# Chap 1.

Landau: a few d.o.f.

$\underbrace{\text{d.o.f.}}$ # of unknowns

↑

time dependence

Classical Mechanics

(고전)

↕

Quantum Mechanics

(양자)

↕

Fields

장

time dependence ↔ infinite # of d.o.f → Fields

(ex) Electromagnetism

$$\vec{E}(t, \vec{r})$$

| | Mechanis | Field |
|---|---|---|
| Classic | ⊛ | Electrodynamics |
| Quan | . | Quantum Field Theory |

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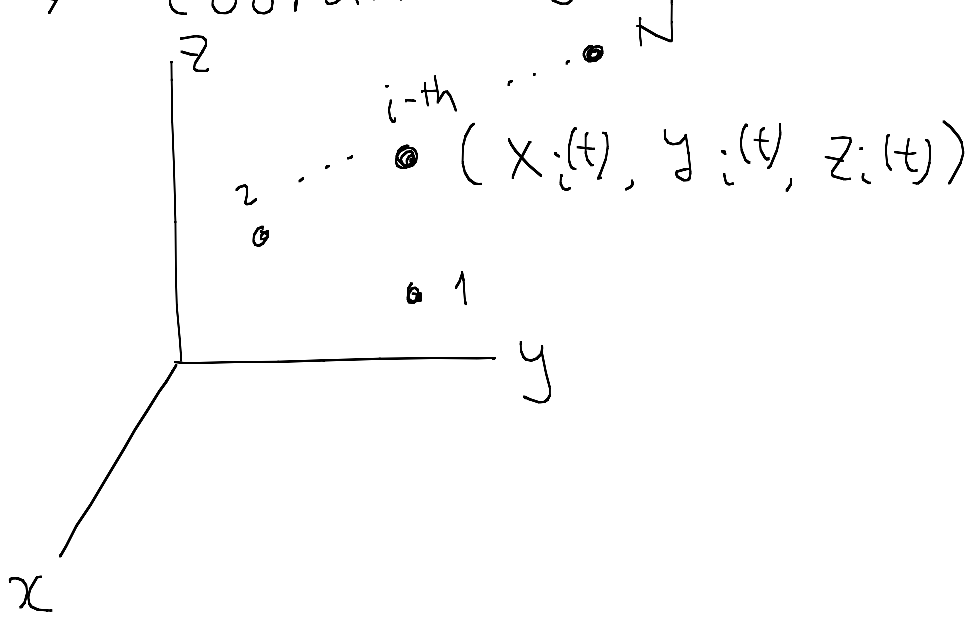
d.o.f. = 3N

CM → coordinates



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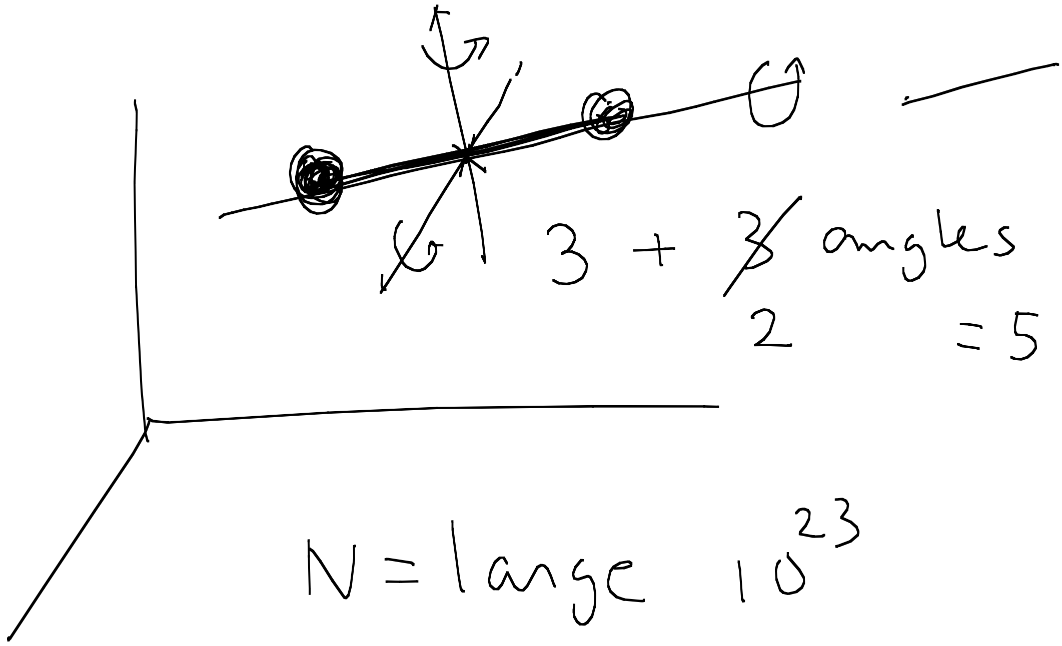
3N → N = 1, 2

constraints



$3 + \cancel{3}_{2}$ angles $= 5$

N = large $10^{23}$

rigid body 강체

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d.o.f = 6

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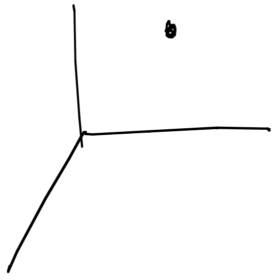
If d.o.f = 1 ( x )

3N →



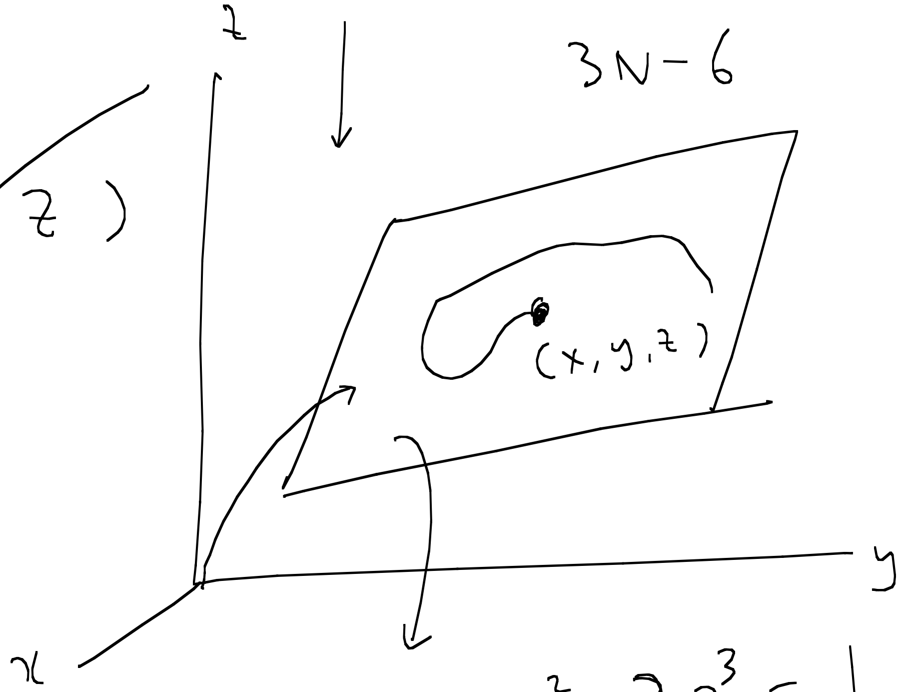
d.o.f < 3N

generalized coordinates

H.W. 6 = 3N − "constraints" of rigid body with N atoms.

