3장.

1 dim motion $\rightarrow$ d.o.f $= 1$

$$L = \frac{1}{2} a(q)\dot{q}^2 - U(q)$$

$q$ : generalized

$$\left( L = \frac{1}{2} m\dot{x}^2 - U(x) \right)$$

$x$ : 1d Cartesian

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$H = \dot{x}p - L$$

$$= m\dot{x}^2 - L = \frac{1}{2} m\dot{x}^2 + U(x) = E = \text{constant}$$

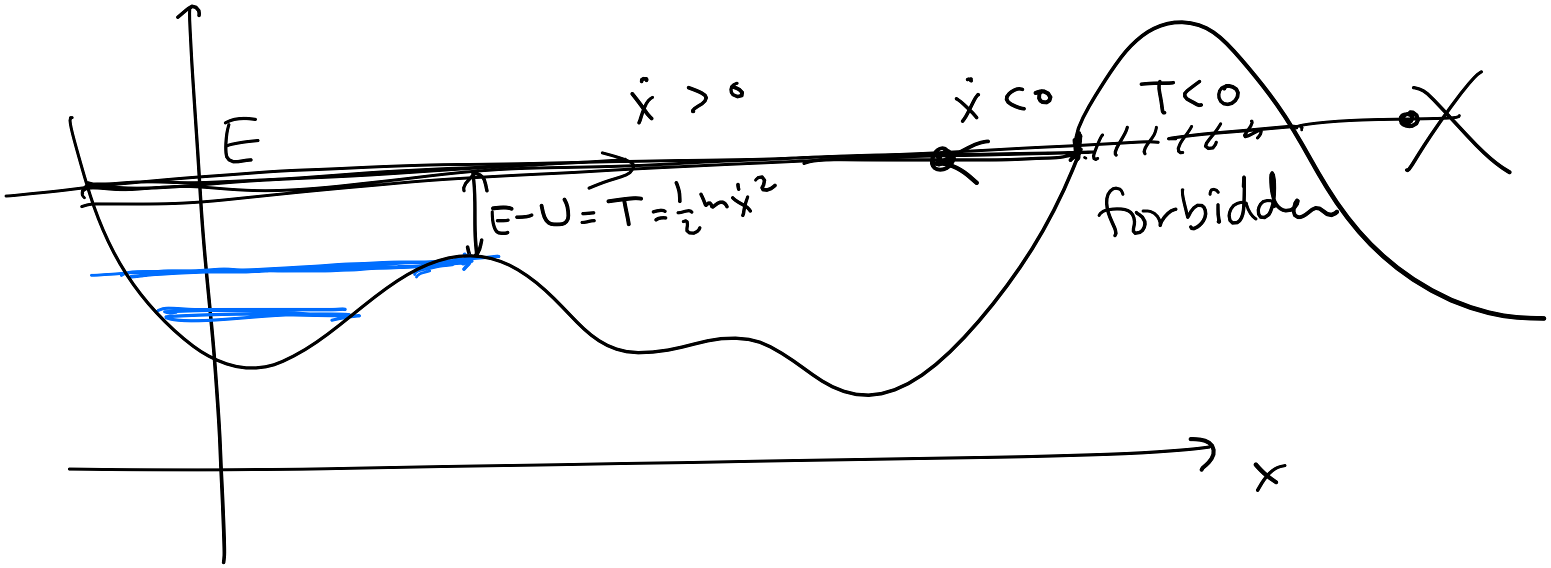
$$p_q = \frac{\partial L}{\partial \dot{q}} = a(q)\dot{q} \rightarrow E = \dot{q}a\dot{q} - L = \frac{a(q)}{2}\dot{q}^2 + U(q)$$

$$\dot{x} = \pm\sqrt{\frac{2(E - U(x))}{m}} = \frac{dx}{dt} \longrightarrow \frac{dx}{dt} = f(x) \rightarrow \int \frac{dx}{f(x)} = \int dt = t$$

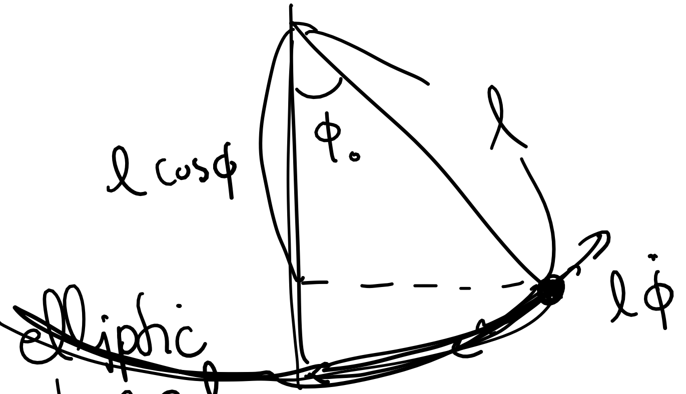
no t-dependence

$$\boxed{t = \sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{E - U(x)}} + C}$$

$$E - U(q) = \frac{1}{2} a \dot{q}^2 \rightarrow \dot{q} = \pm \sqrt{\frac{2}{a(q)}\left(E - U(q)\right)} = \pm f(q)$$




Prob. 1.




$$E = \frac{1}{2} m (l\dot{\phi})^2 - m g l \cos\phi = -m g l \cos\phi_0$$

$$\dot{\phi}(0) = 0 \qquad \phi(0) = \phi_0$$

$$\frac{d\phi}{dt} = \dot{\phi} = \sqrt{2 \frac{g}{l} (\cos\phi - \cos\phi_0)}$$

Incomplete elliptic integral

$$\int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}} = \int \sqrt{\frac{2g}{l}}\, dt = \sqrt{\frac{2g}{l}}\, t \longrightarrow \int_0^{\phi_0} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}} = \sqrt{\frac{2g}{l}} \left(\frac{T}{4}\right)$$

Complete elliptic integral

$$T = 4\sqrt{\frac{l}{2g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}} = 4\sqrt{\frac{l}{2g}}\,\frac{1}{\sqrt{2}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2\frac{\phi_0}{2} - \sin^2\frac{\phi}{2}}}$$

$$\cos\phi = 1 - 2\sin^2\frac{\phi}{2}$$

$$\frac{\sin\phi/2}{\sin\phi_0/2} \equiv \sin\xi$$

$$\frac{\frac{1}{2}\cos\frac{\phi}{2}\,d\phi}{\sin\frac{\phi_0}{2}} = \cos\xi\, d\xi$$

$$\frac{d\phi}{\cos\xi} = \frac{2\sin\frac{\phi_0}{2}\, d\xi}{\cos\frac{\phi}{2}}$$

$$\frac{1}{\sqrt{\sin^2\frac{\phi_0}{2}}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{1 - \underbrace{\frac{\sin^2\frac{\phi}{2}}{\sin^2\frac{\phi_0}{2}}}_{\sin^2\xi}}} = \frac{1}{\sin\frac{\phi_0}{2}} \int_0^{\phi_0} \frac{d\phi}{\cos\xi}$$

$$= \frac{1}{\cancel{\sin\frac{\phi_0}{2}}}\, 2\cancel{\sin\frac{\phi_0}{2}} \int_0^{\frac{\pi}{2}} \frac{d\xi}{\cos\frac{\phi}{2}} = \underbrace{2\int_0^{\frac{\pi}{2}} \frac{d\xi}{\sqrt{1-k^2\sin^2\xi}}}_{K(k)}$$

$$1 = \cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2} = \cos^2\frac{\phi}{2} + \underbrace{\sin^2\frac{\phi_0}{2}}_{k^2}\sin^2\xi$$

$$\sin\frac{\phi_0}{2} \equiv k$$

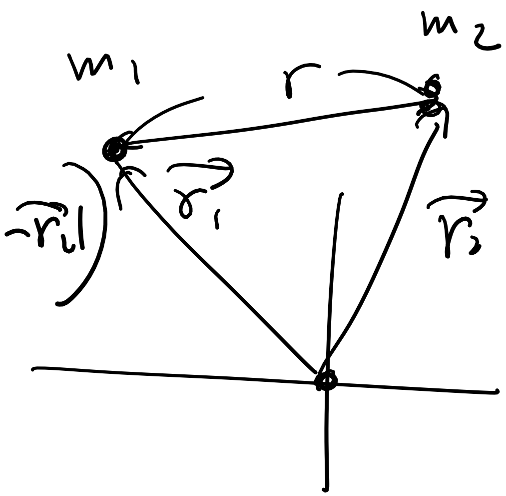
$$K(k^2) = \int_0^{\frac{\pi}{2}} \frac{d\xi}{\sqrt{1-k^2\sin^2\xi}} \longrightarrow T = 4\sqrt{\frac{l}{g}}\, \underbrace{K\left(\sin^2\frac{\phi_0}{2}\right)}$$

$$= 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\phi_0^2 + \cdots\right)$$

if $\phi_0 \ll 1$; $\frac{\pi}{2}\left(1 + \frac{1}{16}\phi_0^2 + \cdots\right)$

Prob 2. on p.27 → H.W.

## §13. reduced mass : 2 body




$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U\left(|\vec{r}_1 - \vec{r}_2|\right)$$

2 body $\xrightarrow{\text{can be reduced to}}$ 1 body

$$r = |\vec{r}_1 - \vec{r}_2|$$

$$\begin{cases} \vec{r}_1 - \vec{r}_2 \equiv \vec{r} \\ m_1 \vec{r}_1 + m_2 \vec{r}_2 = \vec{0} \end{cases}$$

, choose origin = C of M $= \vec{0} \equiv \dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$

$$\vec{r}_1 = \frac{m_2 \vec{r}}{m_1 + m_2} \qquad \vec{r}_2 = \vec{r}_1 - \vec{r} = -\frac{m_1 \vec{r}}{m_1 + m_2}$$

$$\rightarrow \dot{\vec{r}}_1 = \frac{m_2}{m_1 + m_2} \dot{\vec{r}} \quad , \quad \dot{\vec{r}}_2 = -\frac{m_1}{m_1 + m_2} \dot{\vec{r}}$$

$$T = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} = \frac{1}{2} m_1 \left(\frac{m_2}{m_1+m_2}\right)^2 \dot{\vec{r}}^{\,2} + \frac{1}{2} m_2 \left(\frac{m_1}{m_1+m_2}\right)^2 \dot{\vec{r}}^{\,2}$$

$$= \frac{\dot{\vec{r}}^{\,2}}{2} \left[ \frac{m_1 m_2^2}{(m_1+m_2)^2} + \frac{m_2 m_1^2}{(m_1+m_2)^2} \right]$$

