Classical Mechanics

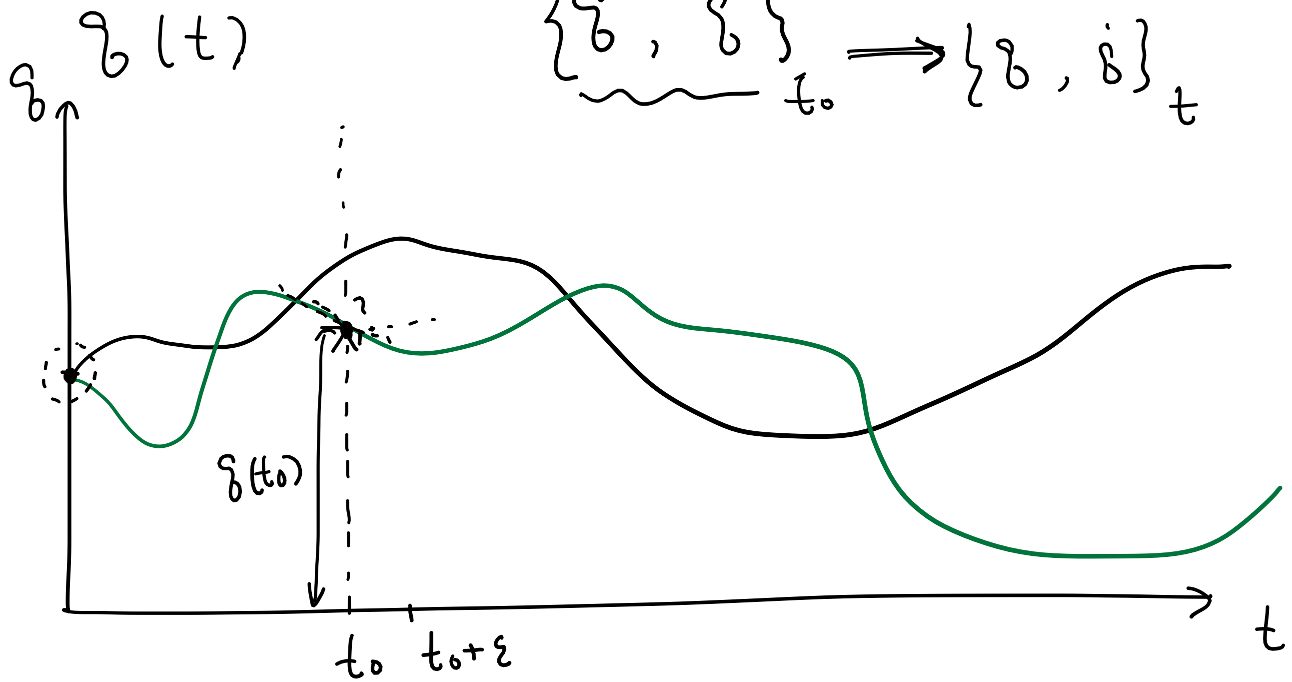
text book: Landau + Lifshitz

Chapter 1.

	mechanics (discrete d.o.f)	field (continuous d.o.f.)
Classic	✓ $q_j(t)$	Electrodynamics (ex) $\vec{E}(\vec{r},t) \to E_{\vec{r}}(t)$
quantum	Q.M.	quantum field theory

Classic vs. Quantum

$$\underbrace{\{q,\ \dot q\}}_{t_0} \Longrightarrow \{q,\ \dot q\}_t \qquad\qquad \{q,\ \dot q\}_{t_0} \ \times$$



$$\dot q = \left.\frac{dq}{dt}\right|_{t_0} = \dot q(t_0)$$

$$\ddot q(t_0)$$

$$\vdots$$

$$\{\ -----\ \}_{t_0+\epsilon} = F\left(\left\{ \boxed{q(t_0),\ \dot q(t_0)},\ \ddot q(t_0), \ldots\ \cancel{q^{(n)}(t_0), \ldots}\ \right\}\right)$$

$$n = 0 \ldots \infty$$

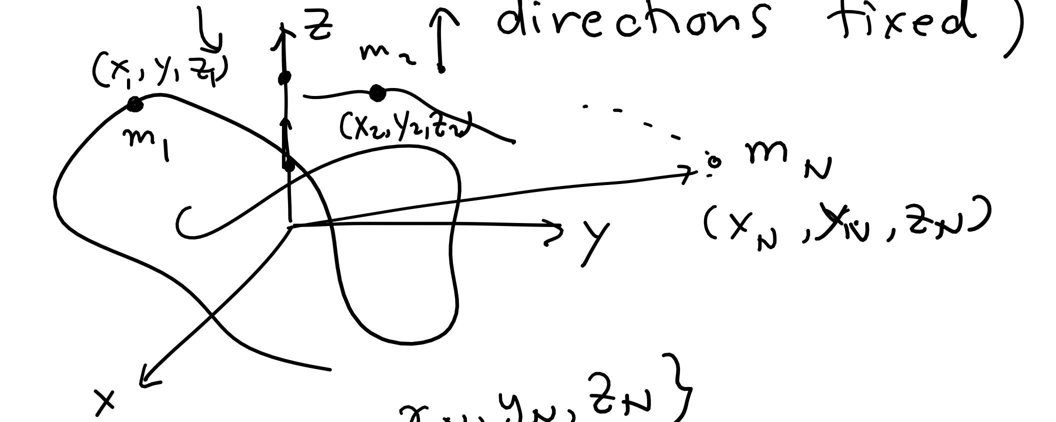
$$\Longrightarrow$$

dis. d.o.f : $q_j(t)$ $j=1, \cdots, n$

d.o.f. 자유도

generalized coordinates

Cartesian (orthogonal 3 axis directions fixed)



$q_j(t) = \{x_1, y_1, z_1 \cdots x_N, y_N, z_N\}$ 3N coordinates

$j = 1, \cdots, 3N$

$\{q_j(t_0), \dot{q}_j(t_0)\} \longmapsto \{q_j(t), \dot{q}_j(t)\}$

initial conditions

$\boxed{\text{Physics}}$ Principle of least action

Laws of physics

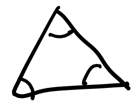
...