3N − 6

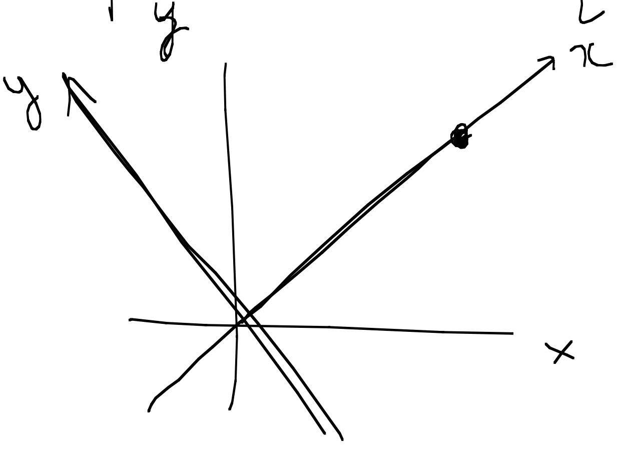
(x, y, z)



$2x^3 - y^2 + 3z^3 = 1$

$y = -1 + 2x + 3z$

top view



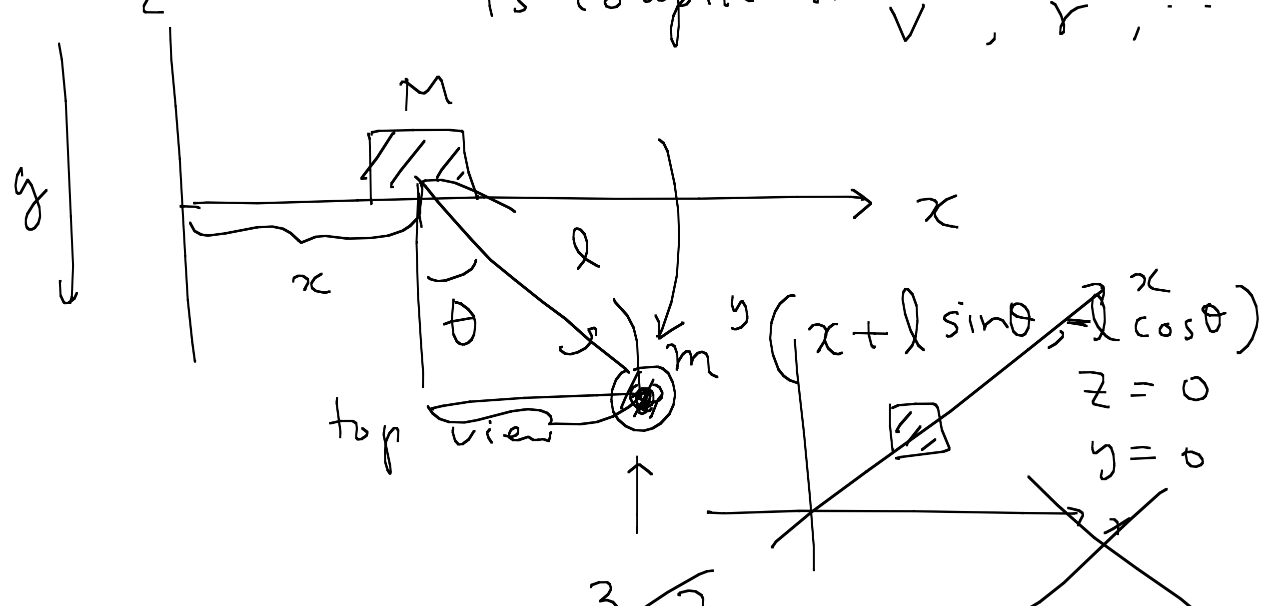
$$q_1, \cdots, \underset{\substack{\uparrow \\ \text{d.o.f}}}{q_s}$$

adv: smaller d.o.f. + potential energy is simple

disadv: not orthogonal → no Pythagoras

↓

Kinetic energy is complicated. $\vec{V}^2, \vec{r}^2, \cdots$



$\cancel{3,2} \to 1$

$$\underline{X = x + \ell\sin\theta} \qquad \underline{Z = -\ell\cos\theta}$$

orthogonal

$$\dot{X} \equiv \frac{dX}{dt}$$

$$\frac{1}{2} m\left(\dot{X}^2 + \dot{Z}^2\right)$$

$$= \frac{1}{2} m\left(\left(\dot{x} + \ell\dot{\theta}\cos\theta\right)^2 + \left(\ell\dot{\theta}\sin\theta\right)^2\right)$$

$$= \frac{1}{2} m\left(\underset{\uparrow}{\dot{x}}^2 + \ell^2\underset{\uparrow}{\dot{\theta}}^2 + 2\ell\underset{\uparrow}{\dot{x}}\underset{\uparrow}{\dot{\theta}}\underline{\cos\underset{\uparrow}{\theta}}\right)$$

# $\boxed{\text{Eq. of Motion}}$

운동 방정식 (뉴턴의 법칙)

⋮

## Lagrangian

$$L = T - V = L(\{\dot{q}_i, q_i\})$$

orthogonal

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = f(\underset{\sim}{\dot{x}_i})$$

general. coord.

$$q_1 \cdots q_s$$

$$T = T(\dot{q}_1, \cdots, \dot{q}_s ; q_1, \cdots, q_s)$$

$$V = V(q_1 \cdots q_s)$$

---

EoM comes from

$\underline{\underline{\text{least}}}$ $\underline{\underline{\text{action}}}$ principle

$$S = \int_{t_1}^{t_2} \underset{\underset{L(q,\, \dot{q},\, t)}{\parallel}}{L}\, dt$$

$$\underline{\underline{\boxed{q(t)}}} \leftarrow ?$$

$s = 1$

$f(x)$ ⟶ ① ~~min. of f~~ ?

