Lagrangian

$\downarrow$

time

t

$\ddot{x} = \dfrac{d^2 x(t)}{dt^2}$

Euler-Lag. Eqs

(Eq. of Motion) $q_a(t)$ $a = 1 \cdots n$

2nd-order differential Eqs; $\ddot{q}_a + f(\dot{q}_b, q_c) = 0$

very difficult to solve analytically

$\updownarrow$

numerically

(cf) field theory

$q_{\vec{r}}(t)$

$\vec{r}$ ← continuos

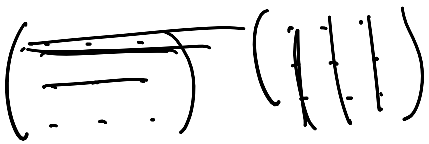
Install MATLAB in your own computer

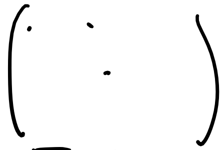
(Borrow CD from software CH92)

MATLAB

행렬연산 (Matrix operation)

A , B ; matrix

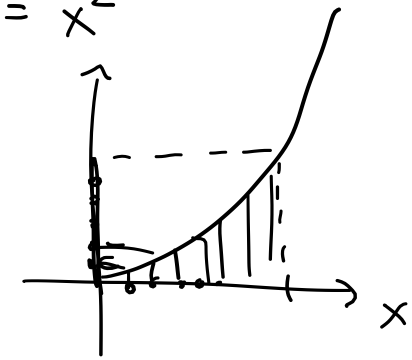


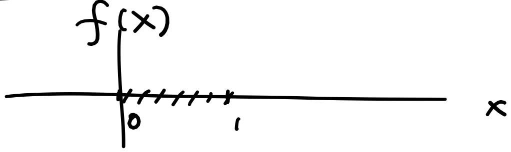
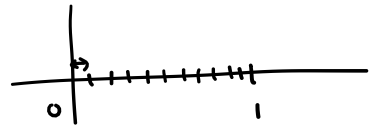
AB = 

Fortran ...

matrix ⊃ scalar, vector,

$y = x^2$



연속인 구간  → 

(0, 0.1, 0.2, ..., 1)

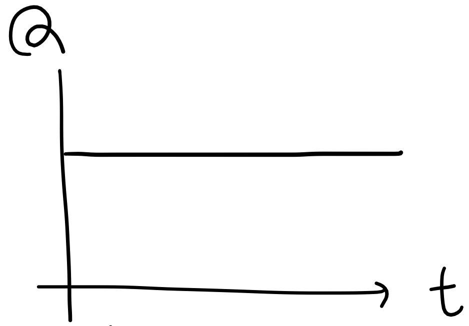
f : 함수(function) → matrix

x → f(x)

↓ ↓

vector vector

Chapt. II. Conservation Laws



· Symmetry / Invariance : no change in L under some transformation

↓

Conservation

1. homogeneity of time.

implicit $F(q(t))$ → $\frac{\partial F}{\partial t} = 0$

(ex) $= q^2 - 3q$

$$\frac{dF}{dt} \neq 0 = \frac{dF}{dq}\frac{dq}{dt} = (2q-3)\dot{q}$$

explicit (ex) $q^2 - 3tq$

partial diff.

$$\frac{\partial F}{\partial t} = -3q$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{dF}{dq}\frac{dq}{dt} = -3q + (2q-3t)\dot{q}$$

If L satisfies $\underset{\sim\sim\sim}{\frac{\partial L}{\partial t} = 0}$

(ex) $L = \frac{m}{2}\dot{q}^2 - \frac{1}{2}kq^2$

dyn. variable ~ given (no dynamical)

$\underset{\sim\sim\sim}{k = k_0 \rightarrow \frac{\partial L}{\partial t} = 0}$

(ex) $k = k_0 + k_1 t^2 \rightarrow \frac{\partial L}{\partial t} \neq 0$

$t \rightarrow t + a \qquad L = L(q, \dot{q})$

(시점에 무관)

$$\frac{dL}{dt} = \cancel{\frac{\partial L}{\partial t}}^{\,0} + \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}$$

(cf) $F(x(t), y(t))$

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\dot{x} + \frac{\partial F}{\partial y}\dot{y}$$

Euler-Lag. $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\underbrace{\ddot{q}}_{\frac{d\dot{q}}{dt}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right)$$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q} - L\right) = 0$$

$$\therefore H \equiv \frac{\partial L}{\partial \dot{q}} \dot{q} - L = \text{conserved.}$$

$\uparrow$ Energy

$$\frac{\partial L}{\partial t} = 0 \longrightarrow \frac{dH}{dt} = 0 \qquad H = \text{conserved} = \text{Energy}$$

If $\quad L = L(q_i, \dot{q}_i)$

$$\frac{dL}{dt} = \cancel{\frac{\partial L}{\partial t}}^{0} + \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i$$

Euler-Lag. $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$

$$= \sum_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \underbrace{\ddot{q}_i}_{\frac{d\dot{q}_i}{dt}} \right] = \frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i\right)$$

$$\therefore \frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L\right) = 0$$

(Ex) $L = T - U = \underbrace{\sum_{a=1}^{n} \frac{1}{2} m_a \vec{v}_a^{\,2} - U(\vec{r}_1, \cdots, \vec{r}_n)}$