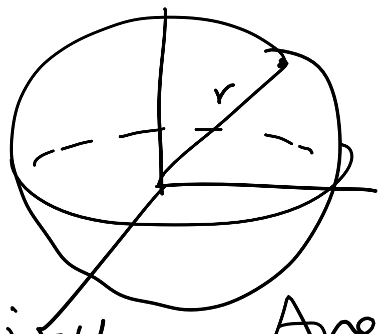
$$= \frac{1}{2} \underbrace{\frac{m_1 m_2}{m_1+m_2}}_{m \text{ "reduced mass"}} \dot{r}^{\,2}$$

$$L = \frac{1}{2} m \dot{\vec{r}}^{\,2} - U(|\vec{r}|)$$

$\therefore$ $\boxed{\underset{\vec{r}_1,\, \vec{r}_2}{\overset{6}{\uparrow}} \text{ d.o.f} \longrightarrow \underset{\vec{r}}{3} \text{ d.o.f}}$

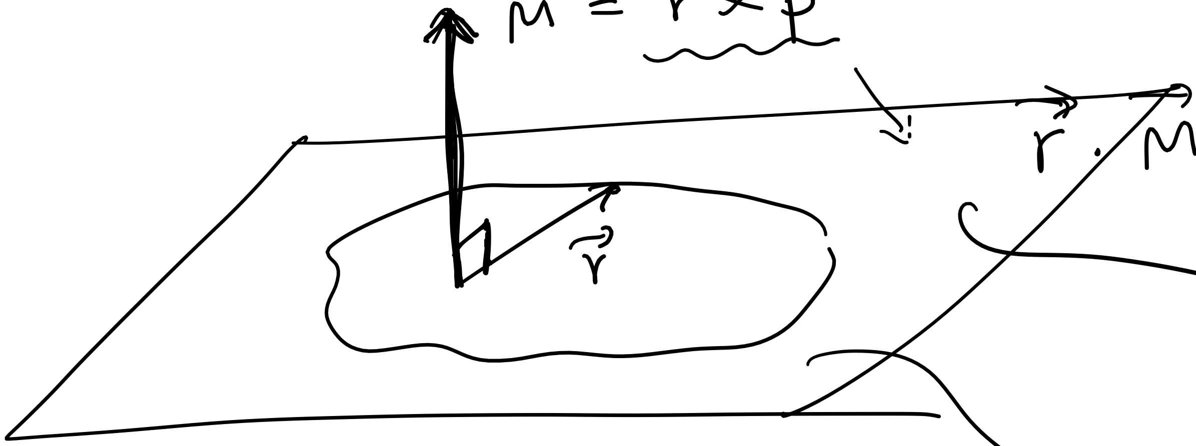
## §14. central field

$$U(\vec{r}) = U(|\vec{r}|) = U(r)$$



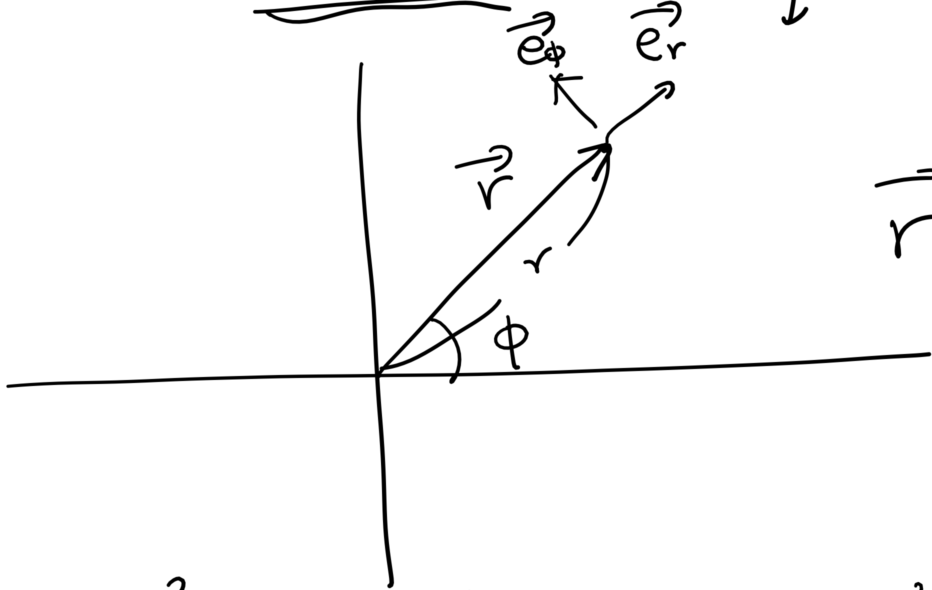

→ spherical symmetry → rotational inv. → Ang. mom. conserved.

$\vec{M} = \vec{r}\times\vec{p}$ = constant.




$\vec{r}\cdot\vec{M} = 0 \longrightarrow \vec{r}\perp\vec{M}$

$U(r) \longrightarrow$ 3 d.o.f → 2 d.o.f $(r,\phi)$




$$\vec{r} = r\vec{e}_r$$

$$\dot{\vec{r}} = \dot{r}\vec{e}_r + r\underbrace{\dot{\vec{e}}_r}_{\dot{\phi}\vec{e}_\phi}$$

$$\dot{\vec{r}}^{\,2} = \dot{r}^2 + r^2\dot{\phi}^2 \quad \therefore \quad L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

$$\rightarrow \quad m\ddot{r} = mr\dot{\phi}^2 - U'(r) = \frac{M^2}{mr^3} - U'(r)$$

$$\dot{\phi} = \frac{M}{mr^2}$$

$$\frac{\partial L}{\partial \phi} = 0 \longrightarrow \frac{\partial L}{\partial \dot\phi} = \text{const.}$$

$$= m r^2 \dot\phi = M$$

$$\therefore \underbrace{\dot\phi = \frac{M}{m r^2}}$$

$$E = T + U = \frac{1}{2} m \left( \dot r^2 + r^2 \dot\phi^2 \right) + U(r)$$

$$= \frac{1}{2} m \left( \dot r^2 + r^2 \frac{M^2}{m^2 r^4} \right) + U(r)$$

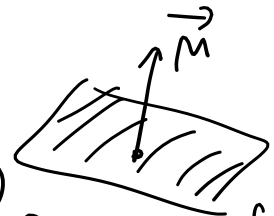
centrifugal potential

$$= \underbrace{\frac{1}{2} m \dot r^2} + \underbrace{\frac{M^2}{2 m r^2} + U(r)}_{U_{eff}(r)}$$

$\longrightarrow$ 1d problem

1 d.o.f.

$$\boxed{(\vec{r}_1, \vec{r}_2) \text{ 6 d.o.f.}} \xrightarrow{\text{CM}} \vec r \text{ 3 d.o.f} \xrightarrow{U(r)} \text{2 d.o.f} \xrightarrow{\text{A.M.}} \text{1 d.o.f.}$$




$(M_z)$

Angular Momentum

A.M.

$U \to U_{eff}$

$$E = \frac{1}{2} m \dot{r}^2 + U_{eff}(r) \to \frac{dr}{dt} = \sqrt{\frac{2}{m}}\sqrt{E - U_{eff}}$$

$$\downarrow$$

$$dt = \frac{dr}{\sqrt{\frac{2}{m}}\sqrt{E - U_{eff}}}$$

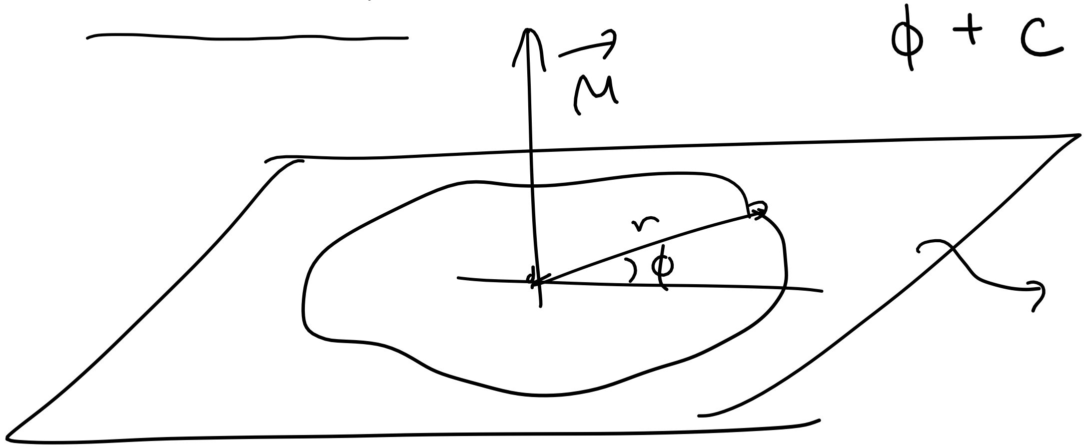
$$\to \int \frac{dr}{\sqrt{E - U_{eff}(r)}} = \sqrt{\frac{2}{m}} \int dt$$

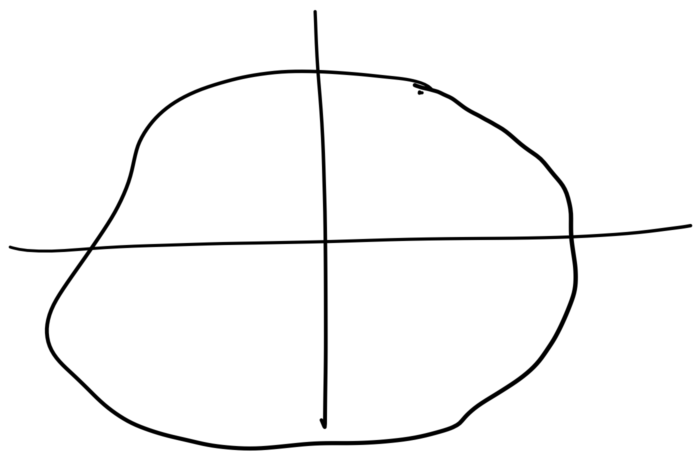
$$\sqrt{\frac{m}{2}} \int \frac{dr}{\sqrt{E - U_{eff}(r)}} = t = \int dt$$

