)} + i\varepsilon} = e^{\frac{i\pi}{2}}\sqrt{\rho^2-m^2}$$

$$e^{\frac{i\pi}{2}} = i, \qquad e^{i\pi}$$



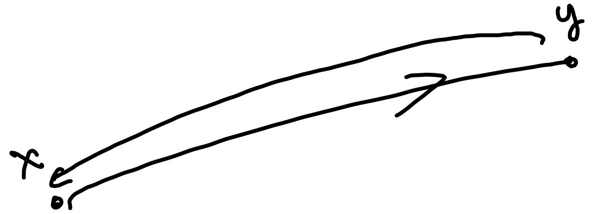
$$\frac{1}{8\pi^2 i r}\int_m^\infty \frac{\overbrace{i\,d\rho}^{dp}\; i\rho}{i\sqrt{\rho^2-m^2}}\, e^{-\rho r} = \frac{1}{8\pi^2 r}\int_m^\infty \frac{\rho}{\sqrt{\rho^2-m^2}}\, e^{-\rho r}\, d\rho$$

$$\underset{mr \gg 1}{\sim} \frac{1}{8\pi^2}\frac{m}{\sqrt{-mr}}\, e^{-mr} \qquad \frac{1}{\sqrt{\rho^2-m^2}}\, e^{-\rho r} = e^{\underbrace{-\rho r - \frac{1}{2}\ln(\rho^2-m^2)}_{-f(\rho)}}$$

$$\neq 0.$$

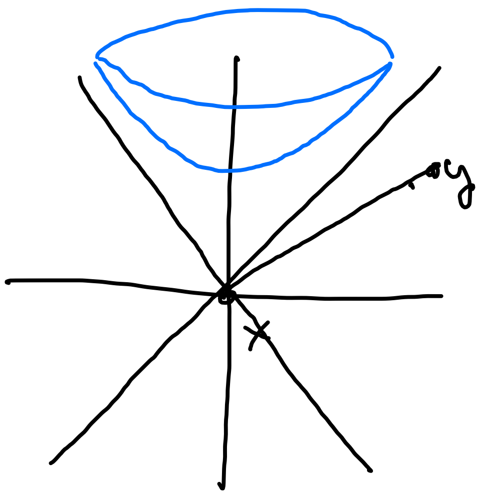
$$f'(\rho) = r + \frac{\rho}{\rho^2-m^2} = 0$$

$$r^2(\rho^2-m^2) = -r\rho \approx -mr \leftarrow \underbrace{\rho^2-m^2+\frac{\rho}{r}}=0 \rightarrow \rho = m + O\left(\frac{1}{mr}\right)$$



$$\langle 0 | [\phi(x), \phi(y)] | 0\rangle = \langle 0|\phi(x)\phi(y)|0\rangle - \langle 0|\phi(y)\phi(x)|0\rangle$$

$$= D(x-y) - D(y-x) = 0$$



$x^0 = y^0$

$[\phi(x), \phi(y)] = 0$

$x^0 = y^0$

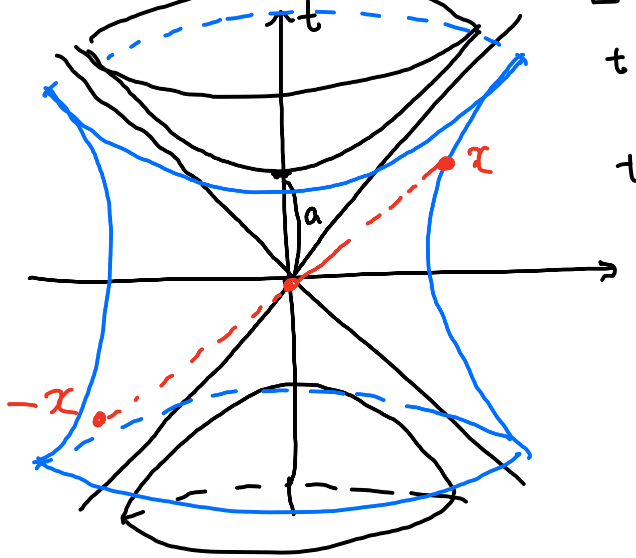
$[\phi, \pi] \neq 0 \quad ([\phi,\phi], [\pi,\pi] = 0)$