$\vec{v}_a \equiv \frac{d\vec{r}_a}{dt} = \dot{\vec{r}}_a$

$= L(\dot{\vec{r}}_a, \vec{r}_a)$

$$\frac{\partial L}{\partial \dot{\vec{r}}_a} = m_a \vec{v}_a \qquad \sum_a \dot{\vec{r}}_a (= \vec{v}_a) \underbrace{\frac{\partial L}{\partial \dot{\vec{r}}_a}}_{m_a \vec{v}_a} = \sum_a m_a \vec{v}_a^{\,2}$$

$$H = \sum_a m_a \vec{v}_a^{\,2} - \left(\sum_a \frac{1}{2} m_a \vec{v}_a^{\,2} - U \right) = \underbrace{\frac{1}{2} \sum_a m_a \vec{v}_a^{\,2}}_{T} + \underbrace{U}_{U}$$

---

homogeneity of space

if $\vec{r}_a \to \vec{r}_a + \vec{\epsilon}$ ← const does not change L ; momentum conservation

$$L = L(\dot{\vec{r}}_a, \vec{r}_a)$$

$$\delta L = L\Big(\underbrace{(\vec{r}_a + \vec{\epsilon})^{\cdot}}_{= \dot{\vec{r}}_a}, \underbrace{\vec{r}_a + \vec{\epsilon}}\Big) - L(\dot{\vec{r}}_a, \vec{r}_a)$$

$$= \sum_a \vec{\epsilon} \cdot \frac{\partial L}{\partial \vec{r}_a} = 0 \to \text{If } \underline{\sum_a \frac{\partial L}{\partial \vec{r}_a} = 0}$$

$U = \frac{1}{2} \vec{r}_1^{\,2}$ $\quad \vec{r}_1 \to \vec{r}_1 + \vec{\epsilon}$

$U \to \frac{1}{2}(\vec{r}_1 + \vec{\epsilon})^2$

$= \frac{1}{2} \vec{r}_1^{\,2} + \underbrace{\vec{r}_1 \cdot \vec{\epsilon} + \frac{1}{2} \vec{\epsilon}^{\,2}}_{\neq 0}$

$f(x+\epsilon) - f(x) = \epsilon \frac{df}{dx}$ small ϵ

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec r}_a} - \frac{\partial L}{\partial \vec r_a} = 0 \longrightarrow \sum_a \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec r}_a} - \underbrace{\sum_a \frac{\partial L}{\partial \vec r_a}}_{=0} = 0$$

$$T = \sum_a \frac{1}{2} m_a \dot{\vec r}_a^{\,2}$$

$$\underbrace{\frac{\partial L}{\partial \dot{\vec r}_a}} = m_a \dot{\vec r}_a = \underbrace{m_a \vec v_a} = \underbrace{\vec p_a}$$

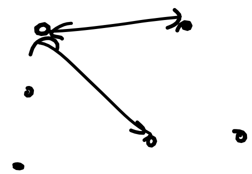
$$\therefore \frac{d}{dt}\left(\sum_a \frac{\partial L}{\partial \dot{\vec r}_a}\right) = 0$$

$$\frac{d}{dt}\left(\sum_a \vec p_a\right) = 0$$

total momentum $\longrightarrow$ $\therefore \sum_a \vec p_a$ = conserved.

---

① $\frac{\partial L}{\partial t} = 0$ (many cases) $\rightarrow$ E conserved



② $\sum_a \frac{\partial L}{\partial \vec r_a} = 0$ (")

$\updownarrow$

$\underbrace{\vec r_a \Rightarrow \vec r_a + \vec \epsilon}$

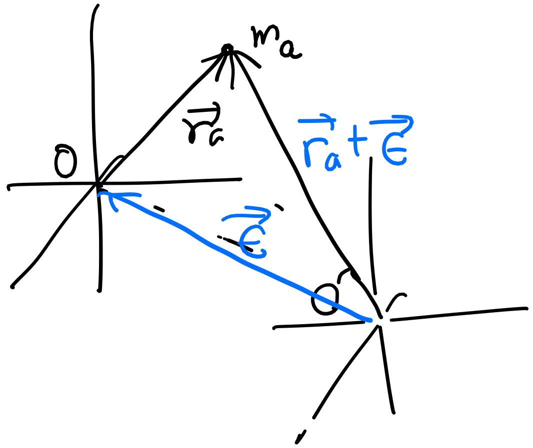
(ex) grav. potential $U = \sum_{a,b} G \frac{m_a m_b}{|\vec r_a - \vec r_b|}$

$\vec r_a - \vec r_b \Rightarrow (\vec r_a + \cancel{\vec \epsilon}) - (\vec r_b + \cancel{\vec \epsilon}) = \vec r_a - \vec r_b$

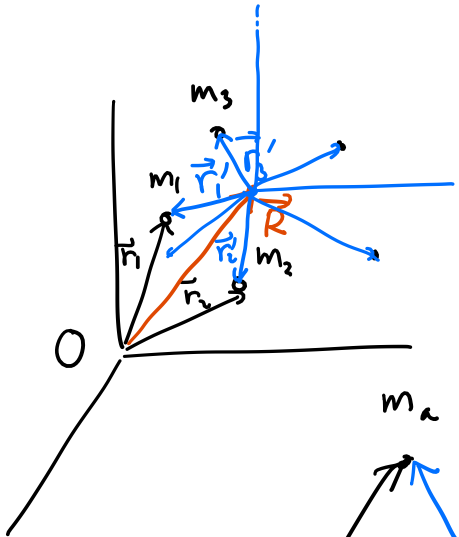
$U = \sum_a \frac{1}{2} k \vec r_a^{\,2}$