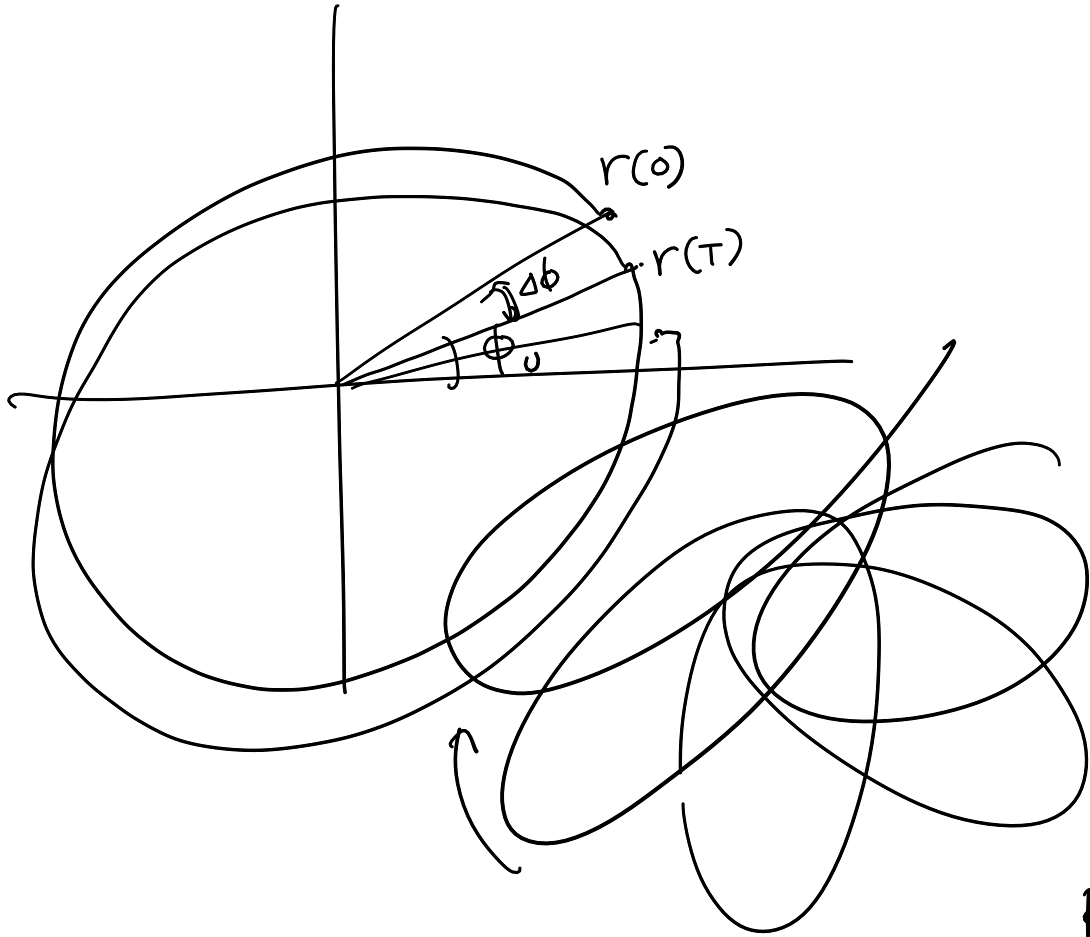
$$\dot{\phi} = \frac{M}{m r^2} = \frac{d\phi}{dt} \to \int d\phi = \frac{M}{m} \int \frac{dt}{r^2} = \frac{M}{m} \frac{1}{\sqrt{\frac{2}{m}}} \int_{r_0}^{r} \frac{dr}{r^2 \sqrt{E - U_{eff}}}$$

$$\int d\phi = \phi + C$$

$$\frac{M}{m} \frac{1}{\sqrt{\frac{2}{m}}} = \frac{M}{\sqrt{2m}}$$



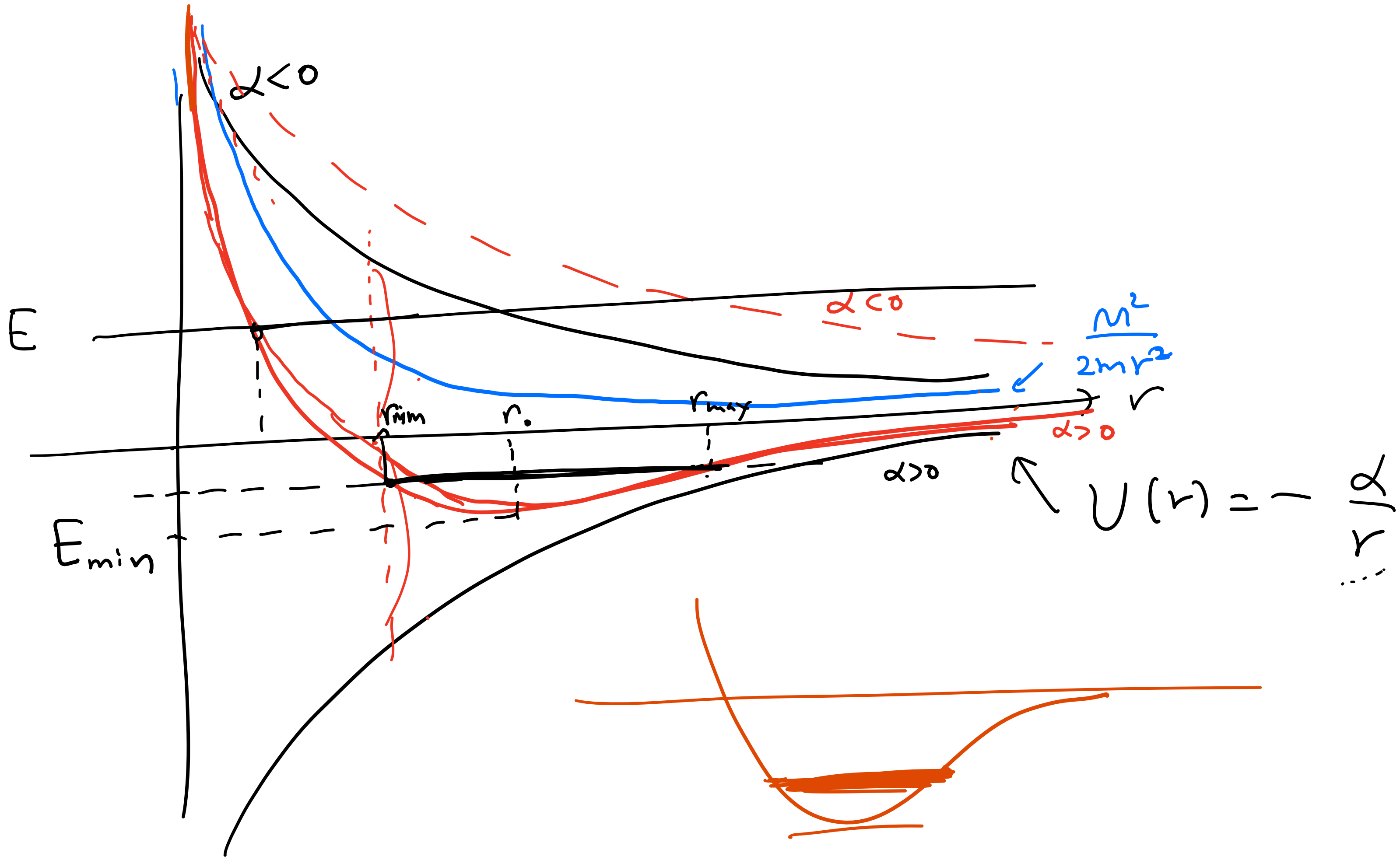



$\Delta\phi$

precession.

$$U(r) = -\frac{\alpha}{r}$$ "Kepler problem"

$$U_{eff}(r) = -\frac{\alpha}{r} + \frac{M^2}{2mr^2}$$




$\boxed{\alpha > 0}$

If $E_{min} < E < 0$ $\longrightarrow$ $r_{min} < r < r_{max}$ $\longrightarrow$ motion of planets

If $E > 0$ $\longrightarrow$ $r > r_{min}$ $\longrightarrow$ motion of comets

$E < E_{min}$ $\longrightarrow$ not allowed.

$$\sqrt{\frac{m}{2}}\int\frac{dr}{\sqrt{E-U_{eff}(r)}} = t$$

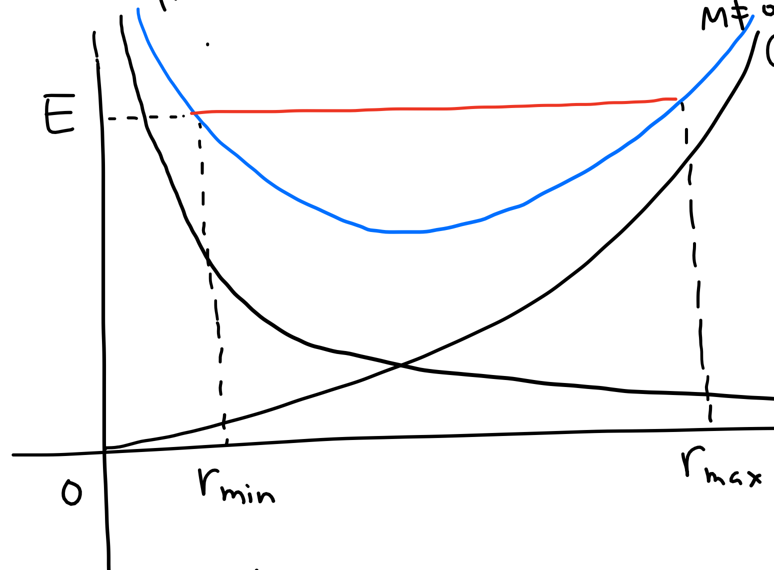
$$E = \underbrace{\frac{1}{2}m\dot{r}^2}_{\geq 0} + U_{eff}(r)$$

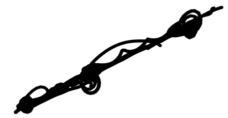
$$U_{eff}(r) = U(r) + \frac{M^2}{2mr^2} \qquad (r \geq 0)$$