② find $x$ which gives min.

function of function ≡ functional

variational method (변분법)

$$L(\underbrace{q(t)}, \dot{q}(t), t)$$

$$\{ q(0), q(0.00\cdots 1), q(0.\cdots 2) \cdots \}$$

$f(x)$

$\underline{f'(x) = 0} \rightarrow x = x_0$

$f''(x_0) > 0$

$$\downarrow$$

$$0 = \frac{\partial S}{\partial q(0)} = \frac{\partial S}{\partial q(0.00\cdots 1)} = \cdots -$$

$$\downarrow$$

Taylor expansion

$$\underbrace{f(x_0 + \varepsilon)} = f(x_0) + \underset{\sim}{\varepsilon} \cancel{f'(x_0)} + \cdots$$

$|\varepsilon| \ll 1$

$$\underline{f(x_0 + \varepsilon) - f(x_0) = 0}$$

$$\delta S = S[q + \delta q(t), \dot{q} + \delta \dot{q}] - S[q, \dot{q}] = 0$$

$$\delta S = \int_{t_1}^{t_2} dt \left[ L(q + \delta q, \dot{q} + \delta \dot{q}, t) - L(q, \cancel{\dot{q}}, t) \right]$$

$$|\delta q| \ll 1$$

$$f(x+\varepsilon_1, y+\varepsilon_2)$$
$$= \underset{\sim}{f(x,y)} + \underline{\varepsilon_1 \frac{\partial f}{\partial x} + \varepsilon_2 \frac{\partial f}{\partial y}}$$
$$+ O(\varepsilon^2)$$

$$L(q+\delta q, \dot q + \delta \dot q, t)$$
$$= L(q, \cancel{\dot q}, t) + \delta q \frac{\partial L}{\partial q(t)} + \delta \dot q \frac{\partial L}{\partial \dot q} + \dots$$

$$\delta S = \int_{t_1}^{t_2} dt \left[ \underline{\underline{\underset{\sim}{\delta q} \frac{\partial L}{\partial q}}} + \delta \dot q \boxed{\frac{\partial L}{\partial \dot q}} \right]$$

$$\uparrow$$

$$\underbrace{\frac{d\,\boxed{\delta q}}{dt}}_{}$$

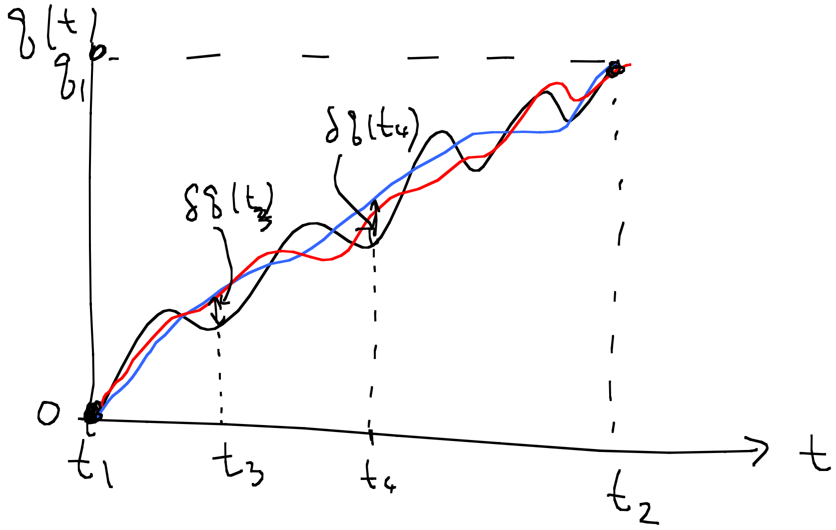
$$\frac{d}{dt}\left(\delta q \frac{\partial L}{\partial \dot q}\right) - \underline{\underline{\underset{\sim}{\delta q} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)}}$$

$$\delta S = \underline{\delta q \frac{\partial L}{\partial \dot q}\Bigg|_{t_1}^{t_2}} + \underbrace{\int_{t_1}^{t_2} dt \; \underset{\text{arbitrary}}{\underline{\underline{\delta q(t)}}} \left[ \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) \right]}_{}$$

$= 0$ for minimizing action
(least action)

$$\underset{\sim}{\delta q(t_2)} \frac{\partial L}{\partial \dot q}(t_2) - \underset{\sim}{\delta q(t_1)} \frac{\partial L}{\partial \dot q}(t_1) = 0$$

(Assume.
$\delta q(t_1) = \delta q(t_2) = 0$)



$$\therefore \quad \delta S = 0 \longmapsto \boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0}$$

Euler-Lagrange Eq.