Center of Mass

$\vec{r}_a \to \vec{r}_a + \vec{\epsilon}$; origin



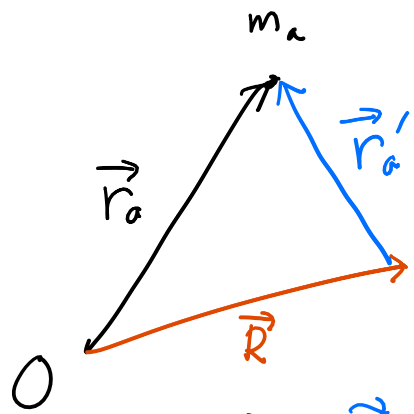
$$\vec{R} = \frac{\sum_a m_a \vec{r}_a}{\sum_a m_a} \quad : \text{ COM}$$



$$\vec{r}_a = \vec{r}_a{}' + \vec{R}$$
$$\vec{v}_a = \vec{v}_a{}' + \vec{V}$$

$$\vec{P} = \sum_a m_a \vec{v}_a = \sum_a m_a (\vec{v}_a{}' + \vec{V})$$

$$= \underbrace{\sum_a m_a \vec{v}_a{}'}_{= \vec{P}'} + \underbrace{\left(\sum_a m_a\right)}_{\mu} \vec{V}$$

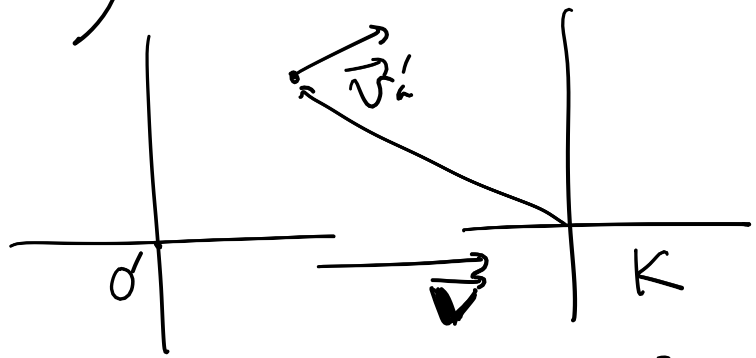


$$\vec{r}_a = \vec{r}_a{}' + \vec{R}$$
$$\vec{v}_a = \vec{v}_a{}' + \dot{\vec{R}} \equiv \vec{V}$$

In COM: $\underbrace{\vec{P}' = 0}$ If not, we can always choose a frame K to make $\vec{P}' = 0$.

moving with constant velocity

(If NOT $\sum_a m_a \vec{v}_a' \neq 0$)



$$\vec{v}_a^{(K)} = \vec{v}_a' - \vec{V}$$

$$\sum_a m_a \vec{v}_a^{(K)} = \sum_a m_a \vec{v}_a' - \sum_a m_a \vec{V} = 0$$

$$\vec{V} = \frac{\sum_a m_a \vec{v}_a'}{\sum_a m_a} = \frac{\vec{P}'}{\sum_a m_a}$$

$$\vec{P} = \mu \vec{V}$$

Any motion can be separated into

$\begin{cases} \text{motion of CM} \\ \quad\text{"} \quad \text{of relative motion} \end{cases}$

$$T = \frac{1}{2}\sum_a m_a \vec{v}_a^{\,2}$$

$$\vec{v}_a = \vec{v}_a{}' + \vec{V}$$

$$= \frac{1}{2}\sum_a m_a \left(\vec{v}_a{}' + \vec{V}\right)^2 = \frac{1}{2}\sum_a m_a \vec{v}_a{}'^{\,2} + \sum_a m_a \vec{v}_a{}' \cdot \vec{V} + \frac{1}{2}\sum_a m_a \vec{V}^2$$

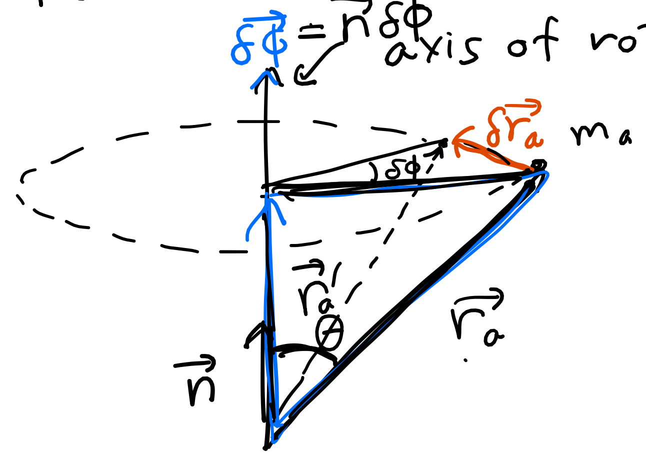
$$= \overbrace{\frac{1}{2}\sum_a m_a \vec{v}_a{}'^{\,2}}^{T'} + \vec{V}\cdot\underbrace{\left(\sum_a m_a \vec{v}_a{}'\right)}_{=0} + \frac{1}{2}\underbrace{\left(\sum_a m_a\right)}_{\mu}\vec{V}^2$$