$$\int d\phi = \frac{M}{\sqrt{2m}}\int_{r_0}^{r}\frac{dr}{r^2\sqrt{E-U_{eff}}}$$

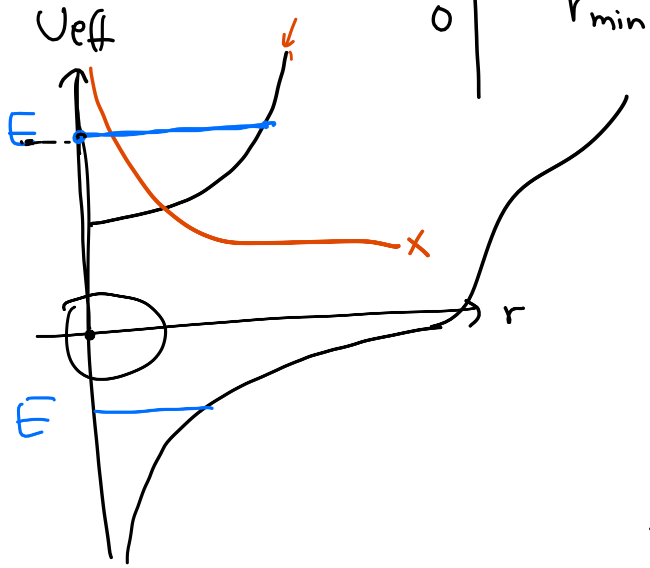
$M \neq 0$

(ex) $U(r) = \frac{k}{2}r^2$

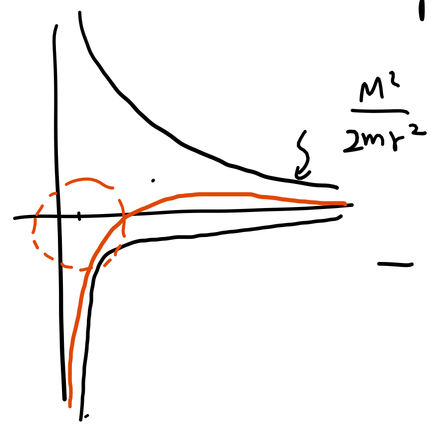


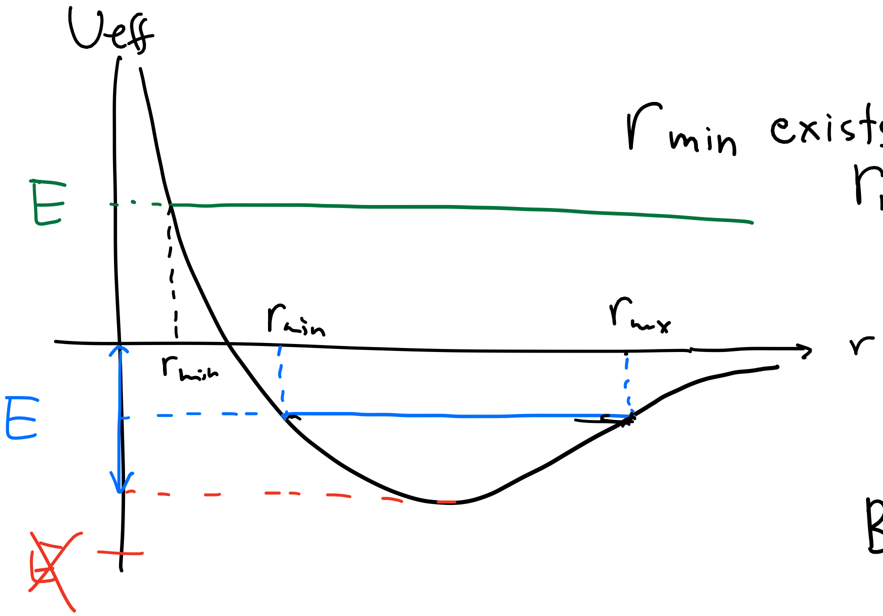
$r=0$ possible if




$$U(r) = -\frac{\alpha}{r^n} \qquad (\alpha > 0) \qquad n \geq 3$$




$$-\frac{\alpha}{r^2} \quad \& \quad \alpha \geq \frac{M^2}{2m}$$

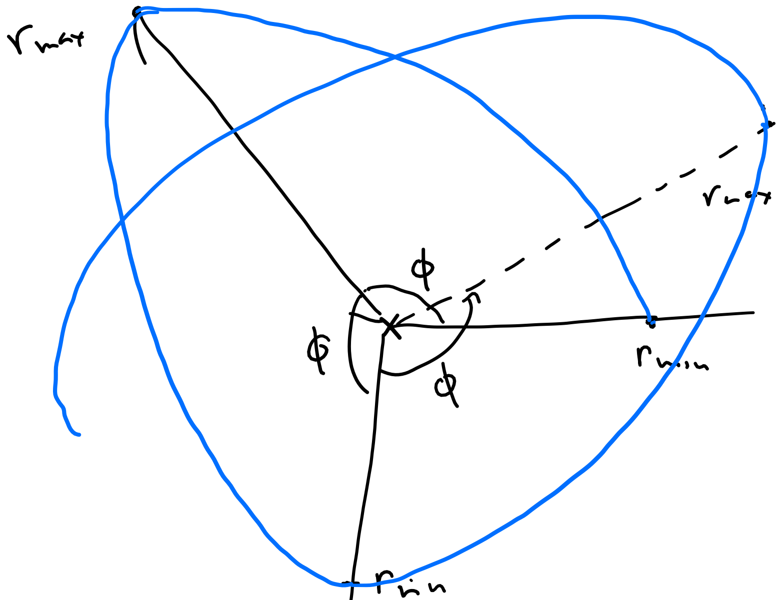


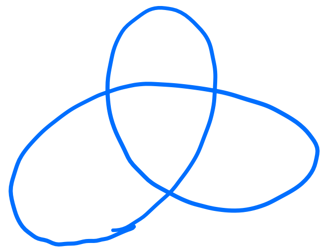

$r_{min}$ exists but $r_{max} = \infty$

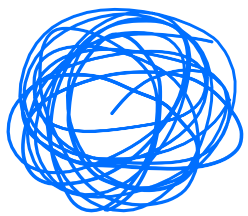
$$E = \underbrace{\frac{1}{2} m \dot{r}^2}_{\geq 0} + U_{eff}(r)$$

$$E \geq U_{eff}(r)$$

Bound State ( $r \to \infty$ is impossible)





$$\phi = \frac{M}{\sqrt{2m}} \int_{r_{min}}^{r_{max}} \frac{dr}{r^2 \sqrt{E - U_{eff}}}$$

$$\phi = 2\pi \underbrace{\frac{m}{n}}_{\text{rational number}} \qquad n, m = \text{integer}$$

closed orbit

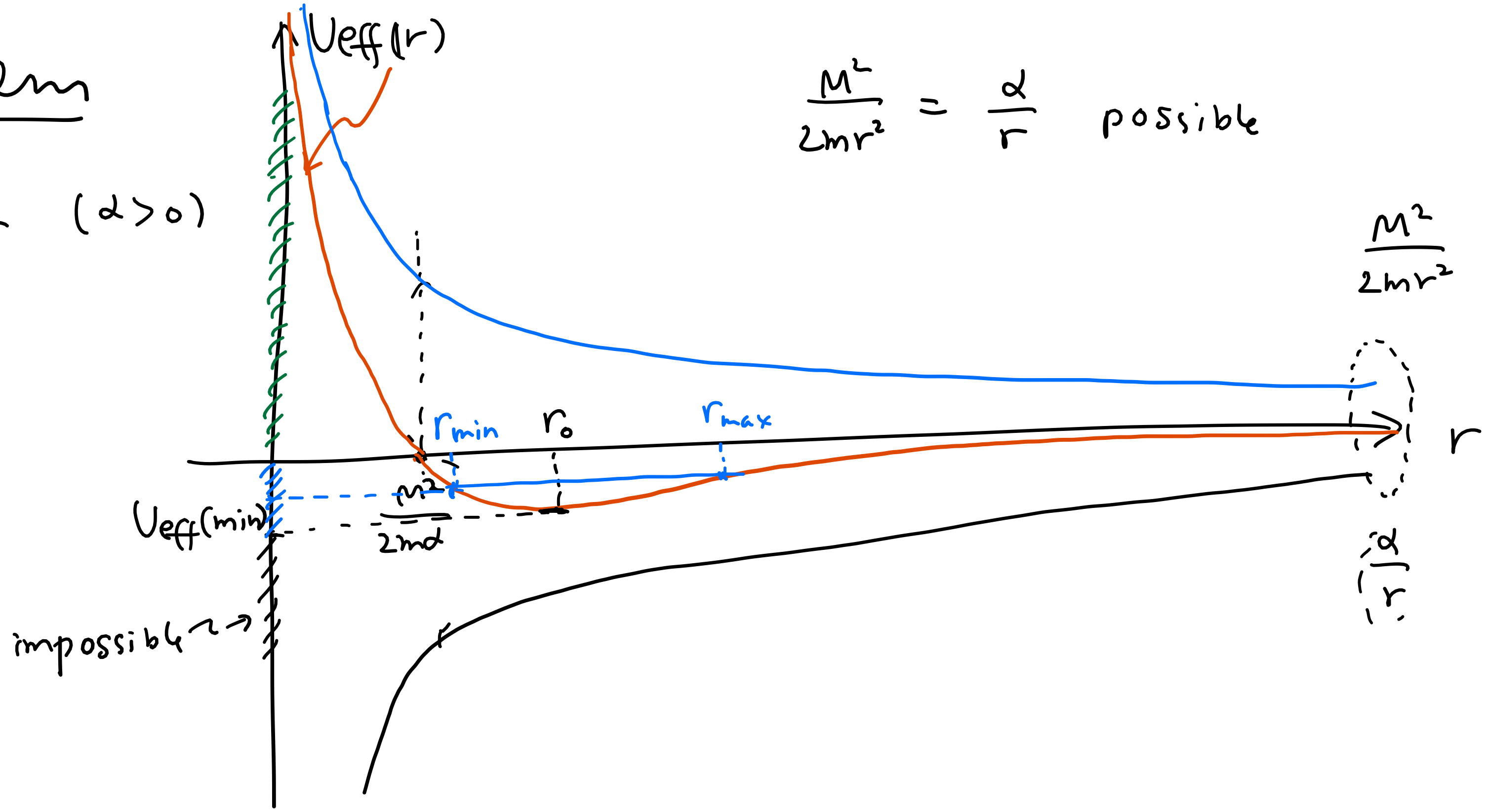
$= 2\pi \,(\text{irrational \#}) \to$ open orbit

(ex) $\pi^2 = \pi \cdot \pi$

# Kepler problem

$U(r) = -\frac{\alpha}{r} \quad (\alpha > 0)$

$\frac{M^2}{2mr^2} = \frac{\alpha}{r}$ possible




① $U_{eff}(min) \leq E < 0 \longrightarrow$ bound state (ellipse) closed orbit

원운동 $(r = r_0)$

((ex) planets) $r = r(\phi)$

② $E \geq 0 \longrightarrow$ comets which never return

$$\int d\phi = \cancel{\frac{M}{\sqrt{2m}}} \int_{r_0}^{r} \frac{dr}{r^2\sqrt{E-U_{eff}}} \rightarrow \sqrt{E-U_{eff}} = \sqrt{E + \frac{\alpha}{r} - \frac{M^2}{2mr^2}}$$

$$= \cancel{\sqrt{\frac{M^2}{2m}}} \underbrace{\sqrt{\frac{2mE}{M^2} + \frac{2m\alpha}{M^2 r} - \frac{1}{r^2}}}$$

$$\frac{2mE}{M^2} + \frac{2m\alpha}{M^2} u - u^2$$

$$\frac{1}{r} \equiv u \rightarrow du = -\frac{1}{r^2} dr$$

$$du = dv$$

$$\shortparallel$$

$$\underbrace{\frac{2mE}{M^2} + \left(\frac{m\alpha}{M^2}\right)^2}_{\equiv A^2 \geqq 0} - \underbrace{\left(u - \frac{m\alpha}{M^2}\right)^2}_{\equiv v}$$