Mathematics

$\boxed{\text{Axiom}}$ (self obvious)

(ex) Euclidean geometry

Theorem (needs proof)



180°

Corollary, ...

Action

Laglangian

$$L(q_j, \dot{q}_j, \cancel{\ddot{q}_j}) \left(= \underset{\substack{\uparrow \\ \text{Kinetic}}}{T} - \underset{\substack{\uparrow \\ \text{potential}}}{V} \right)$$

Energies

q_j: $\{q_1, \cdots, q_n\}$

action

$$S = \int_{t_1}^{t_2} L(q_j, \dot{q}_j)\, dt$$

functional vs function

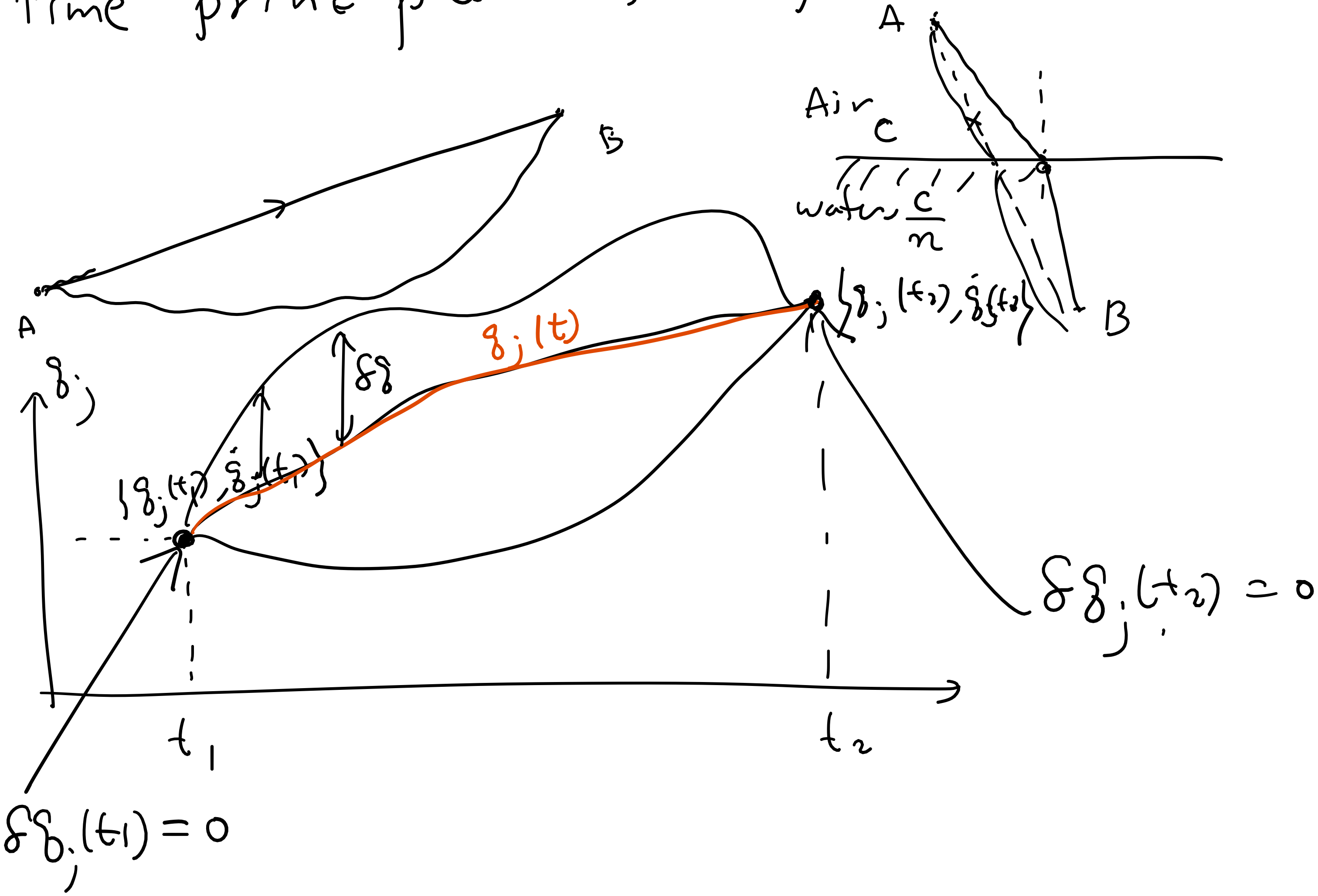
functional	function
$F(\underbrace{x(t)}_{\text{variable}})$	$f(x, y)$
$F(x(0), x(0.0001), \cdots)$	$\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \cdots$

$$\frac{\partial F}{\partial x(t)} \Leftarrow \frac{\partial F}{\partial x(0.0001)}, \frac{\partial F}{\partial x(0.0002)}$$

least action principle = Hamilton's principle

minimize S $\longleftrightarrow$ $q_j(t)$

(cf) "least time principle of light"