$$T = \underbrace{\frac{1}{2}\mu \vec{V}^2}_{\text{K.E of CM}} + \underbrace{T'}_{\text{K.E of Relative to CM.}}$$

$$\vec{P} = \vec{p}\,' + \mu\vec{V}$$

③ Rotational Invariance

$$\vec{r}_a \to \vec{r}_a{}' = \vec{r}_a + \delta\vec{r}_a$$



$\delta\phi \ll 1$

$\delta\vec{r}_a \perp \vec{r}_a,\ \delta\vec{\phi}$

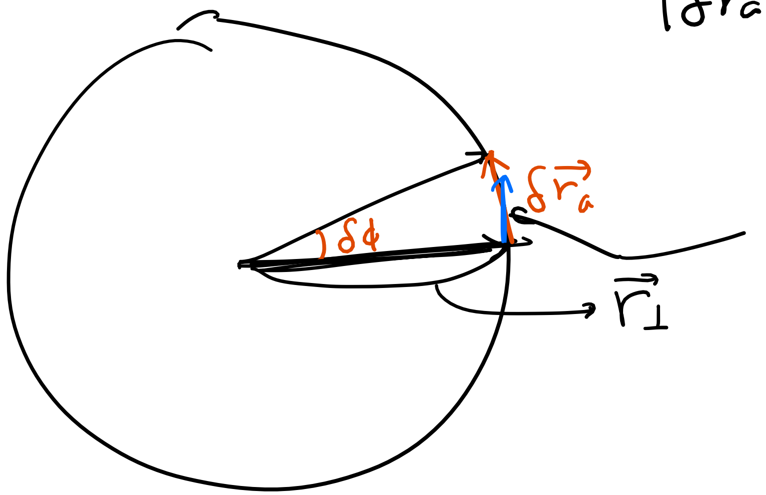
$\Downarrow$

$\delta\vec{r}_a \propto \vec{r}_a \times \delta\vec{\phi}$

$\Downarrow$

$$|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\sin\theta$$

$$|\delta\vec{r}_a| = \underbrace{|\vec{r}_a|\sin\theta}\,\delta\phi$$
$$= |\vec{n}\times\vec{r}_a|\,\delta\phi = |\delta\vec{\phi}\times\vec{r}_a|$$



$|\vec{r}_\perp|\,\delta\phi$

$\shortparallel$

$|\vec{r}_a|\sin\theta$

$$\therefore\ \delta\vec{r}_a = \underbrace{\delta\vec{\phi}}_{=\vec{n}\,\delta\phi}\times\vec{r}_a$$

$$\delta \vec{r}_a = \delta \vec{\phi} \times \vec{r}_a$$

$$\downarrow$$

$$\delta \vec{v}_a = \delta \vec{\phi} \times \vec{v}_a$$

$$L = L(\vec{v}_a, \vec{r}_a) \longrightarrow L(\vec{v}_a{}', \vec{r}_a{}') = L(\vec{v}_a + \delta\vec{v}_a, \vec{r}_a + \delta\vec{r}_a)$$

$$= L(\vec{v}_a, \vec{r}_a) + \underbrace{\sum_a \delta\vec{v}_a \cdot \frac{\partial L}{\partial \vec{v}_a} + \sum_a \delta\vec{r}_a \cdot \frac{\partial L}{\partial \vec{r}_a}}_{\delta L}$$

$$f(x+\epsilon_1, y+\epsilon_2)$$

$$= f(x,y) + \epsilon_1 \frac{\partial f}{\partial x} + \epsilon_2 \frac{\partial f}{\partial y} + \cdots$$

Invariance : $\delta L = 0$

$$\delta L = \sum_a \delta\vec{v}_a \cdot \underbrace{\frac{\partial L}{\partial \vec{v}_a}}_{\frac{\partial L}{\partial \dot{\vec{r}}_a}} + \sum_a \delta\vec{r}_a \cdot \underbrace{\frac{\partial L}{\partial \vec{r}_a}}_{= \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}_a}} = 0 = \sum_a \Big(\underbrace{\delta \vec{v}_a}_{= \frac{d}{dt}(\delta \vec{r}_a)} \cdot \frac{\partial L}{\partial \dot{\vec{r}}_a} + \delta\vec{r}_a \frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{r}}_a} \Big)$$

$$= \frac{d}{dt}\left(\sum_a \delta\vec{r}_a \cdot \frac{\partial L}{\partial \dot{\vec{r}}_a} \right)$$

$$\therefore \sum_a \delta\vec{r}_a \cdot \frac{\partial L}{\partial \dot{\vec{r}}_a} = \text{conserved} = \text{angular mom.}$$

$$\sum_a \underbrace{\delta\vec{r}_a}\cdot\frac{\partial L}{\partial\dot{\vec{r}}_a} = \sum_a \vec{p}_a\cdot(\delta\vec{\phi}\times\vec{r}_a) = \sum_a \underbrace{\delta\vec{\phi}}\cdot(\vec{r}_a\times\vec{p}_a) = \delta\vec{\phi}\cdot\sum_a \vec{r}_a\times\vec{p}_a$$