$U_{eff}(min)$ :

$$U_{eff}(u) = -\alpha u + \frac{M^2}{2m} u^2$$

$$U'_{eff} = -\alpha + \frac{M^2}{m} u = 0$$

$$u_0 = \frac{1}{r_0} = \frac{m\alpha}{M^2}$$

$$U_{eff}(min) = \frac{M^2}{2m}\left(\frac{m\alpha}{M^2}\right)^2 - \alpha\frac{m\alpha}{M^2} = -\frac{m\alpha^2}{2M^2}$$

$$A^2 = \frac{2m}{M^2}\left(\underbrace{E + \frac{m\alpha^2}{2M^2}}_{\geqq 0}\right) \geqq 0$$

$$E \geqq U_{eff}(min) \rightarrow E + \frac{m\alpha^2}{2M^2} \geqq 0$$

$$\therefore \int_0^{\phi} d\phi = \int_{v}^{v_0} \frac{dv}{\sqrt{A^2 - v^2}}$$

$$v_0 = v(r = r_{min}) = \frac{1}{r_{min}} - \frac{m\alpha}{M^2}$$

$$\phi = \int_0^{\phi} d\phi = \int_{v}^{v_0} \frac{dv}{\sqrt{A^2 - v^2}} = \int_{\theta}^{\theta_0} \frac{A\cancel{\cos\theta}\, d\theta}{A\cancel{\cos\theta}} = \theta_0 - \theta \quad \leftarrow$$

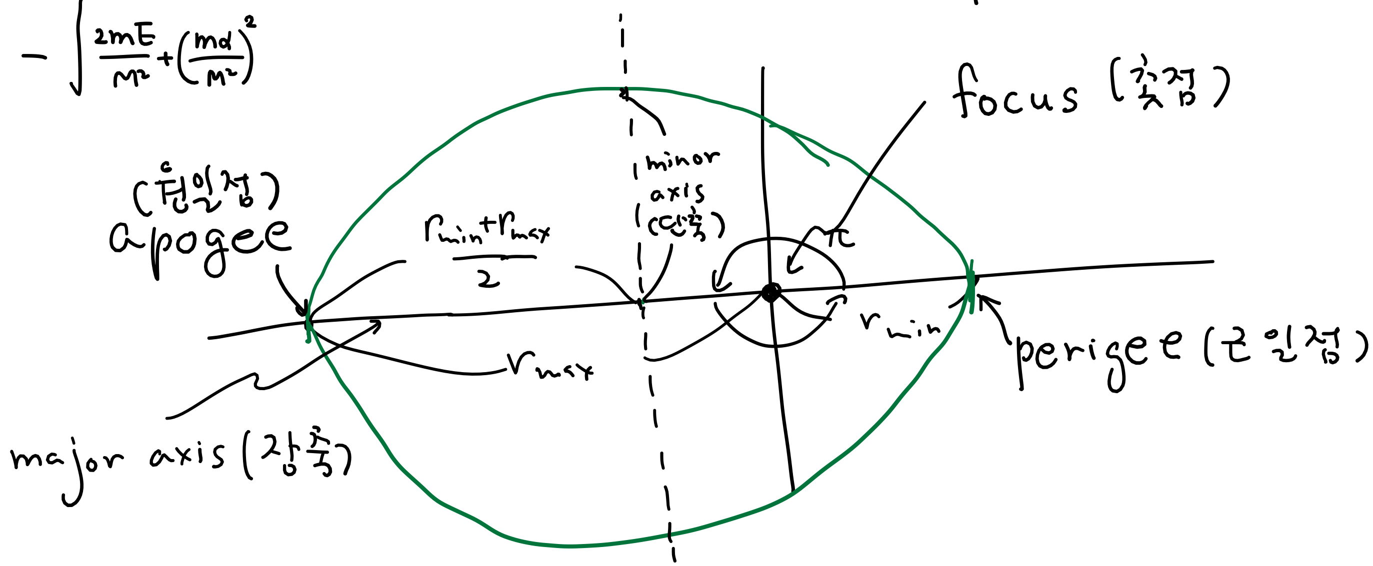
$$v = A\sin\theta \rightarrow dv = A\cos\theta\, d\theta$$

$$\therefore \quad \phi = -\theta + \theta_0 = -\sin^{-1}\left(\frac{v}{A}\right) + \theta_0 = -\sin^{-1}\left(\frac{1}{A}\left(\frac{1}{r} - \frac{m\alpha}{M^2}\right)\right) + \theta_0$$

$$\therefore \quad r = \frac{1}{\underbrace{\frac{m\alpha}{M^2}}_{\equiv \frac{1}{p}} - A\sin(\phi - \theta_0)} = \frac{1}{\underbrace{\frac{m\alpha}{M^2}}_{\equiv \frac{1}{p}} + \underbrace{\sqrt{\frac{2mE}{M^2} + \left(\frac{m\alpha}{M^2}\right)^2}}_{\frac{e}{p} \equiv} \cos\phi}$$

$$\underline{\underline{\phi = 0}} \rightarrow r_{min} \rightarrow \cancel{\phi} - \theta_0 = -\frac{\pi}{2} \rightarrow \theta_0 = \frac{\pi}{2}$$

$$r_{max} = \frac{1}{\frac{m\alpha}{M^2} - \sqrt{\frac{2mE}{M^2} + \left(\frac{m\alpha}{M^2}\right)^2}} \qquad \phi = \pi \qquad \therefore \quad \Delta\phi = \pi - 0 = \pi \quad ✓$$



$$r=\frac{1}{\underbrace{\frac{m\alpha}{M^2}}_{=\frac{1}{p}}+\underbrace{\sqrt{\frac{2mE}{M^2}+\left(\frac{m\alpha}{M^2}\right)^2}}_{\frac{e}{p}\equiv}\cos\phi}$$

negative

$$;\ \underset{\overset{\shortparallel}{U_{eff}(\min)}}{E_{\min}} \leq E<0 \rightarrow \frac{e}{p}<\frac{1}{p} \rightarrow 0\leq e<1$$

eccentricity (이심률)

$$r=\frac{1}{\frac{1}{p}+\frac{e\cos\phi}{p}}=\frac{p}{1+e\cos\phi} \rightarrow \boxed{\frac{p}{r}=1+e\cos\phi}$$

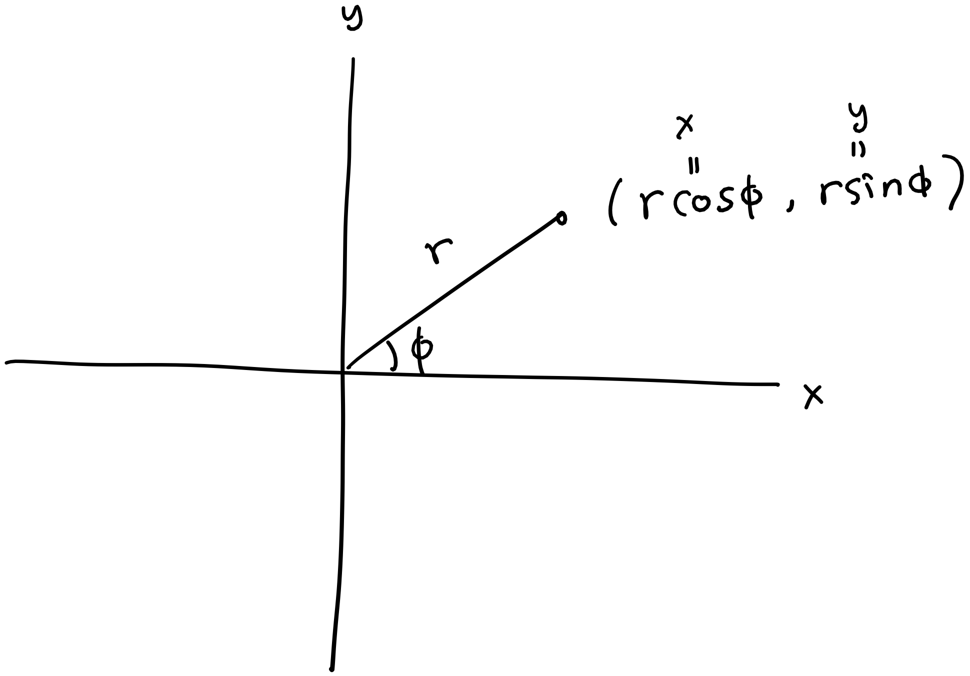
$e=0$

$$E=-\frac{m\alpha^2}{2M^2}=U_{eff}(\min)$$

$$\downarrow$$

$$r=\frac{1}{\frac{m\alpha}{M^2}}=\frac{M^2}{m\alpha}\ \overset{\shortparallel}{\text{const.}}$$

(circular orbit)




$$p=r+ex$$

$$=\sqrt{x^2+y^2}+ex \rightarrow \left(\sqrt{x^2+y^2}\right)^2=(p-ex)^2$$

$$x^2+y^2=p^2-2epx+e^2x^2$$

$$(1-e^2)x^2+2pex+y^2=p^2$$

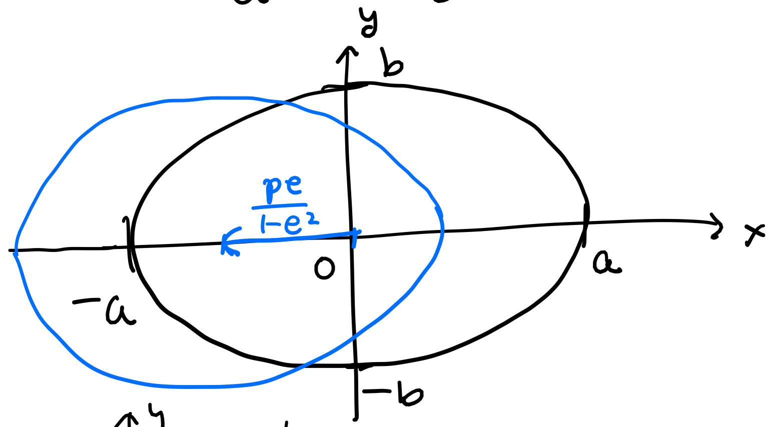
$$(1-e^2)\left[\underbrace{x^2+\frac{2pe}{1-e^2}x+\left(\frac{pe}{1-e^2}\right)^2}-\left(\frac{pe}{1-e^2}\right)^2\right]+y^2=p^2$$