$x^2 = t^2 - \vec{x}^2 = a^2$

$\llcorner x$

space-like

Lorentz transf : $x \to -x$



$t = \pm\sqrt{\vec{x}^2 + a^2}$

$t^2 - \vec{x}^2 = -a^2$

$\vec{x}^2 = t^2 + a^2$

Klein-Gordon propagator

$$\langle 0|[\phi(x),\phi(y)]|0\rangle = \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{2E_{\vec p}}\, e^{-ip\cdot(x-y)}\Big|_{p^0=E_{\vec p}} + \frac{1}{-2E_{\vec p}}\, e^{\overbrace{ip\cdot(x-y)}^{e^{-ip\cdot(x-y)}|_{p^0=-E_{\vec p}}}}\right)$$

✓ ✓

$$e^{i[p^0\cdot(x^0-y^0) + \vec p\cdot(\vec x-\vec y)]}$$

$\vec p \to -\vec p$

$-p^0\big|_{p^0=-E_{\vec p}}$

$\cancel{x^0 \neq y^0}$

$\cancel{\neq E_{\vec p}}$

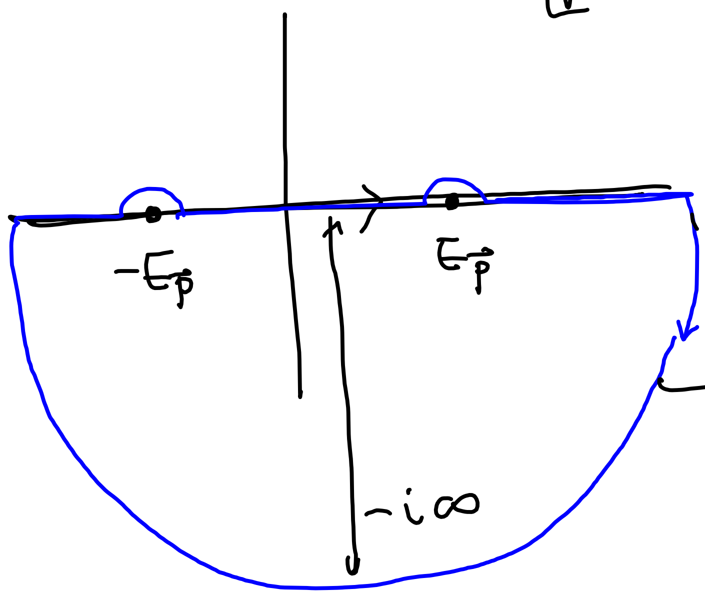
$$= \int \frac{d^3\vec p}{(2\pi)^3}\int \frac{dp^0}{(2\pi i)}\,\frac{-1}{p^2-m^2}\, e^{-ip\cdot(x-y)}$$

$$\frac{-1}{p_0^2-(\vec p^{\,2}+m^2)} = \frac{-1}{p_0^2-E_{\vec p}^2} = \frac{-1}{(p^0-E_{\vec p})(p^0+E_{\vec p})}$$

$p^0 = \pm E_p$

$x^0 > y^0$

p^0



$$-\cdot\frac{-1}{2E_p} \qquad \Big\{ \qquad -\cdot\frac{-1}{-2E_{\vec p}}$$

$$\underbrace{e^{-i\overset{-i\infty}{p^0}\overbrace{(x^0-y^0)}^{\text{positive}}}}_{=0}\; e^{i\vec p\cdot(\vec x-\vec y)}$$

$$\langle 0 | [\phi(x), \phi(y)] | 0 \rangle = [\phi(x), \phi(y)] \underbrace{=}_{x^0 > y^0} D_R(x-y)$$

$$= \underbrace{\int \frac{d^4p}{(2\pi)^4} \underbrace{\frac{i}{p^2 - m^2}}_{} e^{-ip\cdot(x-y)}}_{\tilde{D}_R(p)}$$

$$\tilde{D}_R(p) = \frac{i}{p^2 - m^2}$$

$$(\partial_x^2 + m^2) D_R(x-y) = \underbrace{-i\,\delta^{(4)}(x-y)}$$

$$(\partial^2 + m^2) \int \frac{d^4p}{(2\pi)^4} \cdots e^{-ip\cdot(x-y)} = \int \frac{d^4p}{(2\pi)^4} \underbrace{(-p^2 + m^2)\tilde{D}_R(p)}_{(-i)} e^{-ip\cdot(x-y)}$$

$$= (-i) \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} = -i\delta^{(4)}(x-y)$$