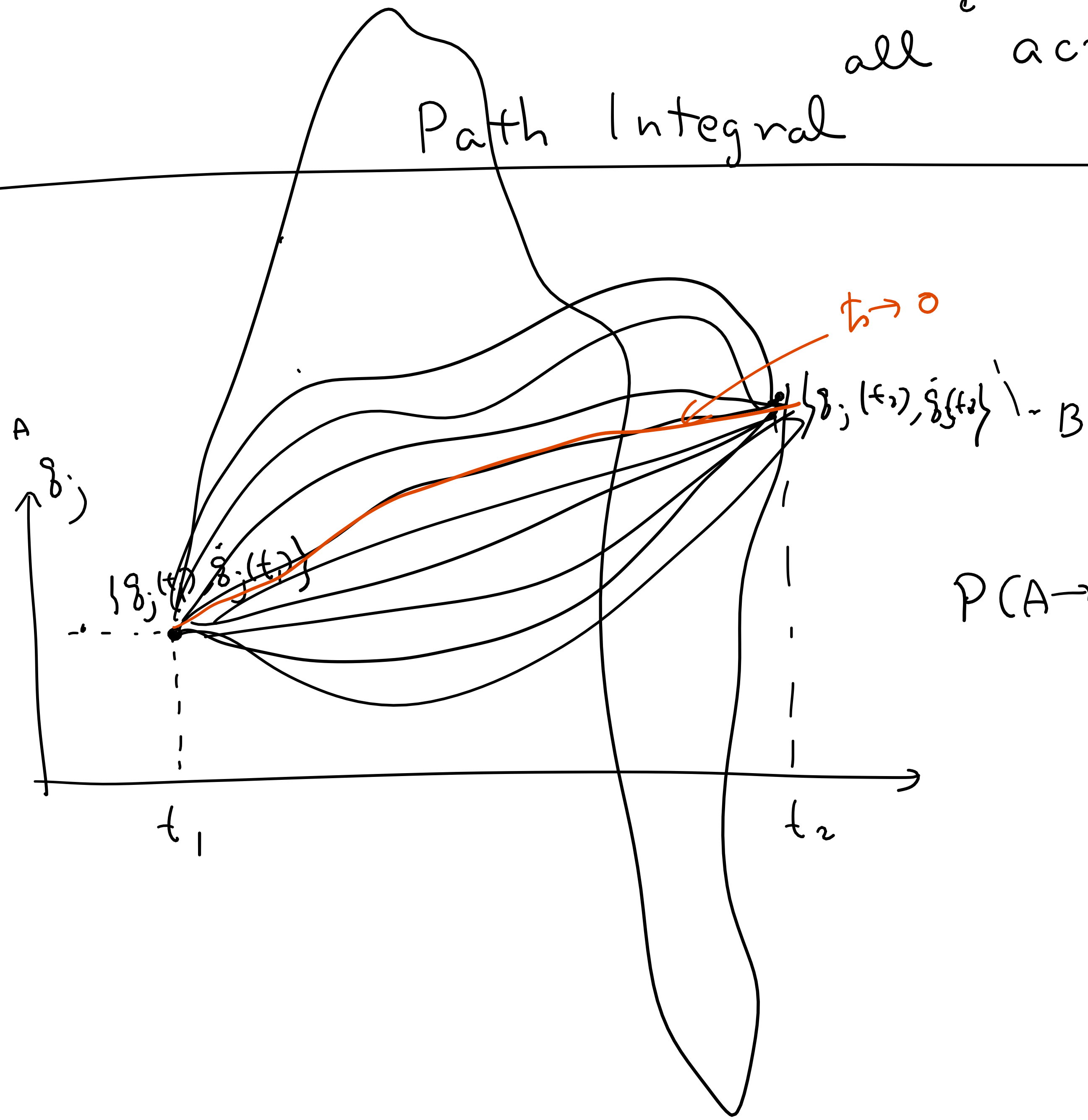


Dirac, Feynman change least action

$\downarrow$

all action $\Rightarrow$ Q.M.

Path Integral



$$P(A \to B) = \int \mathcal{D}q \; e^{\frac{i}{\hbar} S}$$

$\big\downarrow \; \hbar \to 0$

only min. of S matters in path integral

action can be easily generalized from mechanics to field

Cl	$L(q_j, \dot{q}_j)$ S - minimum $S=\int dt\, L$	$\longrightarrow$ $L(q_{\vec{r}}, \dot{q}_{\vec{r}})$ $S = \int dt\, L(q_{\vec{r}}, \dot{q}_{\vec{r}})$
Q	Path $\int Dq\, e^{\frac{i}{\hbar} S}$	