## Rules of solving

① choose good generalized coordinates $q_1 \cdots q_s$

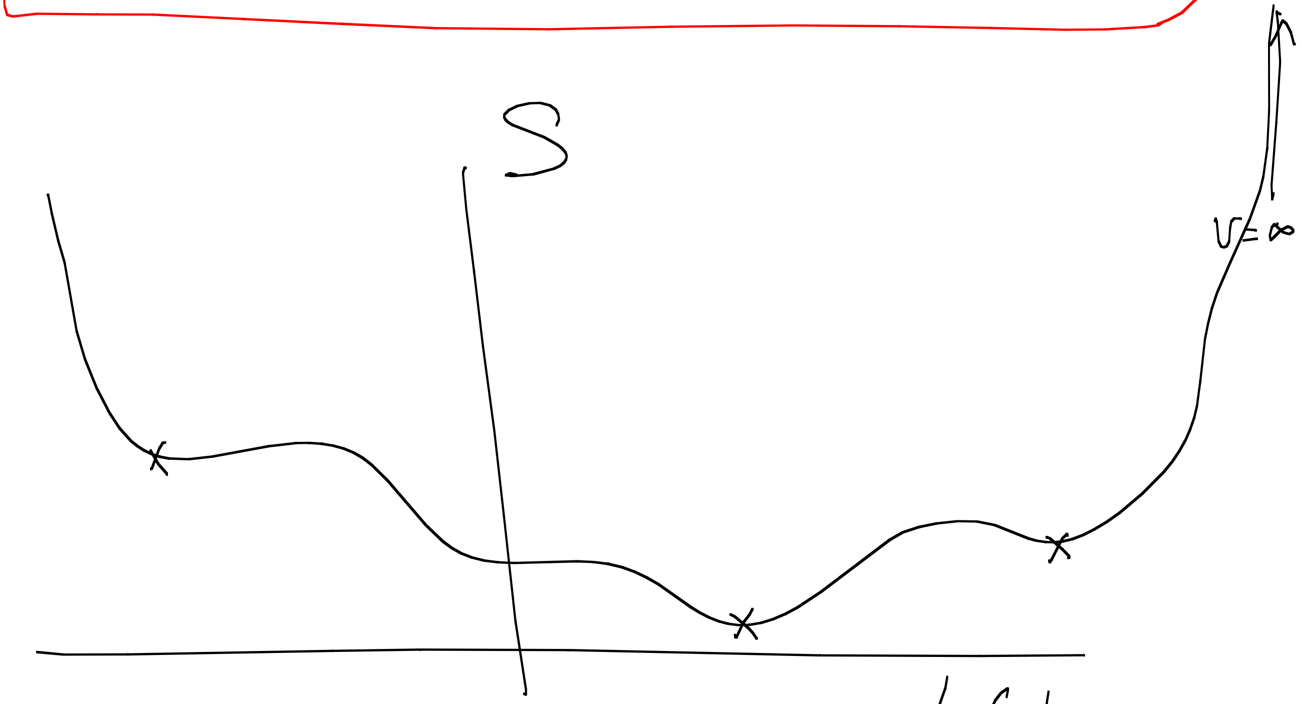
② construct $L = T - V$

for $T$ : need relations with orthogonal coord. $(x, y, z)$

③ Use E-L eq. $s \neq 1$

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \qquad i = 1, \cdots, s$$

④ 2nd order time differential eq.
solve them. (need math skills)



$\{ q \}$

$$L'(q, \dot q, t) = L(q, \dot q, t) + \frac{d f(q, \dot q, t)}{dt}$$

$$S' = \int L' dt = \underset{\overset{\shortparallel}{S}}{\int L \, dt} + \underbrace{\int \frac{df}{dt} dt}_{f(t_2) - f(t_1)}$$

$\uparrow$ indep of $q, \dot q$

$$\delta S' = 0 = \delta S$$

same eom

$\cancel{f(t)}$

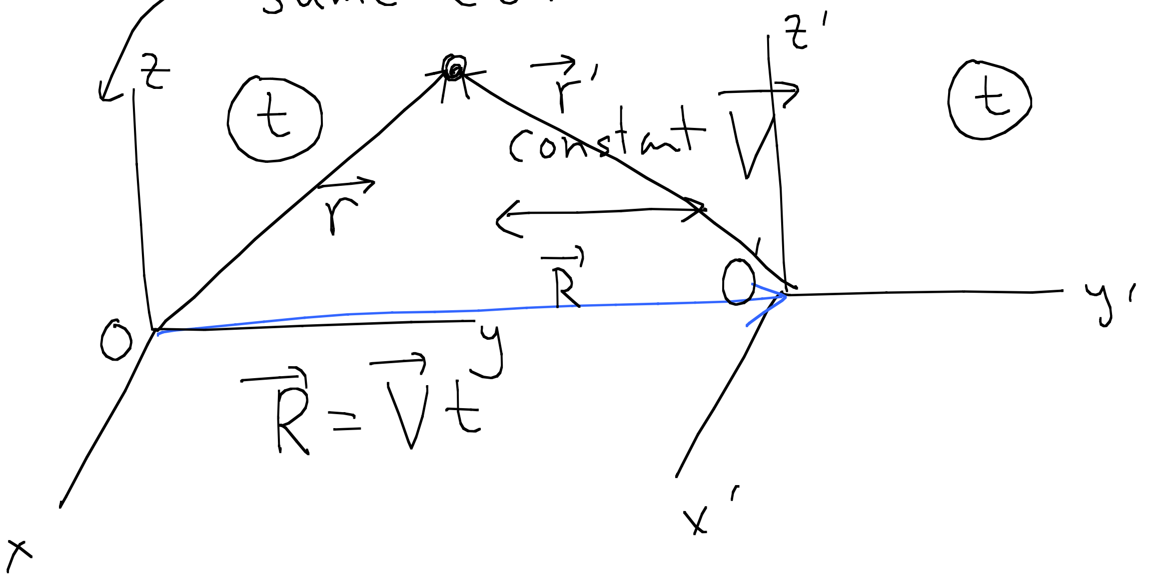
$$L = \frac{1}{2} m \dot q^2 - \frac{1}{2} k q^2 + \cancel{\alpha \cos \omega t}$$

$$+ \cancel{\underbrace{\dot q \sin t + q \cos t}_{\frac{d}{dt}(q \sin t)}} \quad \cancel{\frac{d}{dt}\left(\frac{\alpha}{\omega} \sin \omega t\right)}$$

# Galileo's relativity

reference frames are all relative

same eom for all ref. frame



$$\vec r = \vec r' + \vec R = \vec r' + \vec V t$$

$$\hookrightarrow \dot{\vec r} = \dot{\vec r}' + \vec V$$

$$L = L(\underset{\sim}{\vec r}, \underset{\sim}{\dot{\vec r}}, \underset{\sim}{t}) \longrightarrow \underset{\sim}{L} = \frac{1}{2} m \dot{\vec r}^2$$

$$L' = L'(\vec r', \dot{\vec r}', t)$$

$$L(\vec r' + \vec V t, \dot{\vec r}' + \vec V, t)$$

$$= T(\quad '' \quad) - V(\vec r')$$

$$(V = 0)$$

$$= \frac{1}{2} m (\underbrace{\dot{\vec r}' + \vec V}_{\dot{\vec r} = \vec V})^2 = \frac{1}{2} m \dot{\vec r}'^2 + m \dot{\vec r}' \cdot \vec V + \frac{1}{2} m \vec V^2$$

$$= \frac{1}{2} m \dot{\vec{r}}'^2 + \frac{d}{dt} \underbrace{\left( m \dot{\vec{r}}' \cdot \vec{V} + \frac{1}{2} m \vec{V}^2 t \right)}$$

$$\equiv \frac{1}{2} m \dot{\vec{r}}'^2 = L'(\dot{\vec{r}}')$$

$L' = L$ → same eof □

$$V(\vec{r}' + \vec{V} t) \neq V(\vec{r}')$$

In Quantum mechanics

EoM : Schrödinger eq.