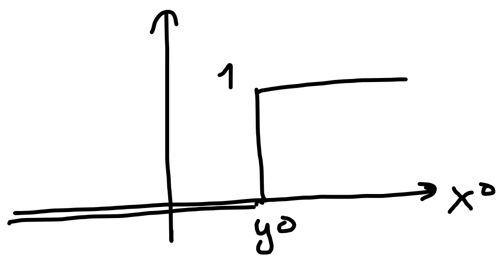
$$(\partial_x^2 + m^2)\left[\underbrace{\theta(x^0-y^0)}\langle 0|\,[\phi(x),\phi(y)]\,|0\rangle\right]$$

step fct ; $\theta(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$

$$\underbrace{\partial_{x_0}^2 - \vec{\nabla}_{\vec{x}}^2}_{\partial_\mu \partial^\mu}$$

$$= \partial_\mu\left[\partial^\mu\theta\,\langle\quad\rangle + \theta\,\partial^\mu\langle\quad\rangle\right] + m^2\theta\,\langle\quad\rangle$$

$$= \underbrace{\partial^2\theta\langle\,\rangle} + \underbrace{2\,\partial_\mu\theta\,\partial^\mu\langle\quad\rangle}_{2\,\delta(x^0-y^0)\,\partial^0\langle\quad\rangle} + \underbrace{\theta}\,(\partial^2+m^2)\langle\quad\rangle$$



$$\| $$

$$\partial_{x_0}^2\theta\,\langle\quad\rangle \qquad \partial_{x^0}\theta = \frac{\partial}{\partial x^0}\theta(x^0-y^0) = \delta(x^0-y^0)$$

$$\|$$

$$\partial_{x^0}\,\delta(x^0-y^0)\,\langle\quad\rangle$$

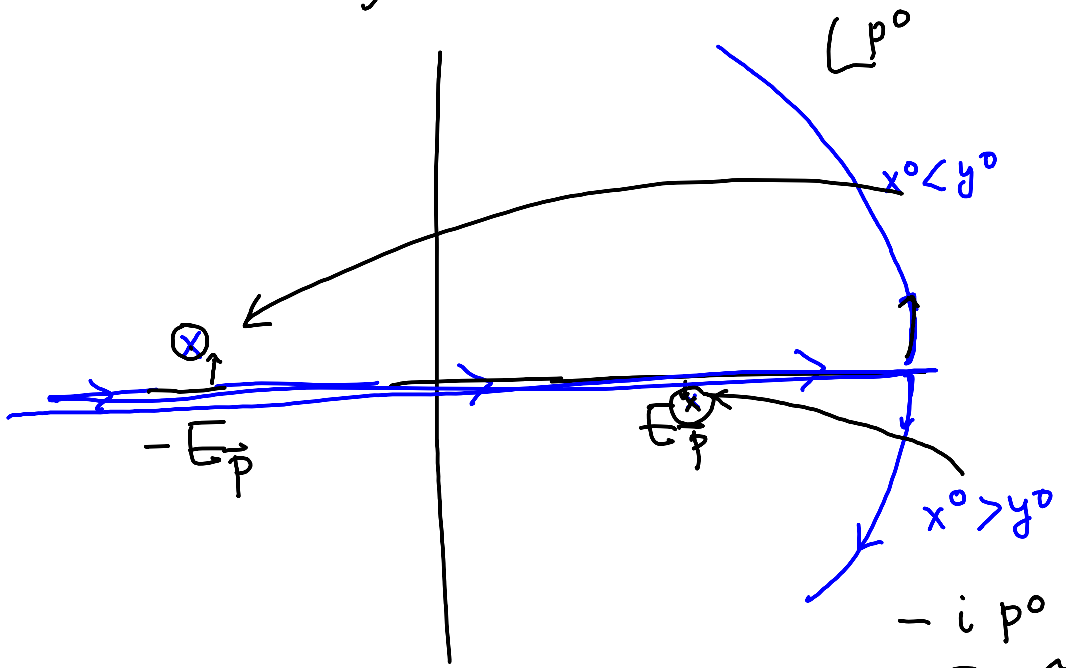
$$= \partial_{x^0}\left[\delta(x^0-y^0)\,\langle\quad\rangle\right] - \delta(x^0-y^0)\,\partial_{x^0}\langle\quad\rangle$$