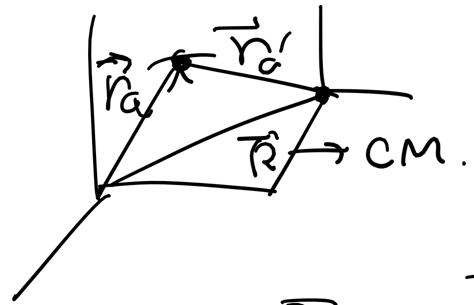
$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{C}\times\vec{A}) \qquad \sum_a \vec{r}_a\times\vec{p}_a = \text{const.}$$

$$\therefore \sum_a \vec{r}_a\times\vec{p}_a = \text{conserved.} = \text{total angular mom.}$$

$$\vec{r}_a\times\vec{p}_a = m_a\,\vec{r}_a\times\vec{v}_a = \overset{m_a\text{의}}{\text{ang. mom.}}$$

transformation	L is inv.	conserved quantity
$t\to t+a$	$\delta L = 0 = \frac{d}{dt}\left(\sum_a \dot{r}_a\cdot\frac{\partial L}{\partial\dot{\vec{r}}_a} - L\right)$	$H = \sum_a \dot{\vec{r}}_a\cdot\frac{\partial L}{\partial\dot{\vec{r}}_a} - L$ Energy $\vec{r}_a$, q_i
$\vec{r}_a\to\vec{r}_a+\vec{\epsilon}$	$\delta L = 0 = \vec{\epsilon}\cdot\frac{d}{dt}\left(\sum_a\frac{\partial L}{\partial\dot{\vec{r}}_a}\right)$	$\frac{\partial L}{\partial\dot{\vec{r}}_a}\equiv\vec{p}_a \to \sum_a\vec{p}_a=\vec{P}=$ conserved total momentum.
$\vec{r}_a\to\vec{r}_a+\delta\vec{r}_a(=\delta\vec{\phi}\times\vec{r}_a)$	$\delta L = 0 = \delta\vec{\phi}\cdot\frac{d}{dt}\left(\sum_a\vec{r}_a\times\vec{p}_a\right)$	$\therefore \sum_a\vec{r}_a\times\vec{p}_a =$ conserved $=$ total Ang. Mom.

$$\vec{M} = \sum_a \vec{r}_a \times \vec{p}_a = \sum_a (\vec{r}_a{}' + \vec{R}) \times (m_a \vec{v}_a)$$



$$\underbrace{\vec{r}_a = \vec{r}_a{}' + \vec{R}}$$

$$\dot{\vec{r}}_a = \vec{v}_a = \underset{\vec{v}_a{}'}{\underset{\parallel}{\dot{\vec{r}}_a{}'}} + \underset{\vec{V}}{\underset{\parallel}{\dot{\vec{R}}}} = \vec{v}_a{}' + \vec{V}$$

$$\sum_a m_a \vec{r}_a{}' \times \vec{v}_a{}' = \vec{M}' + \vec{R} \times (\mu \vec{V})$$

$$\vec{M} = \sum_a m_a (\vec{r}_a{}' + \vec{R}) \times (\vec{v}_a{}' + \vec{V}) = \sum_a m_a \Big(\vec{r}_a{}' \times \vec{v}_a{}' + \cancel{\vec{R} \times \vec{v}_a{}'} + \cancel{\vec{r}_a{}' \times \vec{V}} + \vec{R} \times \vec{V} \Big)$$

$$\left.\begin{array}{ll} \text{2nd term} ; & \vec{R} \times \underbrace{\sum_a m_a \vec{v}_a{}'} \\ \text{3rd} \quad " ; & \underbrace{\Big(\sum_a m_a \vec{r}_a{}'\Big)}_{=0}{}^{\frac{d}{dt}} \times \vec{V} \end{array}\right\} = 0$$

$$\sum_a m_a \vec{r}_a{}' = 0 \overset{\text{why?}}{=} \sum_a m_a (\vec{r}_a - \vec{R})$$

$$= \underbrace{\Big(\sum_a m_a \vec{r}_a\Big)}_{= \sum_a m_a \vec{R}} - \Big(\sum_a m_a\Big) \vec{R} = 0$$

$$\vec{R} = \frac{\sum_a m_a \vec{r}_a}{\sum_a m_a}$$

$$\vec{M} = \underbrace{\vec{M}'} + \underbrace{\vec{R} \times \vec{P}}$$

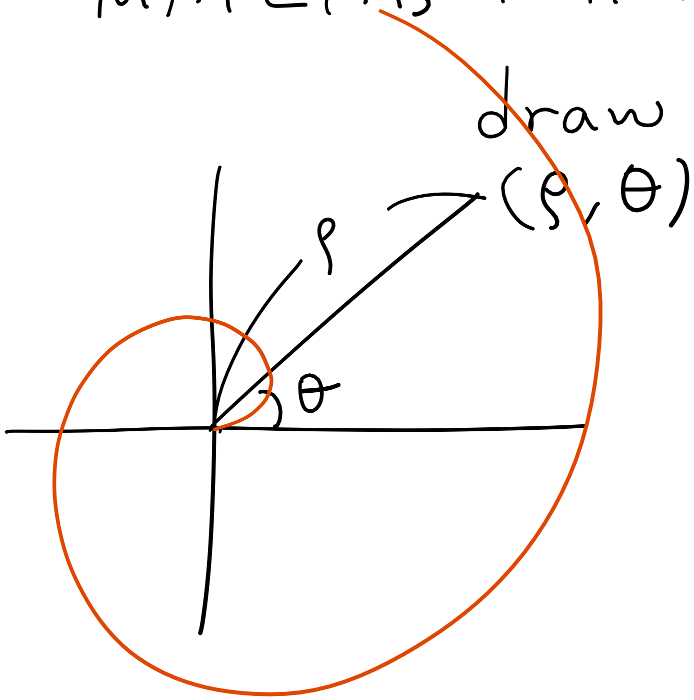
$$(\vec{P} = \mu \vec{V})$$

H.W. # 1, 2, 3 on p. 21

+

MATLAB : H.W.