$$(1-e^2)\left(x+\frac{pe}{1-e^2}\right)^2+y^2=p^2+\frac{p^2e^2}{1-e^2}$$

$$=\frac{p^2}{1-e^2}$$

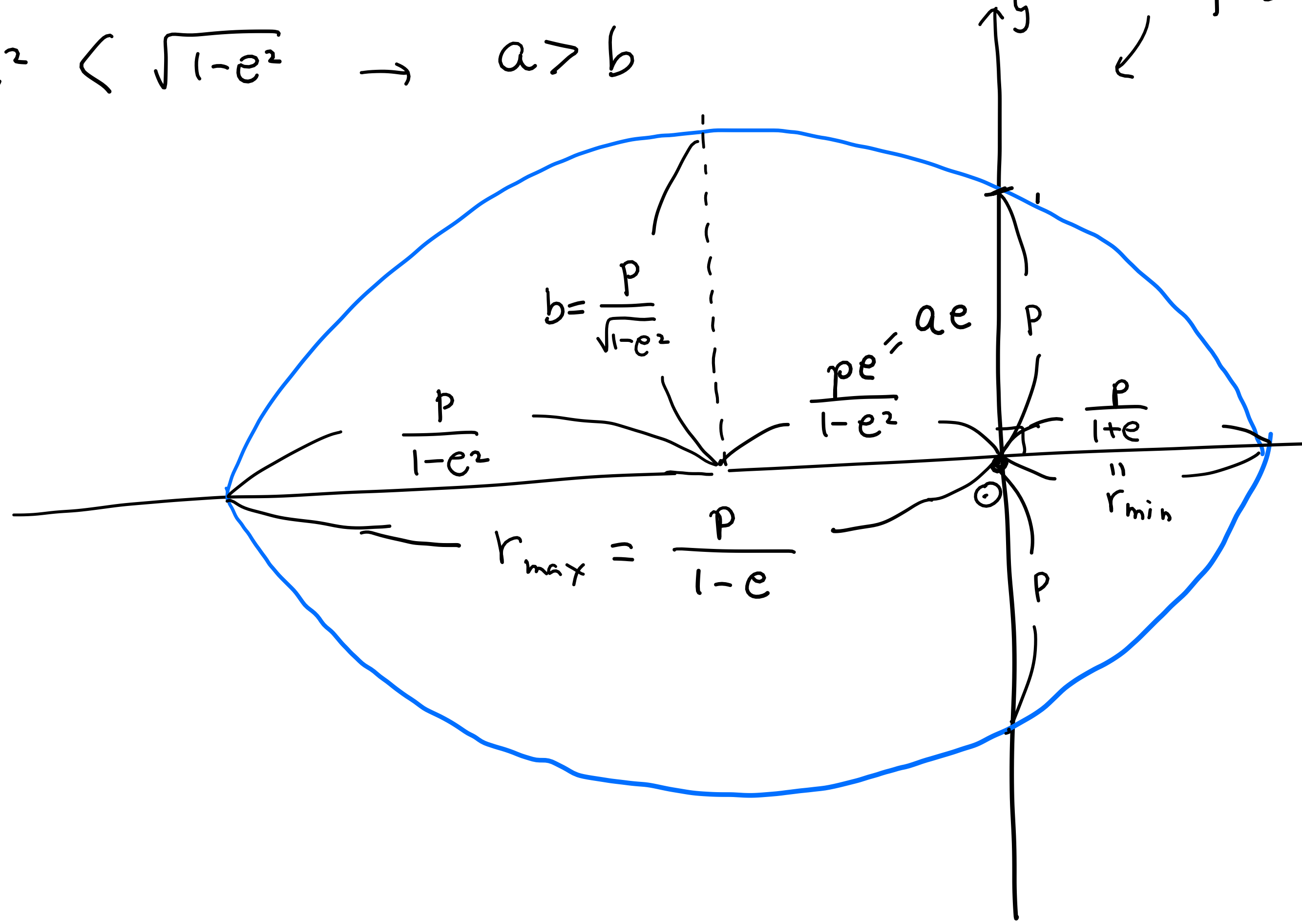
$$\frac{\left(x+\frac{pe}{1-e^2}\right)^2}{\frac{p^2}{(1-e^2)^2}}+\frac{y^2}{\frac{p^2}{(1-e^2)}}=1 \Leftarrow$$

$$\frac{\left(x+\frac{pe}{1-e^2}\right)^2}{\frac{p^2}{(1-e^2)^2}}+\frac{y^2}{\frac{p^2}{(1-e^2)}}=1 \quad\longleftrightarrow\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$




$$a=\frac{p}{1-e^2} \qquad b=\frac{p}{\sqrt{1-e^2}}$$

$$1-e^2<\sqrt{1-e^2}\ \rightarrow\ a>b$$




$$p=\frac{M^2}{m\alpha}$$

$$e=\sqrt{\frac{2mE}{M^2}+\left(\frac{m\alpha}{M^2}\right)^2}\cdot\frac{M^2}{m\alpha}$$

$$=\sqrt{1+\frac{2M^2}{m\alpha^2}E}$$

$$\alpha=GmM_{\odot}$$

$$\frac{p}{r}=1+e\cos\phi$$

$\uparrow$
$\frac{\pi}{2}$

$$a=\frac{\cancel{M^2}/\cancel{m\alpha}}{\frac{2\cancel{M^2}}{m\alpha^2}|E|}=\frac{\alpha}{2|E|}$$

$$b=\frac{M^2/m\alpha}{\sqrt{\frac{2M^2|E|}{m\alpha^2}}}=\frac{M}{\sqrt{2m|E|}}$$

if $$r = \frac{1}{A + B\cos[\beta\phi]}$$

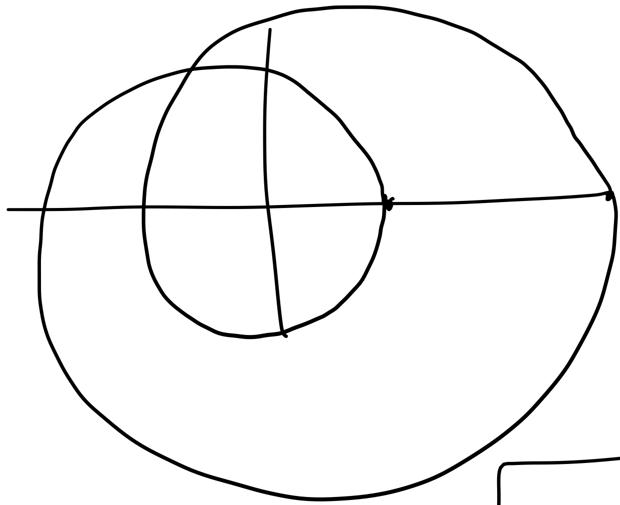
$$\left.\begin{array}{l} \phi = 0 \to r_{min} \\ \phi = \frac{\pi}{\beta} \to r_{max} \end{array}\right\} \quad \Delta\phi = \frac{\pi}{\beta} = 2\pi \cdot \left(\frac{\beta}{2}\right)$$

closed ← rational

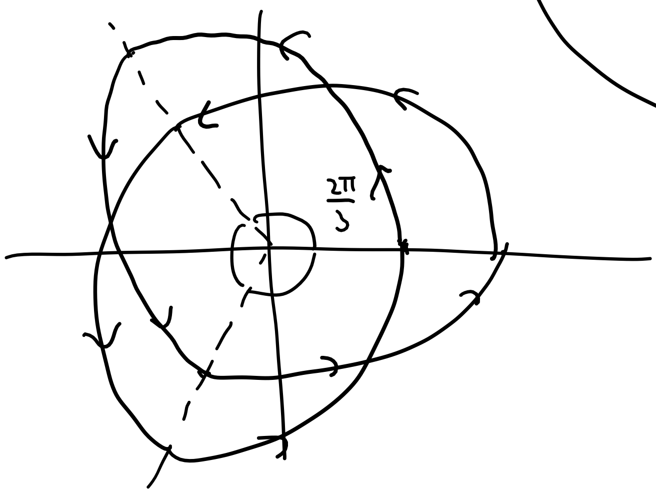
open ← or irrational

Kepler: $U(r) = -\frac{\alpha}{r} \to \beta = 1$

if $\beta = \frac{1}{2}$



if $\beta = \frac{2}{3}$ $\left(\Delta\phi = \frac{2\pi}{3}\right)$




MATLAB

make polar plots for

H.W. $A = 2$, $B = 1$

$\phi = 0 : 10\pi$

1. $\beta =$ rational $= \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}$
2. $\beta =$ ir " $= \sqrt{2}, \frac{\pi}{2}, \sqrt{3}, \sqrt{5}$

$\boxed{\text{H.W. Prob3 on page 40.}}$

(MATLAB) (a) $\delta U = \frac{\beta}{r^2}$ (b) $\delta U = \frac{\gamma}{r^3}$

$(M = m = l = \alpha)$

$$\delta\phi = \phi(r_{max}) - \phi(r_{min}) = 2\pi \cdot f = \frac{\partial}{\partial M}\left(\frac{2m}{M}\int_0^{\pi} r^2\,\delta U\, d\phi\right)$$

(ex) $\delta U = \frac{\beta}{r^2}$ $= \frac{\partial}{\partial M}\left(\frac{2m}{M}\pi\beta\right)$

$\therefore f = -\frac{m\beta}{M^2}$ $\delta\phi = -\frac{2m\pi\beta}{M^2}$

$\doteq -\beta \rightarrow \beta = \frac{1}{3}$

find $\beta, \gamma$

such that $\delta\phi = \frac{2\pi}{3}$

$\boxed{\text{plot trajectory}}$ using MATLAB

# On Mid term Exam

April 25 (at Class) + MATLAB Exam time (immediately after)

Written exam. will be posted.

1 problem is "derivation"

1 problem is from H.W. in the text.

$E>0$

$$\int d\phi = \cancel{\frac{M}{\sqrt{2m}}} \int_{r_0}^{r} \frac{dr}{r^2\sqrt{E-U_{eff}}} \rightarrow \sqrt{E-U_{eff}} = \sqrt{E+\frac{\alpha}{r}-\frac{M^2}{2mr^2}}$$

$$= \cancel{\sqrt{\frac{M^2}{2m}}} \underbrace{\sqrt{\frac{2mE}{M^2}+\frac{2m\alpha}{M^2 r}-\frac{1}{r^2}}}$$

$$\frac{1}{r} \equiv u \rightarrow du = -\frac{1}{r^2}dr$$

$$du = dv$$

$$\frac{2mE}{M^2}+\frac{2m\alpha}{M^2}u - u^2$$

$$\shortparallel$$

$$\underbrace{\frac{2mE}{M^2}}+\underbrace{\left(\frac{m\alpha}{M^2}\right)^2} - \underbrace{\left(u-\frac{m\alpha}{M^2}\right)}_{\equiv\, v}{}^2$$

$$A^2 = \frac{2m}{M^2}\left(E+\frac{m\alpha^2}{2M^2}\right) \geq 0$$

$U_{eff}(\min)$ :

$$U_{eff}(u) = -\alpha u + \frac{M^2}{2m}u^2$$

$$U_{eff}' = -\alpha + \frac{M^2}{m}u = 0$$

$$u_0 = \frac{1}{r_0} = \frac{m\alpha}{M^2}$$

$$U_{eff}(\min) = \frac{M^2}{2m}\left(\frac{m\alpha}{M^2}\right)^2 - \alpha\frac{m\alpha}{M^2} = -\frac{m\alpha^2}{2M^2}$$

$$E \geq U_{eff}(\min) \rightarrow \underline{\underline{E+\frac{m\alpha^2}{2M^2} \geq 0}}$$