$n = 1$

$q_{j=1 \cdots n=1} \to q_1 \Rightarrow q$

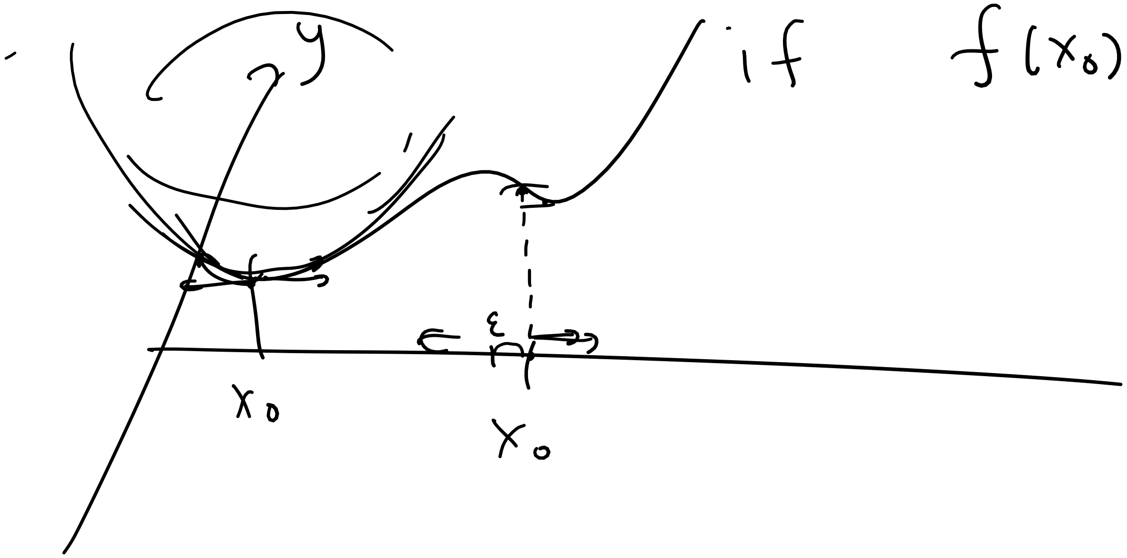
$$L(q, \dot{q}) \longrightarrow S = \int_{t_1}^{t_2} L(q, \dot{q})\, dt$$

minimum action?

(ex) $f(x)$ $\left(\to \underline{f'(x) = 0} \longrightarrow x = x_0 \quad f''(x_0) > 0 \ \checkmark \right)$

$\equiv$

compare $f(x_0)$ vs $f(x_0 + \varepsilon)$ (any ε)



if $f(x_0)$ is always smaller than $f(x_0 + \varepsilon)$

$\to$ x_0 is position of minimum.

$$S' = \int_{t_1}^{t_2} L(q', \dot{q}')\, dt \qquad q'(t) = q(t) + \delta q(t)$$

$$S = \int_{t_1}^{t_2} L(q, \dot{q})\, dt$$

if $S' \geqq S$ always for any δq

$\Longrightarrow$ $q(t)$ is location of min. of S.

$$\delta S = S' - S = \int_{t_1}^{t_2} L(q+\delta q, \dot{q}+\delta\dot{q})\, dt - \int_{t_1}^{t_2} L(q, \dot{q})\, dt$$

$$= \int_{t_1}^{t_2} \left[\underbrace{L(q+\delta q, \dot{q}+\delta\dot{q}) - L(q, \dot{q})} \right] dt$$

Taylor expasion. of function

$$f(x+\epsilon) = f(x) + \epsilon f'(x) \left(+ \frac{\epsilon^2}{2} f''(x) + \cdots \right)$$

$$|\epsilon| \ll |x|$$

$$f(x+\epsilon_1, y+\epsilon_2) = f(x,y) + \epsilon_1 \frac{\partial f}{\partial x} + \epsilon_2 \frac{\partial f}{\partial y} + \underbrace{\cdots}_{\epsilon^2}$$

" of functional

$$F(q+\delta q) = F(q) + \delta q \underbrace{\frac{\partial F}{\partial q}}$$

$(\delta q)^2$

$$L(q+\delta q, \dot{q}+\delta \dot{q}) = \cancel{L(q,\dot{q})} + \delta q \frac{\partial L}{\partial q} + \delta \dot{q} \frac{\partial L}{\partial \dot{q}} + \cdots$$

$$- L(q,\dot{q}) \qquad\qquad - \cancel{L(q,\dot{q})}$$

$$\delta S = \int_{t_1}^{t_2} \left[\delta q \frac{\partial L}{\partial q} + \delta \dot{q} \frac{\partial L}{\partial \dot{q}} \right] dt + \cdots \geqq 0$$

for any δq

$$\int_{t_1}^{t_2}\left[\delta q \frac{\partial L}{\partial q} + \underbrace{\delta \dot{q} \frac{\partial L}{\partial \dot{q}}}\right] dt$$

$$f'g = (fg)' - fg'$$

$$\delta \dot{q} = \frac{d}{dt}(\delta q)$$

$$\delta \dot{q} \frac{\partial L}{\partial \dot{q}} = \frac{d}{dt}(\delta q) \frac{\partial L}{\partial \dot{q}}$$

$$= \frac{d}{dt}\left[\delta q \frac{\partial L}{\partial \dot{q}}\right] - \delta q \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

$$\dot{f} \equiv \frac{df}{dt}$$

$$= \int_{t_1}^{t_2}\left[\underbrace{\delta q} \frac{\partial L}{\partial q} + \underbrace{\frac{d}{dt}\left[\delta q \frac{\partial L}{\partial \dot{q}}\right]} - \underbrace{\delta q} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] dt$$

$$= \cancel{\underbrace{\delta q \frac{\partial L}{\partial \dot{q}}\Bigg|_{t_1}^{t_2}}_{\underbrace{\delta q(t_2)}_{=0} \frac{\partial L}{\partial \dot{q}}(t_2) - \underbrace{\delta q(t_1)}_{=0} \frac{\partial L}{\partial \dot{q}}(t_1)}} + \int_{t_1}^{t_2}\left[\delta q(t)\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right)\right] dt$$

$$\delta S = \int_{t_1}^{t_2} \left[\delta q(t) \underbrace{\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right)}_{\substack{\| \\ 0}} \right] dt + \cdots \geq 0$$

for **any** $\delta q(t)$

↑ only possibility

Euler-Lagrange Eq.