draw polar coordinate



(x, y)

$$(\rho(\theta), \theta)$$

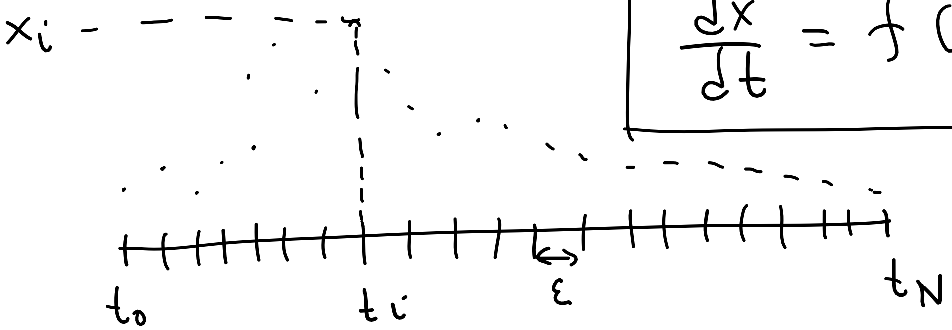
$$\rho(\theta) = \theta^2$$

1st order Ordinary Differential Eq in MATLAB

(상미방)

$$\frac{dx}{dt} = f(x) \rightarrow \int \frac{dx}{f(x)} = \int dt = t$$

$$\boxed{\frac{dx}{dt} = f(x, t)} \not\rightarrow$$



$$\dashrightarrow \left(\frac{dx}{dt}\right)_i = \frac{x_{i+1} - x_i}{\varepsilon}$$

$x(t_0) = x_0$

$$x_{i+1} = x_i + \varepsilon f(x, t)$$

ode45

Lorenz ($\neq$ Lorentz)

$$x(0) = 1.00000121\boxed{\,\,}$$

10^{-9}

$$x(10^3) = \text{-----} \quad \boxed{10^{-9}}$$

Chaos

sensitive dependen on initial conditin

nonlinear eq of motion

x_1, x_2, x_3

$$\frac{dx_1}{dt} = \sigma(x_2 - x_1)$$

$$\frac{dx_2}{dt} = x_1(\rho - x_3) - x_2$$

$$\frac{dx_3}{dt} = x_1 x_2 - \beta x_3$$

$\sigma = 10$

$\rho = 28$

$\beta = \frac{8}{3}$

2nd order DE.

$$\frac{d^2x}{dt^2} = f(t, x, \dot{x})$$

(ex) forced damped HO

$$m\ddot{x} + kx + \gamma\dot{x} = A\cos\omega t$$

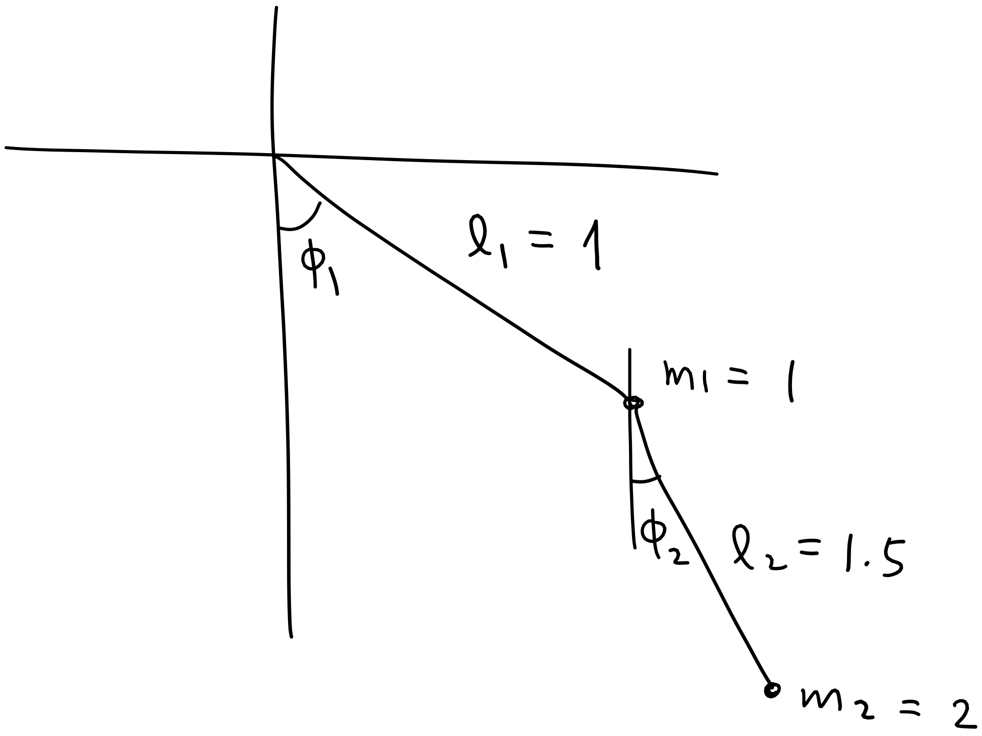
$$+ \beta x^3$$

$x_1 = x$

$x_2 = \dot{x} = \dot{x}_1$

$$\rightarrow \quad \frac{d^2x}{dt^2} = \frac{d}{dx}\Big(\overbrace{\frac{dx}{dt}}^{x_2}\Big) = \boxed{\begin{aligned} \frac{dx_2}{dt} &= f(t, x_1, \overset{\dot{x}}{x_2}) \\ \frac{dx_1}{dt} &= x_2 \end{aligned}}$$