$$\therefore \int_0^{\phi} d\phi = \int_{v}^{v_0} \frac{dv}{\sqrt{A^2-v^2}}$$

$$v_0 = v(r=r_{min}) = \frac{1}{r_{min}} - \frac{m\alpha}{M^2}$$

$$\phi = \int_0^{\phi} d\phi = \int_{v}^{v_0} \frac{dv}{\sqrt{A^2 - v^2}} = \int_{\theta}^{\theta_0} \frac{A \cancel{\cos\theta}\, d\theta}{A \cancel{\cos\theta}} = \theta_0 - \theta \leftarrow$$

$$v = A\sin\theta \rightarrow dv = A\cos\theta\, d\theta$$

$$\therefore \quad \phi = -\theta + \theta_0 = -\sin^{-1}\left(\frac{v}{A}\right) + \theta_0 = -\sin^{-1}\left(\frac{1}{A}\left(\frac{1}{r} - \frac{m\alpha}{M^2}\right)\right) + \theta_0$$

$$\therefore \quad r = \frac{1}{\underbrace{\frac{m\alpha}{M^2}}_{\equiv \frac{1}{p}} - A\sin(\phi - \theta_0)} = \frac{1}{\underbrace{\frac{m\alpha}{M^2}}_{\equiv \frac{1}{p}} + \underbrace{\sqrt{\frac{2mE}{M^2} + \left(\frac{m\alpha}{M^2}\right)^2}}_{\frac{e}{p} \equiv} \cos\phi}$$

$$\underline{\underline{\phi = 0}} \rightarrow r_{min} \rightarrow \cancel{\phi} - \theta_0 = -\frac{\pi}{2} \rightarrow \theta_0 = \frac{\pi}{2}$$

$$\frac{e}{p} = \sqrt{\frac{2mE}{M^2} + \left(\frac{m\alpha}{M^2}\right)^2} \geq \frac{m\alpha}{M^2} = \frac{1}{p} \rightarrow e \geq 1$$

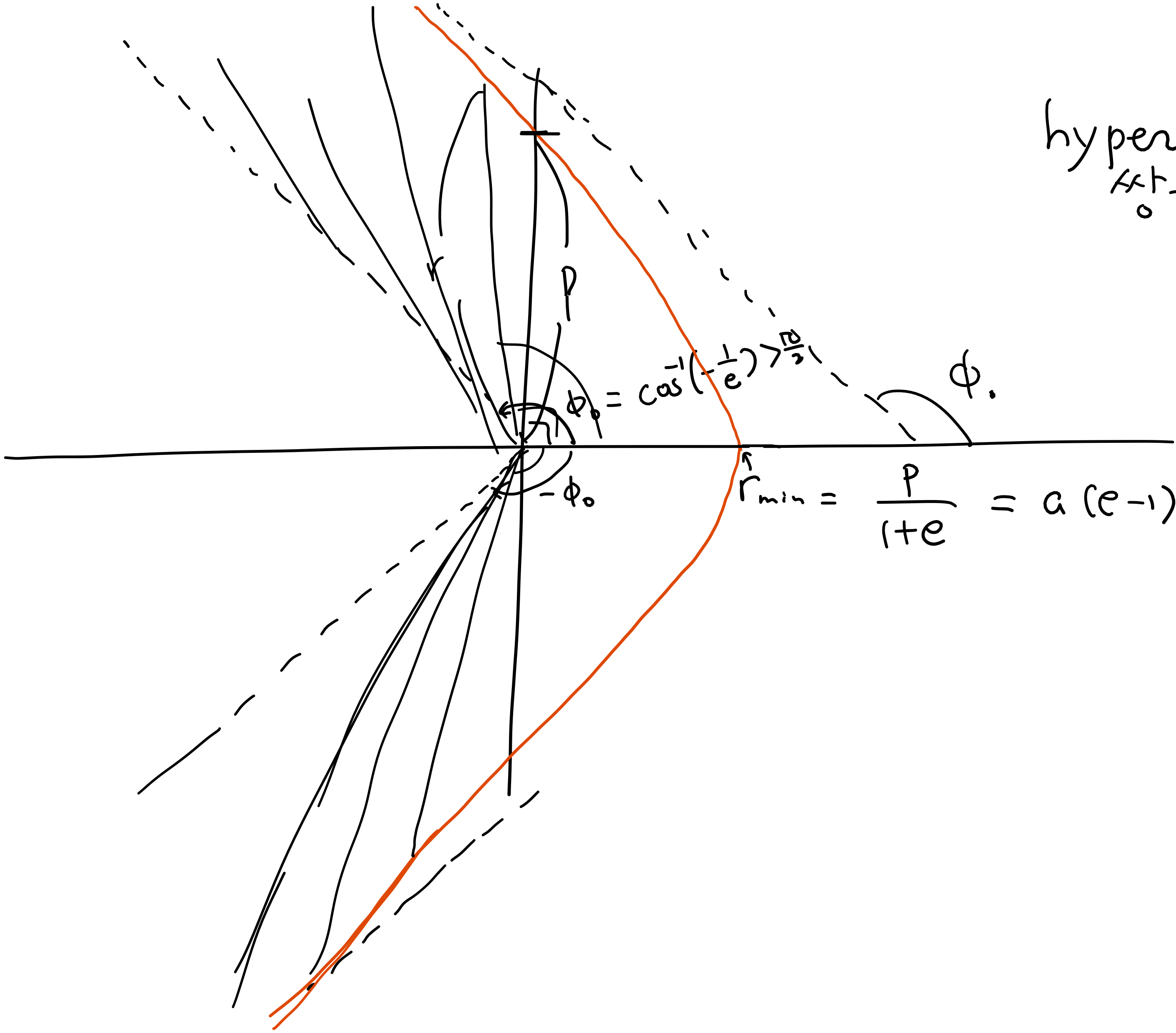
$$E \geq 0$$

$$r = \frac{p}{1 + e\cos\phi} \rightarrow r_{min} = \frac{p}{1+e}, \quad r_{max} \cancel{=} \frac{p}{1-e} \quad (\text{if } e < 1)$$

$e > 1$

$$\text{at} \quad \cos\phi_0 = -\frac{1}{e} \rightarrow r = \infty$$

hyperbolic curve
쌍곡선




$\phi_0 = \cos^{-1}\left(-\frac{1}{e}\right) > \frac{\pi}{2}$

$r_{min} = \frac{p}{1+e} = a(e-1)$

$a = \frac{p}{e^2-1} = \frac{\alpha}{2E}$

$$\sqrt{\frac{m}{2}}\int\frac{r\,dr}{\sqrt{-\underbrace{|E|}_{E}r^2+\alpha r-\frac{M^2}{2m}}}=t=\sqrt{\frac{m}{2|E|}}\int\frac{r\,dr}{\Big(\underbrace{\underbrace{\frac{\alpha}{|E|}}_{2a}r-\underbrace{\frac{M^2}{2m|E|}}_{b^2}-r^2}\Big)^{\frac{1}{2}}}\qquad a=\frac{\alpha}{2|E|}$$

(1) $E<0$

$$\left.\begin{aligned}a^2&=\frac{p^2}{(1-e^2)^2}\\ b^2&=\frac{p^2}{1-e^2}\end{aligned}\right\}\quad \frac{b^2}{a^2}=1-e^2$$

$$2ar-b^2-r^2=-b^2-(\underbrace{r^2-2ar+a^2}_{(r-a)^2}-a^2)$$

$$=\underbrace{a^2-b^2}_{a^2(1-\frac{b^2}{a^2})=a^2e^2}-(r-a)^2$$

$$t=\sqrt{\frac{m}{2|E|}}\int\frac{r\,dr}{\sqrt{a^2e^2-(r-a)^2}}$$

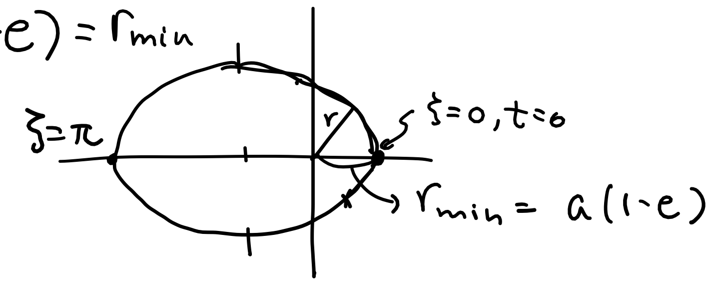
$$r-a\equiv-ae\cos\xi\rightarrow \underline{r=a(1-e\cos\xi)}$$

$$dr=ae\sin\xi\,d\xi$$

$$=\sqrt{\frac{m}{2|E|}}\int\frac{\overbrace{a(1-e\cos\xi)}^{r}\ \overbrace{a\cancel{e\sin\xi}\,d\xi}^{dr}}{a\cancel{e\sin\xi}}=\sqrt{\frac{m}{2|E|}}\,a\int(1-e\cos\xi)d\xi$$

$$=\quad ''\quad(\xi-e\sin\xi)+\underset{0}{\overset{\shortparallel}{c}}$$

$\xi=0\longrightarrow t=\cdots 0=0\qquad r=a(1-e)=r_{min}$



$$\phi = \frac{1}{\sqrt{|E|}}\frac{M}{\sqrt{2m}}\int \frac{dr}{r\sqrt{a^2e^2-(r-e)^2}} = {}^{\prime\prime}\int \frac{\cancel{ae}\sin\xi\, d\xi}{a(1-e\cos\xi)\,\cancel{ae\sin\xi}}$$

$$= \frac{M}{\sqrt{2m|E|}}\frac{1}{a}\int\frac{d\xi}{1-e\cos\xi}$$

---

$$M = mr^2\dot{\phi} \rightarrow \frac{d\phi}{dt} = \frac{M}{mr^2} = \frac{d\phi}{d\xi}\frac{d\xi}{dt} \rightarrow \frac{d\phi}{d\xi} = \frac{M}{m}\;\frac{1}{r^2}\frac{dt}{d\xi} = \frac{M}{m}\sqrt{\frac{m}{2|E|}}\frac{a}{a^2}\frac{\cancel{1-e\cos\xi}}{(1-e\cos\xi)^{\cancel{2}}}$$

$$r = a(1-e\cos\xi)$$

$$t = \sqrt{\frac{m}{2|E|}}\,a(\xi - e\sin\xi)$$

---

$$r = \frac{p}{1+e\cos\phi} \qquad \cos\phi = \frac{1}{e}\left(\frac{p}{r}-1\right) = \frac{1}{e}\left(\frac{p-a(1-e\cos\xi)}{a(1-e\cos\xi)}\right)$$

$$= \frac{1}{\cancel{ae}}\left(\frac{\overbrace{a(1-e^2)-a}^{-ae^2} + ae\cos\xi}{1-e\cos\xi}\right)$$