$$\underline{H} \psi(\vec{r}, t) = i\hbar \frac{\partial \psi}{\partial t}$$

↳ <u>Hamiltonian</u>

Mechanics : $\boxed{H(q_i, p_i) = p_i \dot{q}_i - L}$

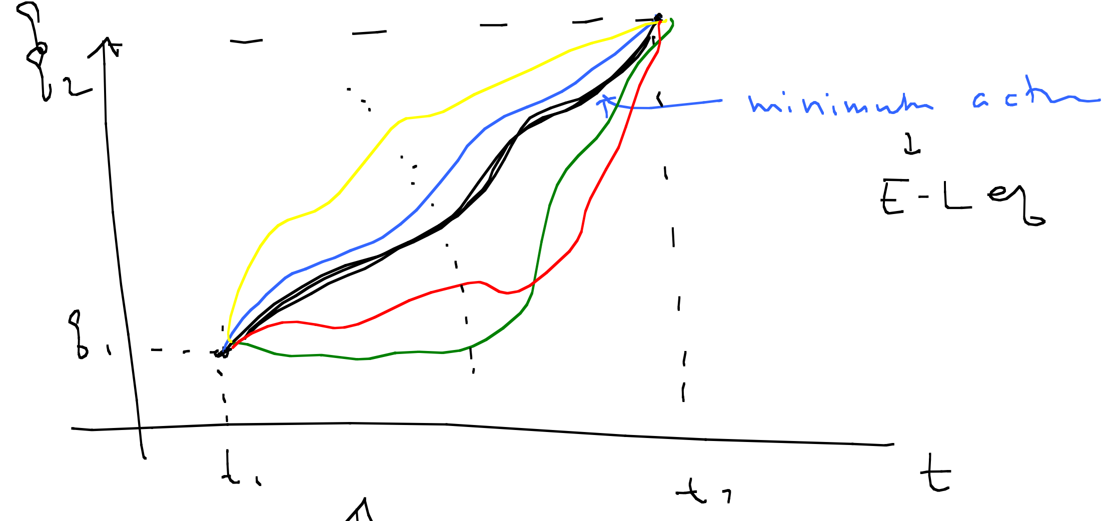
$$p_i \equiv \frac{\partial L(q_i, \dot{q}_i, t)}{\partial \dot{q}_i}$$

Hamilton eq.

$$\boxed{\frac{\partial H}{\partial p_i} = \dot{q}_i , \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i}$$

# Lagrangian of QM ?

Dirac & Feynman "Path Integral"



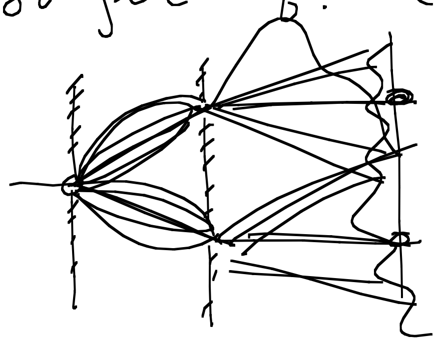
↑

∞ paths : class. mech.
choose only one
Quant mech. (Feynman)
consider all paths equally
but $\hbar \to 0$ (classical limit)
weights on each path
↓

$$\psi(q_1|q_2) = \int e^{\frac{S[\text{path}]}{i\hbar}} \mathcal{D}[\text{path}]$$

$$|||$$

Schrödinger eq. (H)



orthogonal N particles without Constraints

↓

$m_a \quad \vec{r}_a(t)$

$$L = T\left(= \sum_{a=1}^{N} \frac{1}{2} m_a \dot{\vec{r}}_a^{\,2}\right) - V\left(= V(\vec{r}_1 \cdots \vec{r}_N)\right)$$

$$\frac{\partial L}{\partial \vec{r}_a} - \underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}_a}\right)}_{\to\, m_a \ddot{\vec{r}}_a} = 0 \qquad a = 1 \cdots N$$

↑

$$\left(\frac{\partial L}{\partial x_a}, \frac{\partial L}{\partial y_a}, \frac{\partial L}{\partial z_a}\right) = \underbrace{\vec{\nabla}_a} L = -\vec{\nabla}_a V$$

(ex) $$V = \sum_{a=1}^{N} \frac{1}{2} k_a \underbrace{\boxed{\vec{r}_a}^{\,2}}_{(x_a^2 + y_a^2 + z_a^2)}$$

$$\frac{\partial V}{\partial \vec{r}_a} = k_a \vec{r}_a \qquad \left| \quad \begin{aligned} \frac{\partial V}{\partial x_a} &= k_a x_a \\ \frac{\partial V}{\partial y_a} &= k_a y_a \\ \frac{\partial V}{\partial z_a} &= k_a z_a \end{aligned} \right.$$

$$k_a \vec{r}_a = k_a \underbrace{(x_a, y_a, z_a)} = (k_a x_a, k_a y_a, k_a z_a)$$

$$\Rightarrow \quad \underbrace{m_a}\,\underbrace{\ddot{\vec{r}}_a} = -\vec{\nabla}_a V = \underbrace{\vec{F}_a}$$

Newton's Law

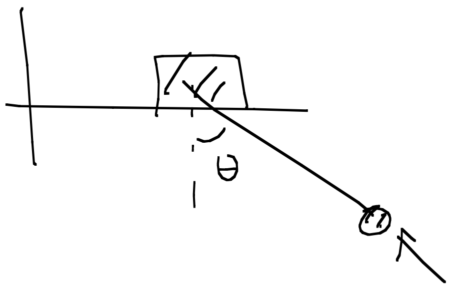
with constraints

$\Rightarrow$ generalized coord. $q_1 \cdots q_s$ orthogonal.

$$\underbrace{X_i}_{\underbrace{\vec r_1}_{3}\ \underbrace{\vec r_2}_{3}\ \cdots\ \underbrace{\vec r_N}_{3}} \qquad i = 1 \cdots 3N$$

$$X_i = f_i(q_1 \cdots q_s)$$

ex)



$$X = x + l \sin\theta$$

$$Z = -l \cos\theta$$

$$\dot X_i = \sum_{k=1}^{s} \frac{\partial f_i}{\partial q_k} \dot q_k$$

$$T = \sum_{i=1}^{3N} \frac{1}{2} m_i \dot X_i^2$$

$$V = V(q_1 \cdots q_s)$$

$$L = T - V$$

$$= \sum_{i=1}^{3N} \underbrace{\frac{1}{2} m_i} \left( \sum_{k=1}^{s} \frac{\partial f_i}{\partial q_k} \dot q_k \right)^2$$

||

$$\left(\sum_{\textcircled{k}=1}^{s} \frac{\partial f_i}{\partial q_k}\,\dot{q}_k\right)\left(\sum_{\textcircled{l}=1}^{s} \frac{\partial f_i}{\partial q_l}\,\dot{q}_l\right)$$

dummy

$$= \frac{1}{2}\sum_{k,l=1}^{s}\left(\underbrace{\sum_{i=1}^{3N} \frac{\partial f_i}{\partial q_{\textcircled{k}}}\,\frac{\partial f_i}{\partial q_{\textcircled{l}}}\, m_i}_{\equiv\, a_{kl}(q_1,\cdots,q_s)}\right)\dot{q}_k\,\dot{q}_l$$

$$L = \frac{1}{2}\sum_{k,l=1}^{s} a_{kl}(q_1 \cdots q_s)\,\dot{q}_k\,\dot{q}_l - V(q_1 \cdots q_s)$$