$\Rightarrow$ ode45



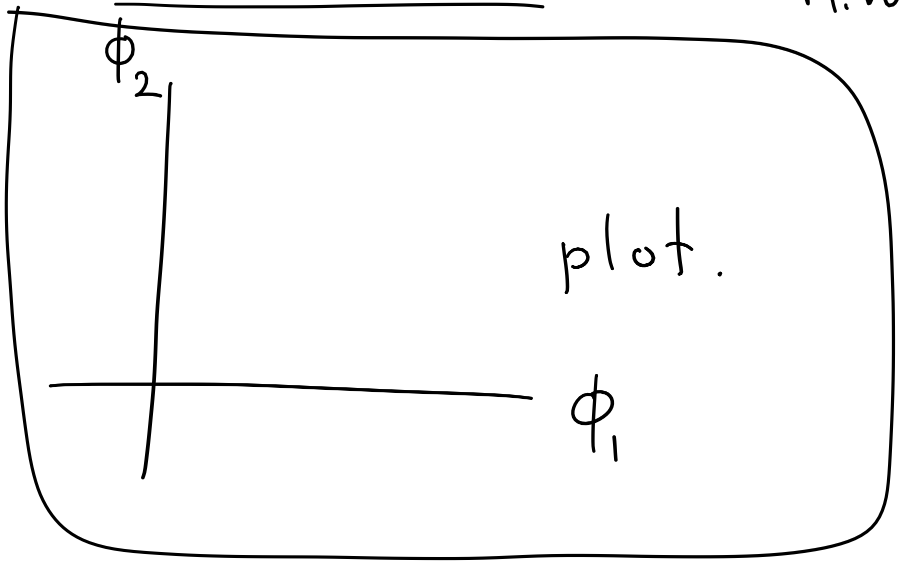
Prob 1 on p.11

$t = 0 \sim 2\pi$

H.W.

$\dot\phi_1(0) = \dot\phi_2(0) = 0$

$\phi_1(0) = \frac{\pi}{4}$, $\phi_2(0) = \frac{\pi}{3}$



Hint.

$\ddot\phi_1 = f_1(\phi_1, \phi_2)$ $\quad \psi_1 = \phi_1$, $\psi_2 = \dot\phi_1$

$\ddot\phi_2 = f_2(\phi_1, \phi_2)$ $\quad \psi_3 = \phi_2$, $\psi_4 = \dot\phi_2$

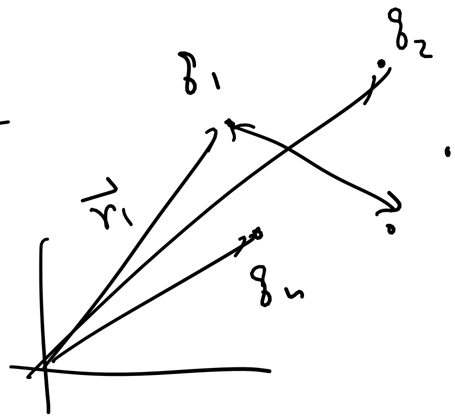
Mechanical Similarity

(ex) $U = -\frac{c}{r}$, $U = \frac{1}{2} c r^2$, ...

$U(\alpha \vec{r}) = \alpha^k U(\vec{r})$ $k = -1, 2$

$U(\vec{r}_1, \vec{r}_2, \cdots \vec{r}_n) \Rightarrow \boxed{U(\alpha \vec{r}_1, \alpha \vec{r}_2, \cdots, \alpha \vec{r}_n) = \alpha^k U(\vec{r}_1, \cdots \vec{r}_n)}$

(ex)



$U = U(\vec{r}_1, \cdots, \vec{r}_n) = \sum_{i>j} \frac{k q_i q_j}{|\vec{r}_i - \vec{r}_j|} \rightarrow k = -1$

$\vec{r}_i \rightarrow \alpha \vec{r}_i$ & $t \rightarrow \beta t$ $\rightarrow \vec{v}_i = \frac{d\vec{r}_i}{dt} \Rightarrow \frac{\alpha}{\beta} \vec{v}_i$

$T = \sum_i \frac{1}{2} m_i \vec{v}_i^{\,2} \longrightarrow \frac{\alpha^2}{\beta^2} T$

$L = T - U$

$\Rightarrow \frac{\alpha^2}{\beta^2} T - \alpha^k U \overset{\text{If}}{=} \frac{\alpha^2}{\beta^2} (T - U) = \frac{\alpha^2}{\beta^2} \cdot L$