$$\partial_{x^0}\left(\langle 0|\,\delta(x^0-y^0)\,\underbrace{[\phi(x),\phi(y)]}_{=0}\,|0\rangle\right) = 0$$

$$= \delta(x^0-y^0)\,\underbrace{\langle 0|\,[\underset{\pi(x)}{\underset{\shortparallel}{\dot\phi}}(x),\phi(y)]\,|0\rangle}_{-i\delta^{(3)}(\vec{x}-\vec{y})} + \theta(x^0-y^0)\,\langle 0|\,[\underbrace{\cancel{(\partial^2+m^2)}\phi(x)}_{\overset{\shortparallel}{0}},\phi(y)]\,|0\rangle$$

$$= -i\,\delta^{(4)}(x-y) \checkmark$$

Feynman Propagator



$$D_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{\overbrace{-i p\cdot(x-y)}^{-i p^0 (x^0 - y^0) + \cdots}}$$

$$p^2 - m^2 \to p_0^2 - E_{\vec p}^2 + i\varepsilon = (p^0 - E_{\vec p} + i\epsilon)(p^0 + E_{\vec p} - i\epsilon)$$

$$p^0 = E_{\vec p} - i\epsilon$$

$$= p_0^2 - (\underbrace{E_{\vec p} - i\epsilon}_{E_{\vec p}^2 - i\epsilon})^2$$

$$\langle 0 | \phi(x)\phi(y) | 0 \rangle$$

$$\equiv$$

$$D_F(x-y) = \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & y^0 > x^0 \end{cases} = \langle 0 | T[\phi(x)\phi(y)] | 0 \rangle$$

$\uparrow$ time ordered $\leftrightarrow$ normal ordered

$$D(y-x) = \langle 0 | \phi(y)\phi(x) | 0 \rangle$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + j(x)\,\phi(x)$$

$\therefore$ free theory + classical source

"~~quantize~~"

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial_\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -m^2\phi + j - \partial^2\phi = 0$$

$j(x) = 0 \qquad \phi_0(x) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}\,e^{-ip\cdot x} + a^\dagger_{\vec{p}}\,e^{ip\cdot x}\right)$

$j(x) \neq 0 \qquad \phi(x)$ satisfy $\quad (\partial^2 + m^2)\phi = j(x)$

$$(\partial^2 + m^2) D_R = -i\,\delta^{(4)}(x-y)$$

$$\phi(x) = \phi_0(x) + i\int\left[\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right)\right] j(y)\, d^4y \qquad \tilde{j}(p)^\dagger = \tilde{j}(-p)$$

$p^0 \equiv E_{\vec{p}}$

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\,e^{-ip\cdot x}\underbrace{\left[\int d^4y\, j(y)\, e^{ip\cdot y}\right]}_{\tilde{j}(p)}$$

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(\underbrace{\left[a_{\vec{p}} + \frac{i}{\sqrt{2E_{\vec{p}}}}\tilde{j}(p)\right]}_{a_{\vec{p}}} e^{-ip\cdot x} + \underbrace{\left[a^\dagger_{\vec{p}} - \frac{i}{\sqrt{2E_{\vec{p}}}}\tilde{j}(-p)\right]}_{a^\dagger_{p}} e^{ip\cdot x}\right)$$

$\tilde{j}(-p)$ " $\tilde{j}(p)^\dagger$

$$j=0 \rightarrow \qquad H = \int \frac{d^3\vec{p}}{(2\pi)^3} E_{\vec{p}} \; a^\dagger_{\vec{p}} a_{\vec{p}}$$

$$j \neq 0 \qquad H = \int \frac{d^3 p}{(2\pi)^3} E_{\vec{p}} \left(a^\dagger_{\vec{p}} - \frac{i}{\sqrt{2E_{\vec{p}}}} \tilde{j}^*(p) \right) \left(a_{\vec{p}} + \frac{i}{\sqrt{2E_{\vec{p}}}} \tilde{j}(p) \right)$$

$$\langle 0 | H | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{E_{\vec{p}}}{2E_{\vec{p}}} |\tilde{j}(p)|^2 \qquad n_p$$

$$a|0\rangle = 0 \qquad \langle 0 | a^\dagger = 0$$

$$N = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} |\tilde{j}(p)|^2$$