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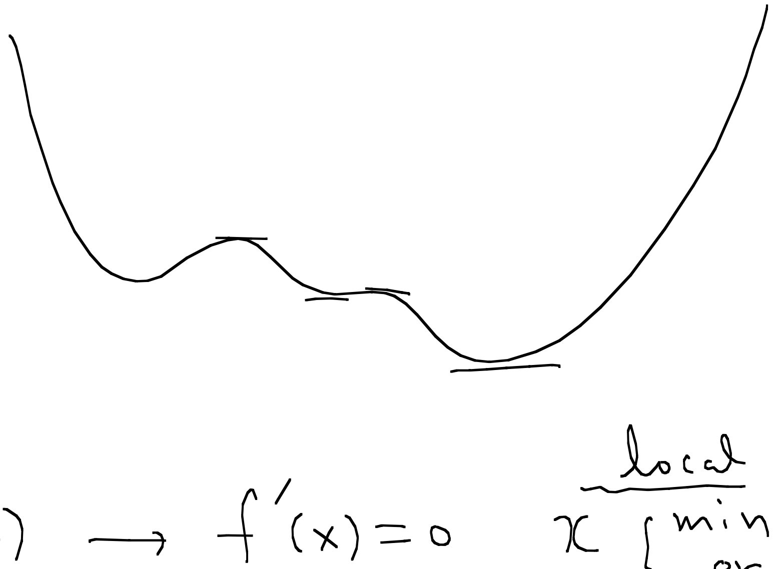

$$f(x) \longrightarrow f'(x) = 0 \quad x \overset{\text{local}}{\left\{\begin{matrix}\text{min}\\ \text{or}\\ \text{max}\end{matrix}\right\}}$$

$\boxed{\text{H.W.}}$ least action condition

$S = \int L(q, \dot{q}, t)\,dt$ is locally min. if $\frac{\delta S}{\delta q} = 0 \left( \rightarrow \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \right)$

But it may be locally max.

Derive condition that it is min.

$$\left( \frac{\delta^2 S}{\delta q^2} > 0 \right)$$

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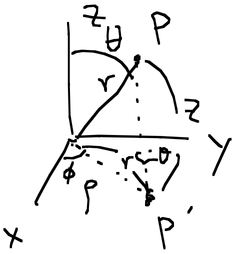
other coordinate systems



- cylindrical : $(\rho, \phi, z)$

  $x = \rho\cos\phi \quad y = \rho \sin\phi$

- spherical : $(r, \theta, \varphi)$ $\quad \varphi = \phi$

Cylindrical

$\dot{x} = \dot{\rho}\cos\phi - \rho\dot{\phi}\sin\phi$

$\dot{y} = \dot{\rho}\sin\phi + \rho\dot{\phi}\cos\phi$

$\dot{z}$

$\vec{v}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2$

$$\frac{m}{2}\vec{v}^2 = \frac{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)m}{2} = \frac{m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)}{2}$$

Spherical

$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$

$\dot{x} = \dot{r}\sin\theta\cos\varphi + r\dot{\theta}\cos\theta\cos\varphi - r\dot{\varphi}\sin\theta\sin\varphi$

$\dot{y} = \dot{r}\sin\theta\sin\varphi + r\dot{\theta}\cos\theta\sin\varphi + r\dot{\varphi}\sin\theta\cos\varphi$

$$\dot{z} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

$$\boxed{\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2\sin^2\theta}$$

$$+ \cancel{2\dot{r}r\dot{\theta}\sin\theta\cos\theta} \cancel{- 2r\dot{r}\dot{\theta}\cos\theta\sin\theta}$$

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2\sin^2\theta\right)$$

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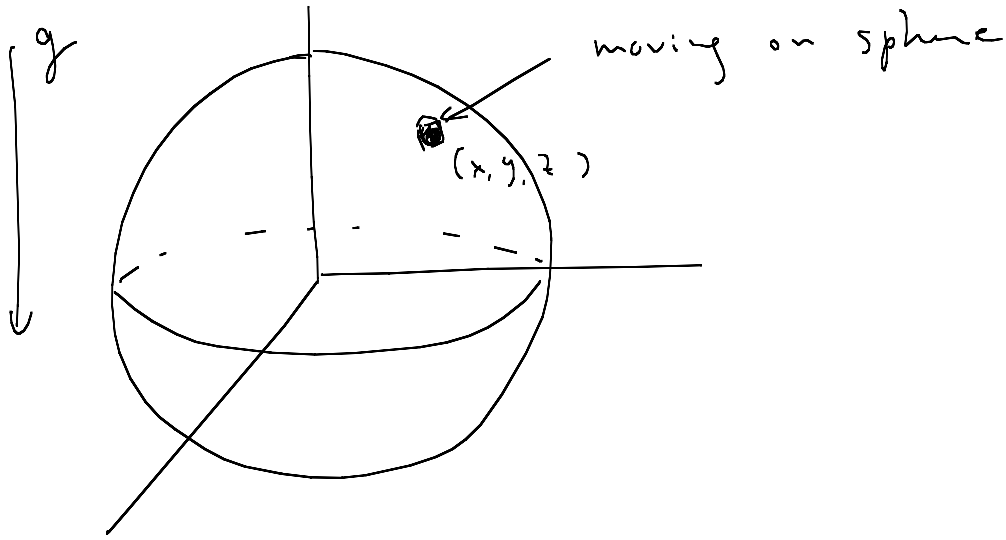
(ex)



Cartesian coordinates : $(x, y, z)$

constraints → $\underline{x^2 + y^2 + z^2 = R^2}$

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

$$z = \sqrt{R^2 - x^2 - y^2}$$

$$\dot{z} = \frac{-x\dot{x} - y\dot{y}}{\sqrt{R^2 - x^2 - y^2}}$$

$$\boxed{L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \frac{(x\dot{x} + y\dot{y})^2}{R^2 - x^2 - y^2}\right) - mg\sqrt{R^2 - x^2 - y^2}}$$

spherical $\qquad r = R$

$$L = \frac{1}{2}m\left(R^2\dot{\theta}^2 + R^2\dot{\varphi}^2\sin^2\theta\right) - mgR\cos\theta$$

$x, y, z \longrightarrow 3$ d.o.f.