E of M. (Newton's Eq)

$$\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0} \Longleftrightarrow \text{least action principle}$$

$n > 1$

$$\boxed{\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0} \quad j = 1, \cdots, n$$

(H.W. derive)

$$L'(q,\dot{q}) \equiv L(q_j,\dot{q}_j) + \underbrace{\frac{d f(q_j,t)}{dt}}$$

$$\Rightarrow \quad L' \equiv L \qquad \text{same eq of motions !}$$

$$S' = \int dt\, L' = \int_{t_1}^{t_2} dt \left(L + \underbrace{\frac{df}{dt}}\right) = \underbrace{\int dt\, L}_{S} + \underbrace{f(t_2) - f(t_1)}_{\text{indep of } q_j}$$

$$\underline{\delta S' = \delta S}$$

$$L' = L(q_j,\dot{q}_j) + \underline{F(t)}$$

$$\downarrow$$

$$S' = S + \underbrace{\int_{t_1}^{t_2} \underbrace{F(t)}\, dt}_{\text{number}} \longrightarrow \delta S' = \delta S$$

(ex) $L = \frac{m\dot{q}^2}{2} - \frac{k}{2}q^2 + \frac{d}{dt}\left(q\sin(\omega_0 t)\right)$

$$m\ddot{q} - kq = 0$$

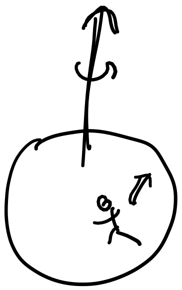
$\boxed{+\,q\sin(\omega_0 t)}$ ✓ no.

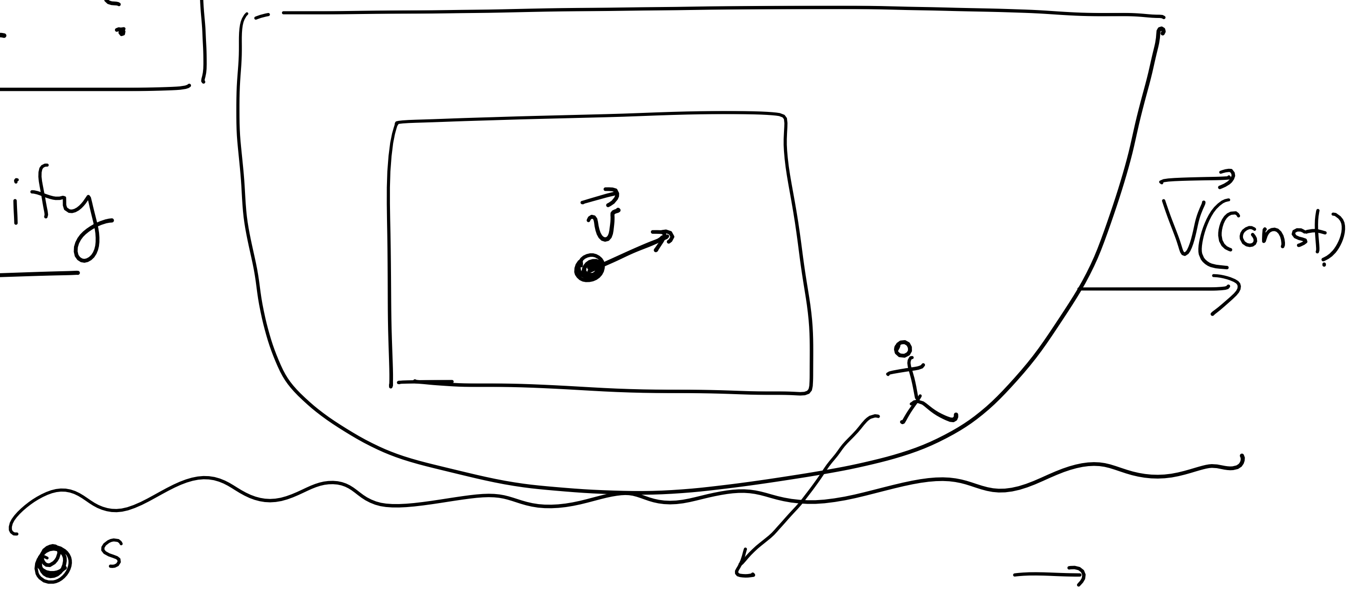
$\boxed{+\,e^{-kt}\sin^2(\omega_0 t)}$ ← explicit time dependence

implicit time dep.

How to find L ?

Galilean relativity





$$L(\vec{v}) \equiv L(\vec{v} + \vec{V})$$

① $\underline{\text{isotropic}}$ $\quad L(\vec{v}) = L(|\vec{v}|)$

$$= L(v^2)$$

$$L(\vec{v}+\vec{V}) = L(|\vec{v}+\vec{V}|^2) = L(\overbrace{v^2}^{x} + \overbrace{2\vec{v}\cdot\vec{V}}^{\epsilon} + \cancel{V^2})$$

if $\vec{V}$ is small

$$\approx L(v^2) + \underbrace{2\vec{v}\cdot\vec{V}\left(\frac{\partial L}{\partial v^2}\right)}_{\text{total derivation}}$$

② Galilean relativity

$$\equiv L(v^2)$$

Galilean inv.