E-L. $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\vec r}_i} - \dfrac{\partial L}{\partial \vec r_i} = 0$

$$c\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec r}_i} - \frac{\partial L}{\partial \vec r_i}\right) = 0$$

$$\frac{d}{dt}\frac{\partial (cL)}{\partial \dot{\vec r}_i} - \frac{\partial (cL)}{\partial \vec r_i} = 0 \quad \longrightarrow \quad L \equiv c \cdot L$$

$$L \to L + c$$

$$L\,,\quad L + c_1\,,\quad c_2(L + c_1)\ \ldots$$

If $\dfrac{\alpha^2}{\beta^2} = \alpha^k \longrightarrow$ same eq of motion $\vec r \to \underset{\vec r\,'}{\alpha \vec r}$

$\beta = \alpha^{1-\frac{k}{2}}$

$t \to \underset{t'}{\beta t} = \alpha^{1-\frac{k}{2}}\, t$

$|\vec r\,'| \equiv l'$

$|\vec r| \equiv l$

$$\underbrace{\frac{t'}{t} = \alpha^{1-\frac{k}{2}} = \left(\frac{l'}{l}\right)^{1-\frac{k}{2}}}$$

$\dfrac{l'}{l} = \alpha$

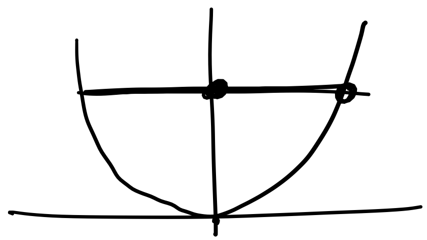
$$v = \frac{l}{t} \longrightarrow \frac{v'}{v} = \frac{l'/t'}{l/t} = \frac{l'}{l}\cdot\left(\frac{t}{t'}\right) = \frac{l'}{l}\left(\frac{l'}{l}\right)^{\frac{k}{2}-1} = \left(\frac{l'}{l}\right)^{\frac{k}{2}}$$

$$E = \frac{1}{2}mv^2 \longrightarrow \frac{E'}{E} = \frac{v'^2}{v^2} = \left(\frac{l'}{l}\right)^{k}$$

$$M = mvl \longrightarrow \frac{M'}{M} = \frac{v'l'}{vl} = \left(\frac{l'}{l}\right)^{1+\frac{k}{2}}$$

(ex) H.O. $k=2$, Coulomb $k=-1$ (Gravitational) $\quad \frac{t'}{t} = \left(\frac{l'}{l}\right)^{\frac{3}{2}} \rightarrow$ kepler 3rd.

Virial theorem : 평균 T 와 평균 U 의 관계



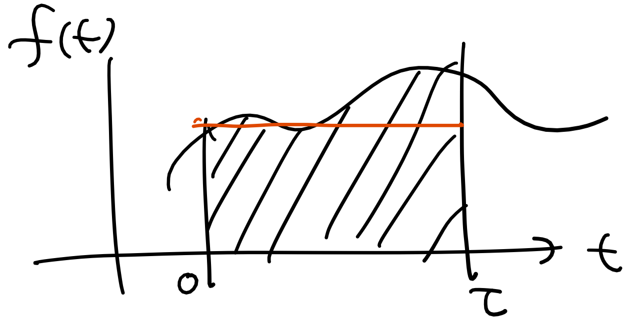
Kinetic Energy ;

$$2T = \sum_a m_a \vec{v}_a = \sum_a \vec{p}_a \cdot \underbrace{\vec{v}_a}_{\frac{d\vec{r}_a}{dt}} = \underbrace{\frac{d}{dt}\left(\sum_a \vec{p}_a \cdot \vec{r}_a\right)} - \sum_a \dot{\vec{p}}_a \cdot \vec{r}_a$$

$$\dot{\vec{p}}_a = m_a \dot{\vec{v}}_a = \vec{F}_a = -\frac{\partial U}{\partial \vec{r}_a}$$

time average

$$\overline{f(t)} = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau f(t)\, dt$$



$$\overline{\frac{dg(t)}{dt}} = \underline{\underline{0}} = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \left(\frac{dg}{dt}\right) dt = \lim_{\tau \to \infty} \frac{1}{\tau}\left(\underbrace{g(\tau) - g(0)}_{\text{finite}}\right)$$

$$2\overline{T} = 0 + \sum_a \overline{\frac{\partial U}{\partial \vec{r}_a} \cdot \vec{r}_a} \longrightarrow kU = k\overline{U}$$

(ex) $\underline{U = \sum_a c\, \vec{r}_a^{\;2 \to k}}$ $\longrightarrow$ $\frac{\partial U}{\partial \vec{r}_a} = \overset{\to k}{2}\, c\, \vec{r}_a^{\;k-1}$

$$\sum_a \frac{\partial U}{\partial \vec{r}_a} \cdot \vec{r}_a = \underset{k}{2}c \sum_a r_a^{2 \to k} = \underset{k}{2}U$$

$$\bar{T} = \frac{k}{2}\bar{U} \longrightarrow \bar{E} = \bar{T} + \bar{U} = \left(1 + \frac{k}{2}\right)\bar{U}$$

$$\bar{E} = E$$

$$\longrightarrow \bar{U} = \frac{E}{1+\frac{k}{2}}$$

const.

$$\bar{T} = \frac{\frac{k}{2}E}{1+\frac{k}{2}}$$

$$\bar{T} = \frac{E}{1+\frac{2}{k}}$$

(ex) $k=2 \rightarrow \bar{T} = \bar{U} = \frac{E}{2}$

$k=-1 \rightarrow \bar{U} = 2E, \quad \bar{T} = -E$

$\bar{T} \geq 0 \rightarrow \underline{E<0}$