$\downarrow$

with 1 const $\longrightarrow$ 2 " ; $\theta, \varphi$

$\uparrow$

generalized

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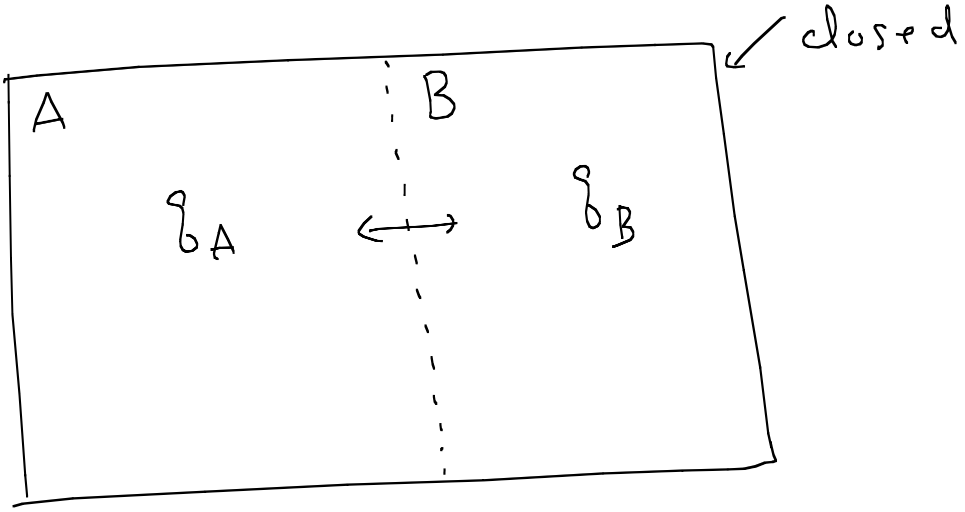

$$L = T_A(q_A, \dot{q}_A) + T_B(q_B, \dot{q}_B) - \underline{U(q_A, q_B)}$$

if $U(q_A, q_B) = U_A(q_A) + U_B(q_B) \longrightarrow$ A & B are closed to each other

if $U(q_A, q_B) \neq U_A(q_A) + U_B(q_B)$

① $q_B$ is given : (ex) $q_B(t) = q_0 \sin \omega t$

(external)

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$L = L + \frac{df}{dt}$

$$L = T_A(q_A, \dot{q}_A) + \cancel{T_B(t)} - U(q_A, \underbrace{q_B(t)}_{\text{given function}})$$

$$\hookrightarrow \frac{\partial L}{\partial q_A} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_A} = 0$$

② $\xi_B$ is not given, but determined by e.o.m.

$$L = L(\xi_A, \xi_B, \dot\xi_A, \dot\xi_B)$$

$$\frac{\partial L}{\partial \xi_A} - \frac{d}{dt}\frac{\partial L}{\partial \dot\xi_A} = 0 = \frac{\partial L}{\partial \xi_B} - \frac{d}{dt}\frac{\partial L}{\partial \dot\xi_B}$$

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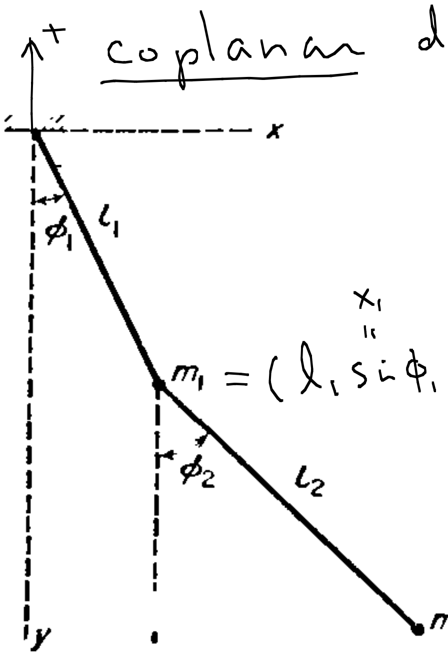
**Prob1** coplanar double pendulum



$$m_1 = (\overset{x_1}{l_1 \sin\phi_1}, \overset{y_1}{-l_1\cos\phi_1})$$

$$m_2\,(\overset{x_2}{x_1 + l_2\sin\phi_2}, \overset{y_2}{y_1 - l_2\cos\phi_2})$$

$$\dot x_2 = \dot x_1 + l_2\dot\phi_2\cos\phi_2 \qquad \dot y_2 = \dot y_1 + l_2\dot\phi_2 \sin\phi_2$$

$$T = \frac{m_1}{2}(\dot x_1^2 + \dot y_1^2) + \frac{m_2}{2}(\dot x_2^2 + \dot y_2^2)$$

$$= \frac{m_1}{2}(l_1^2\dot\phi_1^2) + \frac{m_2}{2}(l_1^2\dot\phi_1^2) + \frac{m_2}{2}(l_2^2\dot\phi_2^2)$$

$$\underbrace{+\frac{m_2}{2}\left(2\underbrace{\dot x_1}_{l_1\dot\phi_1\cos\phi_1} l_2\dot\phi_2\cos\phi_2 + 2\underset{\overset{\uparrow}{l_1\dot\phi_1\sin\phi_1}}{\dot y_1}\, l_2\dot\phi_2\sin\phi_2\right)}_{m_2 l_1 l_2\dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2)}$$

$$T = \left(\frac{m_1+m_2}{2}\right)l_1^2\dot\phi_1^2 + \frac{m_2 l_2^2}{2}\dot\phi_2^2 + m_2 l_1 l_2 \dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2)$$

$$V = m_1 g y_1 + m_2 g y_2 \qquad y_2 = y_1 - l_2\cos\phi_2$$

$$= -(m_1+m_2) g l_1 \cos\phi_1 - m_2 g l_2 \cos\phi_2$$

$$L = T - V$$

$$= \left(\frac{m_1+m_2}{2}\right) l_1^2 \dot{\phi}_1^2 + \frac{m_2 l_2^2}{2} \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1-\phi_2)$$

$$+ (m_1+m_2) g l_1 \cos\phi_1 + m_2 g l_2 \cos\phi_2$$

$$\frac{\partial L}{\partial \phi_1} = - m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \sin(\phi_1-\phi_2) - (m_1+m_2) g l_1 \sin\phi_1$$

$$\frac{\partial L}{\partial \dot{\phi}_1} = (m_1+m_2) l_1^2 \dot{\phi}_1 + m_2 l_1 l_2 \dot{\phi}_2 \cos(\phi_1-\phi_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_1}\right) = (m_1+m_2) l_1^2 \ddot{\phi}_1 + m_2 l_1 l_2 \ddot{\phi}_2 \cos(\phi_1-\phi_2)$$

$$- m_2 l_1 l_2 \dot{\phi}_2 (\dot{\phi}_1 - \dot{\phi}_2) \sin(\phi_1-\phi_2)$$

$$\frac{\partial L}{\partial \phi_2} = + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \sin(\phi_1-\phi_2) - m_2 g l_2 \sin\phi_2$$

$$\frac{\partial L}{\partial \dot{\phi}_2} = m_2 l_2^2 \dot{\phi}_2 + m_2 l_1 l_2 \dot{\phi}_1 \cos(\phi_1-\phi_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_2}\right) = m_2 l_2^2 \ddot{\phi}_2 + m_2 l_1 l_2 \ddot{\phi}_1 \cos(\phi_1-\phi_2)$$

$$- m_2 l_1 l_2 \dot{\phi}_1 (\dot{\phi}_1 - \dot{\phi}_2) \sin(\phi_1-\phi_2)$$

assume $\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2 \ll 1$ (small osc.)