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$\vec{v}\cdot\vec{V}\left(\frac{\partial L}{\partial v^2}\right) = \frac{d}{dt}\left(\vec{r}\cdot\underbrace{\vec{V}\left(\frac{\partial L}{\partial v^2}\right)}_{\text{constant.}}\right)$$

$$= \frac{d\vec{r}}{dt}\cdot\vec{V}\left(\frac{\partial L}{\partial v^2}\right)$$

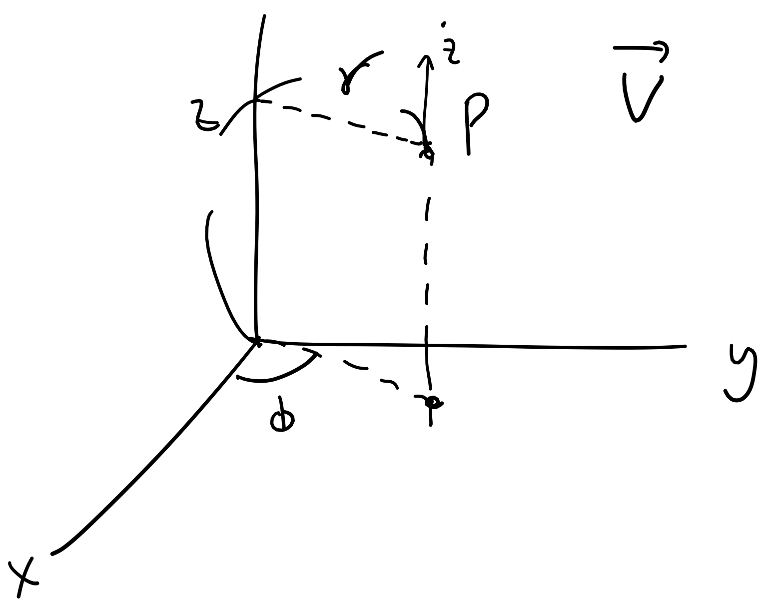
$$\frac{\partial L}{\partial v^2} = \text{const.} = \frac{m}{2}$$

$$\Rightarrow L = \frac{m}{2}v^2 \;!$$

$$L(\dot{q}) = \frac{m}{2}\dot{q}^2$$

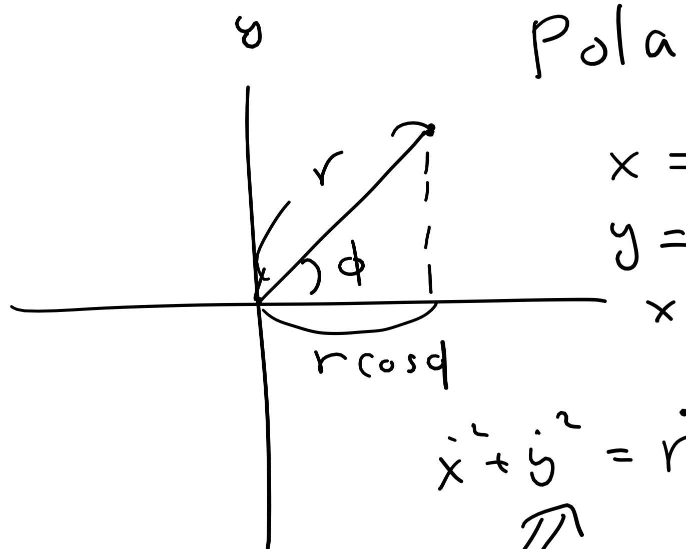
(ex) $L(\underbrace{\dot{x}, \dot{y}, \dot{z}}_{\vec{V}}) = \frac{m}{2}\vec{V}^2 = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

Cylindrical $L(r, \phi, z, \dot{r}, \dot{\phi}, \dot{z}) = \frac{m}{2}\vec{V}^2 = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2)$



$\vec{V}$

Polar



$x = r\cos\phi$

$y = r\sin\phi$

$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\phi}^2$

$\dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi$

$\dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi$

$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$, $\frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}$

$\frac{\partial L}{\partial \dot{z}} = m\dot{z}$

E.M in Cylindrical coord. with $V=0$

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0 \qquad L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right)$$

$q_j = r$; $\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \longrightarrow m r \dot{\phi}^2 - \frac{d}{dt}(m\dot{r}) = 0$

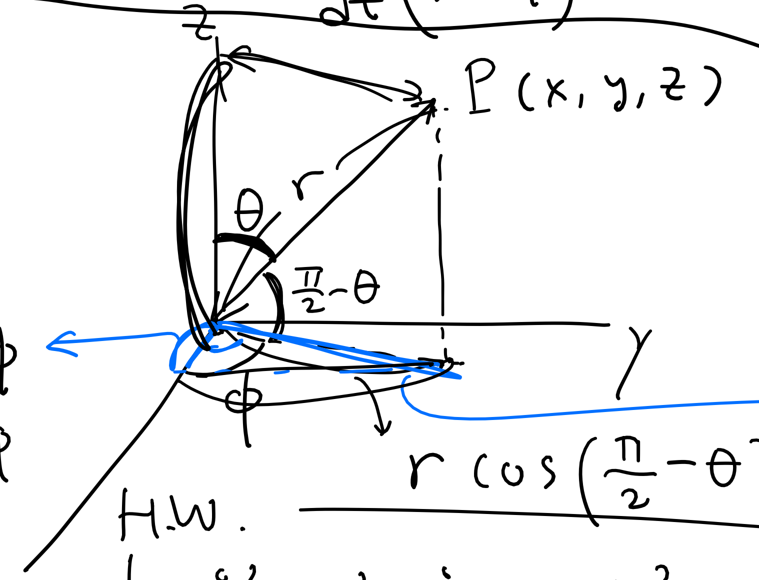
$$\Rightarrow \ddot{r} - r\dot{\phi}^2 = 0$$

$q_j = \phi$; $\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0 \qquad \frac{\partial L}{\partial \phi} = 0 \longrightarrow \frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot{\phi}}}_{m r^2 \dot{\phi}} = 0$

$q_j = z$; $\frac{\partial L}{\partial z} = 0 \rightarrow \frac{d}{dt}(m\dot{z}) = 0 \rightarrow \ddot{z} = 0$

$\therefore m\frac{d}{dt}\left(r^2\dot{\phi}\right) = 0 \Rightarrow m r^2 \dot{\phi} = \text{const} = l$

Spherical Coord.