$$\phi = \cos^{-1}\left(\frac{\cos\xi - e}{1-e\cos\xi}\right)$$

$E>0$

$$\sqrt{\frac{m}{2}}\int \frac{r\,dr}{\sqrt{E r^2 + \alpha r - \frac{M^2}{2m}}} = t = \sqrt{\frac{m}{2E}}\int \frac{r\,dr}{\left(\underbrace{\frac{\alpha}{E}}_{2a} r - \frac{M^2}{2mE} + r^2\right)^{\frac{1}{2}}}$$

$$a = \frac{\alpha}{2E}$$

$$(r+a)^2 - \underbrace{\left(a^2 + \frac{M^2}{2mE}\right)}_{=a^2e^2}$$

$$e^2 = 1 + \frac{2EM^2}{m\alpha^2}$$

$$\frac{\alpha^2}{4E^2} + \frac{M^2}{2mE} = \underbrace{\frac{\alpha^2}{4E^2}}_{a^2}\left(1 + \underbrace{\frac{2EM^2}{m\alpha^2}}_{e^2}\right)$$

$$t = \sqrt{\frac{m}{2E}}\int \frac{r\,dr}{\sqrt{(r+a)^2 - a^2e^2}}$$

$$r + a \equiv ae\cosh\eta$$

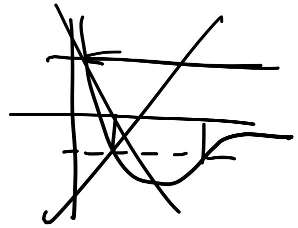
$$dr = ae\sinh\eta\,d\eta$$

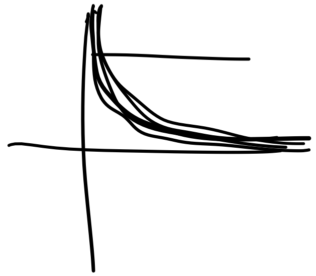
$$\sqrt{(r+a)^2 - a^2e^2} = \sqrt{a^2e^2(\cosh^2\eta - 1)} = \sqrt{a^2e^2\sinh^2\eta} = ae\sinh\eta$$

$$\therefore \begin{cases} t = \sqrt{\frac{m}{2E}}\int \frac{a(e\cosh\eta - 1)\,ae\cancel{\sinh\eta}\,d\eta}{ae\cancel{\sinh\eta}} = \sqrt{\frac{m}{2E}}\,a\left(e\sinh\eta - \eta\right) + C \\ r = a(e\cosh\eta - 1) \end{cases}$$

$$U(r) = \frac{\alpha}{r} \qquad (\alpha > 0)$$

$$E > 0$$





$$\phi = \frac{M}{\sqrt{2m}} \int_{r_{min}}^{r} \frac{dr}{r^2 \sqrt{E - \underbrace{\left(\frac{\alpha}{r} + \frac{M^2}{2mr^2}\right)}_{U_{eff}}}}$$

$$\frac{1}{r} \equiv u \rightarrow -\frac{1}{r^2} dr = du$$

$$= -\frac{M}{\sqrt{2m}} \int \frac{du}{\sqrt{E - \left(\alpha u + \frac{M^2}{2m} u^2\right)}} = -\int \frac{du}{\sqrt{\frac{2mE}{M^2} - \frac{2m\alpha}{M^2} u - u^2}}$$

$$= -\int \frac{A \cancel{\cos} \psi \, d\psi}{\sqrt{A^2 - \cancel{A^2} \sin^2 \psi}} = -(\psi - \psi_0) = -\int \frac{du}{\sqrt{\underbrace{\left(\frac{2mE}{M^2} + \frac{m^2\alpha^2}{M^4}\right)}_{A^2} - \underbrace{\left(u + \frac{m\alpha}{M^2}\right)^2}_{(A \sin\psi)^2}}}$$

$$\frac{e}{p} \equiv A = \sqrt{\frac{2mE}{M^2} + \left(\frac{m\alpha}{M^2}\right)^2} > \frac{m\alpha}{M^2} \equiv \frac{1}{p} \qquad E > 0 \qquad du = A \cos\psi \, d\psi$$

$$\psi_0 - \phi = \psi = \sin^{-1}\left(\frac{1/r + \frac{m\alpha}{M^2}}{A}\right)$$

$$\downarrow$$

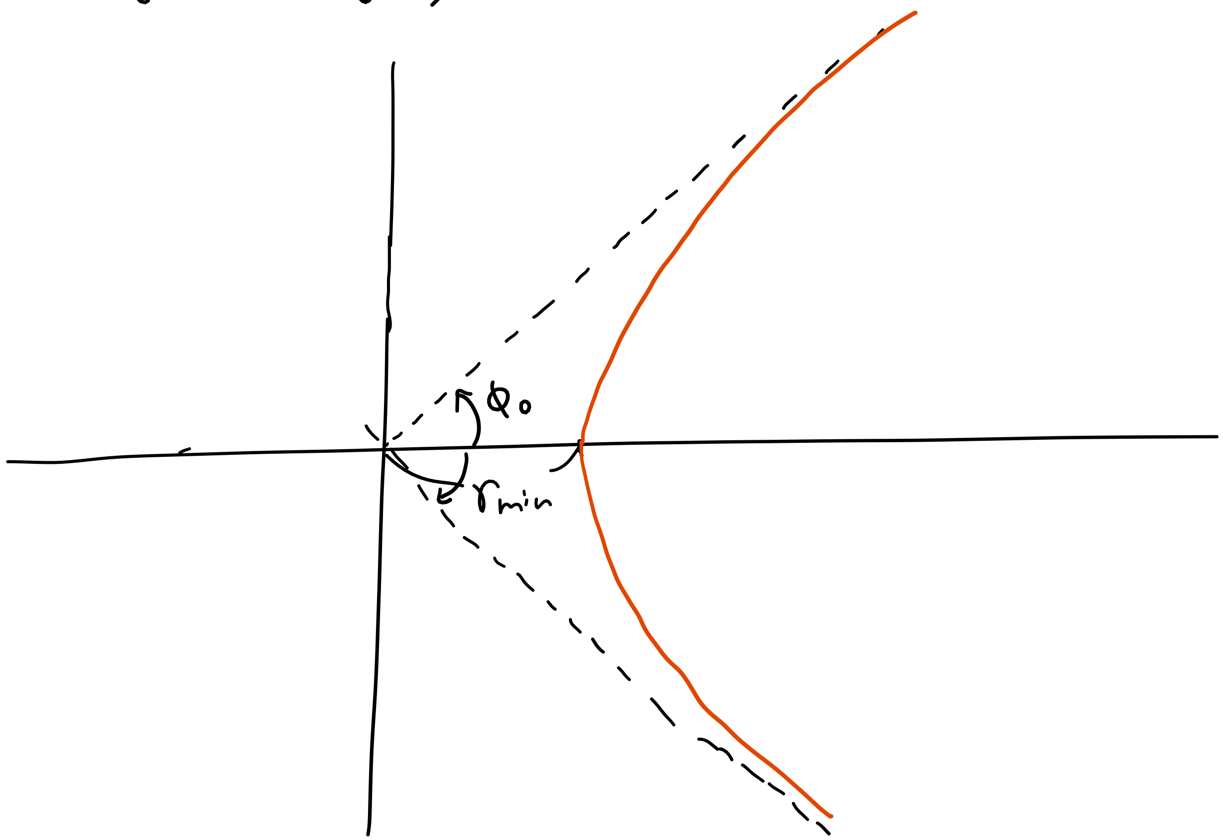
$$\frac{1}{r} = A \sin(\psi_0 - \phi) - \frac{m\alpha}{M^2} \qquad \psi_0 = \frac{\pi}{2}$$

$$= A \cos\phi - \frac{m\alpha}{M^2} \longrightarrow \boxed{r = \frac{p}{e \cos\phi - 1}} \quad (e > 1)$$

$$e\cos\phi_0 = 1 \qquad r = \infty$$

$$\phi_0 = \cos^{-1}\left(\frac{1}{e}\right) < \frac{\pi}{2}$$

$$r_{min} \overset{\phi=0}{=} \frac{p}{e-1} = \frac{a(e^2-1)}{e-1} = a(e+1)$$



$$\underbrace{U = \frac{\alpha}{r}} \quad (\alpha > 0,\ \alpha < 0) \rightarrow \vec{F} = -\vec{\nabla} U = + \frac{\alpha \hat{r}}{r^2} = \frac{\alpha \vec{r}}{r^3}$$

$$\vec{Q} = \vec{v} \times \vec{M} + \alpha \frac{\vec{r}}{r} = \text{const.}$$

$$\vec{M} = m\vec{r} \times \vec{v}$$

$$\downarrow \qquad (\underset{\vec{v}}{\underset{\shortparallel}{\dot{\vec{r}}}} \times \vec{v} = 0)$$

$$\dot{\vec{M}} = m\vec{r} \times \dot{\vec{v}}$$

$$= m\vec{r} \times \vec{a} = \vec{r} \times \underset{\sim}{\vec{F}} = 0$$

$$\vec{F} \propto \vec{r}$$

$$\frac{d\vec{Q}}{dt} = \dot{\vec{v}} \times \underset{m(\vec{r} \times \vec{v})}{\underset{\shortparallel}{\vec{M}}} + \vec{v} \times \underset{\sim}{\overset{0}{\overset{\shortparallel}{\dot{\vec{M}}}}} + \alpha \left( \frac{\dot{\vec{r}}}{r} - \frac{\vec{r}}{r^2} \frac{dr}{dt} \right)$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$$= m \left( \vec{r} \left( \underset{\sim}{\dot{\vec{v}}} \cdot \vec{v} \right) - \vec{v} \left( \underset{\sim}{\dot{\vec{v}}} \cdot \vec{r} \right) \right) + \alpha \left( \frac{\vec{v}}{r} - \frac{\vec{r}}{r^2} \left( \frac{\vec{r} \cdot \vec{v}}{r} \right) \right)$$

$$r = \sqrt{\vec{r} \cdot \vec{r}} \qquad \frac{dr}{dt} = \frac{\cancel{2} \vec{r} \cdot \dot{\vec{r}}}{\cancel{2}\sqrt{\vec{r} \cdot \vec{r}}} = \frac{\vec{r} \cdot \dot{\vec{r}}}{r}$$

$$m\dot{\vec{v}} = m\vec{a} = \vec{F} = \frac{\alpha \vec{r}}{r^3}$$

$$= \vec{r}\, \cancel{\frac{\alpha \vec{r} \cdot \vec{v}}{r^3}} - \vec{v}\, \cancel{\frac{\alpha\, \vec{r} \cdot \vec{r}}{r^3}} \overset{= r^2}{} + \cancel{\alpha \frac{\vec{v}}{r}} - \cancel{\frac{\alpha \vec{r}\, (\vec{r} \cdot \vec{v})}{r^3}} = 0$$