$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$r\cos\left(\frac{\pi}{2} - \theta\right) = r\sin\theta$

H.W.

$$\boxed{\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta}$$

$V=0$ in spherical

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right)$$

E.o.M.

$q=r$ $\qquad r\dot{\theta}^2 - \ddot{r} = 0$

$q=\theta$ $\qquad r^2\dot{\phi}^2\sin\theta\cos\theta - \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$

$q=\phi$ $\qquad \frac{\partial L}{\partial \phi} = 0 \longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0 \rightarrow \frac{\partial L}{\partial \dot{\phi}} = \text{const}$

$$\parallel$$

$$m r^2 \dot{\phi}\sin^2\theta = l$$

Potential Energy

$$U = U(q_j, \dot{q}_j)$$

(ex) gravity $\rightarrow$ $U = \underbrace{mgz}_{\text{Car, Cyl.}}$, $\underbrace{mgr\cos\theta}_{\text{sphe.}}$

$$L = \underset{\sim}{T} - U$$

(ex) $U = mgz$ in Cyl.

$r, \phi \rightarrow$ same.

$z \rightarrow \dfrac{\partial L}{\partial z} = -\dfrac{\partial U}{\partial z} = -mg \rightarrow -mg - \dfrac{d}{dt}\overbrace{\dfrac{\partial L}{\partial \dot{z}}}^{m\dot{z}} = 0$

$\downarrow$

$-mg - m\ddot{z} = 0$

or $\boxed{\ddot{z} = -g}$

$z = \underset{\sim}{z_0 + \dot{z}_0 t} - \frac{1}{2}gt^2$

General case.

$$L = T - U \qquad T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$x = f_1(q_1, q_2, q_3)$$
$$y = f_2(\quad '' \quad)$$
$$z = f_3(\quad '' \quad)$$

(ex) $x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$$\Downarrow$$

$$x_a = f_a(q_1, \cdots, q_s) \qquad a = 1 \cdots s$$

$$T = \frac{1}{2}m\sum_{a=1}^{s}\dot{x}_a^2 \qquad \dot{x}_a = \frac{\partial f_a}{\partial q_1}\dot{q}_1 + \frac{\partial f_a}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial f_a}{\partial q_s}\dot{q}_s$$

$$= \frac{1}{2}m\sum_{a=1}^{s}\left(\sum_{j=1}^{s}\frac{\partial f_a}{\partial q_j}\dot{q}_j\right)^2 \qquad = \sum_{j=1}^{s}\frac{\partial f_a}{\partial q_j}\dot{q}_j$$

$$\sum_a \left(\sum_{j=1}^{s} \frac{\partial f_a}{\partial q_j} \dot{q}_j \right)^2 = \sum_a \left(\underbrace{\sum_{j=1}^{s}} \frac{\partial f_a}{\partial q_j} \dot{q}_j \right) \left(\underbrace{\sum_{i=1}^{s}} \frac{\partial f_a}{\partial q_i} \dot{q}_i \right)$$

$$= \sum_{i,j=1}^{s} \underbrace{\left[\sum_a \left[\left(\frac{\partial f_a}{\partial q_j} \right)^{=A} \left(\frac{\partial f_a}{\partial q_i} \right)^{B} \right] \right]} \dot{q}_i \dot{q}_j = \underbrace{\sum_{i=1}^{s} \sum_{j=1}^{s}}_{\sum_{i,j}} a_{ij}(\underline{q_1 \cdots q_s}) \dot{q}_i \dot{q}_j$$

$/\!/$ depend on i j only $\equiv \underbrace{a_{ij}(q)}$

$$\left(\frac{\partial f_a}{\partial q_i} \right) \left(\frac{\partial f_a}{\partial q_j} \right)$$

$$AB = BA \rightarrow a_{ij} = a_{ji}$$

$$T = \frac{1}{2}$$

$$U = U(q_1, \cdots, q_s)$$

$$L = \frac{1}{2} \sum_{i,j} \underbrace{a_{ij}} \underbrace{\dot{q}_i \dot{q}_j} - \underbrace{U(q)}$$

$k = 1 \cdots s$

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = 0$$

$$\underbrace{\frac{\partial L}{\partial \dot{q}_k}} = \frac{1}{2} \sum_{ij} a_{ij} \underbrace{\frac{\partial (\dot{q}_i \dot{q}_j)}{\partial \dot{q}_k}}$$

(ex) $$\frac{\partial (\dot{q}_1 \dot{q}_2)}{\partial \dot{q}_3} = 0$$

$$\frac{\partial(\dot{q}_i \dot{q}_j)}{\partial \dot{q}_k} = \begin{cases} (i=k) & \dot{q}_j \\ (j=k) & \dot{q}_i \end{cases} = \underbrace{\delta_{ik}}\dot{q}_j + \delta_{jk}\dot{q}_i$$

Kronecker delta symbol

$$\delta_{ik} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

$$\sum_i \delta_{ik} n_i = n_k$$

$$\frac{1}{2}\sum_{i,j} a_{ij} \boxed{\frac{\partial(\dot{q}_i \dot{q}_j)}{\partial \dot{q}_k}} = \frac{1}{2}\sum_{i,j}\left[a_{ij}\left(\delta_{ik}\dot{q}_j + \delta_{jk}\dot{q}_i\right)\right]$$

$$= \frac{1}{2}\left(\sum_j a_{kj}\dot{q}_j + \sum_{i\to j} a_{ik}\dot{q}_i\right)$$

$$\sum_{i\to j} a_{ik}\dot{q}_i = \sum_j \underbrace{a_{jk}}_{= a_{kj}} \dot{q}_j$$

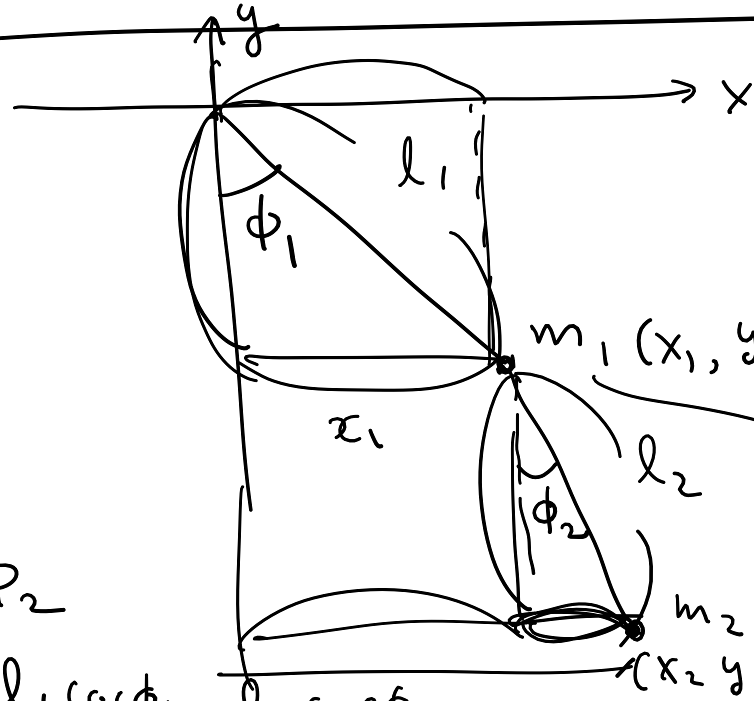
$$\frac{\partial L}{\partial \dot{q}_k} = \sum_j a_{kj}\dot{q}_j$$

$$\frac{d}{dt}\left(\sum_j a_{kj}^{(q)}\dot q_j\right) - \frac{\partial L}{\partial q_k} = 0$$

$$L = \frac{1}{2}\sum_{i,j} a_{ij}^{(q)}\dot q_i \dot q_j - U(q)$$

$$\frac{\partial L}{\partial q_k} = \boxed{\frac{1}{2}\sum_{ij}\left(\frac{\partial a_{ij}}{\partial q_k}\right)\dot q_i \dot q_j - \frac{\partial U}{\partial q_k} = \frac{d}{dt}\sum_j a_{kj}\dot q_j}$$

(Ex) Prob 1.



$x_1 = l_1 \sin\phi_1$

$y_1 = -l_1\cos\phi_1$

$x_2 = \underbrace{l_1 \sin\phi_1}_{x_1} + l_2 \sin\phi_2$

$y_2 = -l_1\cos\phi_1 - l_2\cos\phi_2$

$q_j = \phi_1, \phi_2$

$\dot x_1 = l_1\dot\phi_1\cos\phi_1$

$\dot y_1 = l_1\dot\phi_1 \sin\phi_1$

$\frac{m_1}{2}(\dot x_1^2 + \dot y_1^2) = \frac{m_1}{2} l_1^2\dot\phi_1^2$

$\dot x_2 = l_1\dot\phi_1\cos\phi_1 + l_2\dot\phi_2\cos\phi_2$

$\dot y_2 = l_1\dot\phi_1 \sin\phi_1 + l_2\dot\phi_2\sin\phi_2$

$$\dot{x}_2 = l_1\dot{\phi}_1\cos\phi_1 + l_2\dot{\phi}_2\cos\phi_2$$
$$\dot{y}_2 = l_1\dot{\phi}_1\sin\phi_1 + l_2\dot{\phi}_2\sin\phi_2$$

$$T_2 = \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$
$$= \frac{m_2}{2}\Big(l_1^2\dot{\phi}_1^2 + l_2^2\dot{\phi}_2^2$$
$$+ 2l_1l_2\dot{\phi}_1\dot{\phi}_2\cos\phi_1\cos\phi_2 + 2l_1l_2\dot{\phi}_1\dot{\phi}_2\sin\phi_1\sin\phi_2\Big)$$

$$2l_1l_2\dot{\phi}_1\dot{\phi}_2\underbrace{\left(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2\right)}_{\cos(\phi_1-\phi_2)}$$

$$T = T_1 + T_2 = \frac{m_2}{2}\left(l_1^2\dot{\phi}_1^2 + l_2^2\dot{\phi}_2^2 + 2l_1l_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)\right) + \frac{m_1}{2}l_1^2\dot{\phi}_1^2$$

$$U = m_1 g y_1 + m_2 g y_2 = -m_1 g l_1\cos\phi_1 - m_2 g\left(l_1\cos\phi_1 + l_2\cos\phi_2\right)$$

$$L = T - U \quad //$$

H. W. derive eq. of motion for ϕ_1 & ϕ_2