$$(m_1+m_2) l_1^2 \ddot{\phi}_1 + m_2 l_1 l_2 \ddot{\phi}_2 = -(m_1+m_2) g l_1 \phi_1$$

$$m_2 l_2^2 \ddot{\phi}_2 + m_2 l_1 l_2 \ddot{\phi}_1 = - m_2 g l_2 \phi_2$$

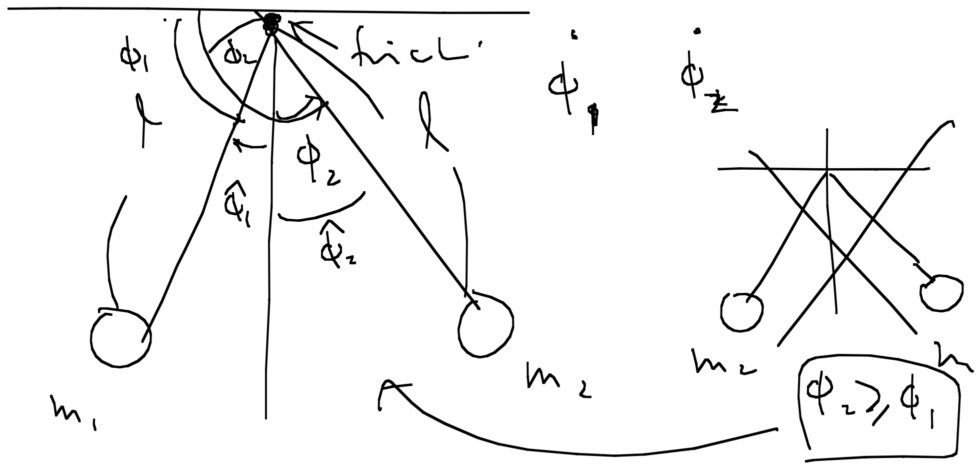
$$\underbrace{\begin{pmatrix} (m_1+m_2) l_1^2 & m_2 l_1 l_2 \\ m_2 l_2^2 & m_2 l_1 l_2 \end{pmatrix}}_{A} \begin{pmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{pmatrix} = -g \underbrace{\begin{pmatrix} (m_1+m_2) l_1 & 0 \\ 0 & m_2 l_2 \end{pmatrix}}_{\mathbb{B}} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$\phi_1 = a e^{i\omega t}$ $\phi_2 = b e^{i\omega t}$

$$-\omega^2 \underbrace{\begin{pmatrix} (m_1+m_2) l_1^2 & m_2 l_1 l_2 \\ m_2 l_2^2 & m_2 l_1 l_2 \end{pmatrix}}_{A} \begin{pmatrix} a \\ b \end{pmatrix} = -g \underbrace{\begin{pmatrix} (m_1+m_2) l_1 & 0 \\ 0 & m_2 l_2 \end{pmatrix}}_{\mathbb{B}} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$(\omega^2 A + \mathbb{B}) \begin{pmatrix} a \\ b \end{pmatrix} = 0 \longrightarrow \det(\omega^2 A + \mathbb{B}) = 0$$

$a, b \neq 0$



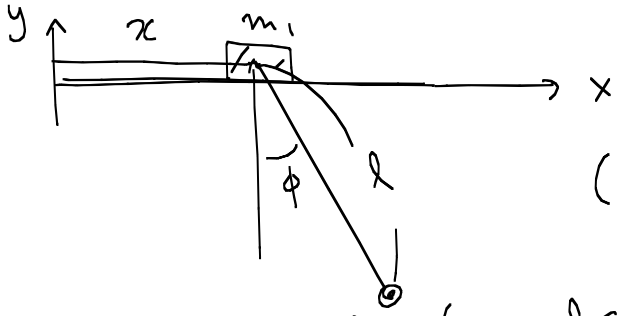
$$L = \frac{1}{2} m_1 l^2 \dot{\hat\phi}_1^2 + \frac{1}{2} m_2 l^2 \dot{\hat\phi}_2^2 + m_1 l \cos\hat\phi_1 + m_2 l \cos\hat\phi_2 - \mu(\dot\phi_1 + \dot\phi_2)$$

$\hat\phi_2 = \overset{-\hat\phi_1}{\phi_2 - \phi_1}$

$\hat\phi_1 = \frac{\pi}{2} - \phi_1$

---

[Prob2]



$(\dot x + l\dot\phi\cos\phi, l\dot\phi\sin\phi)$

$\uparrow$

$m_2 \; (x + l\sin\phi, -l\cos\phi)$

$$T = \frac{1}{2} m_1 \dot x^2 + \frac{1}{2} m_2 (\dot x^2 + l^2 \dot\phi^2 + 2l\dot x\dot\phi\cos\phi)$$

$$V = -m_2 g l \cos\phi$$

$$L = T - V =$$

$$= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left( \dot{x}^2 + l^2 \dot{\phi}^2 + 2 l \dot{x} \dot{\phi} \cos\phi \right) + m_2 g l \cos\phi$$

$$\underline{\underline{\frac{\partial L}{\partial x} = 0}} \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \rightarrow \frac{\partial L}{\partial \dot{x}} = \text{const.} = C_0$$

$$= m_1 \dot{x} + m_2 \dot{x} + m_2 l \dot{\phi} \cos\phi$$

$$= \frac{d}{dt}\left((m_1 + m_2) x + m_2 l \sin\phi\right)$$

$$\therefore \boxed{(m_1 + m_2) x + m_2 l \sin\phi = c_0 t + c_1}$$

$$\frac{\partial L}{\partial \phi} = -m_2 l \dot{x} \dot{\phi} \sin\phi - m_2 g l \sin\phi$$

$$\frac{\partial L}{\partial \dot{\phi}} = m_2 l^2 \dot{\phi} + m_2 l \dot{x} \cos\phi$$

$$m_2 l^2 \ddot{\phi} + m_2 l \left( \ddot{x} \cos\phi - \cancel{\dot{x} \dot{\phi}} \sin\phi \right)$$

$$+ m_2 l \cancel{\dot{x} \dot{\phi}} \sin\phi + m_2 g l \sin\phi = 0$$

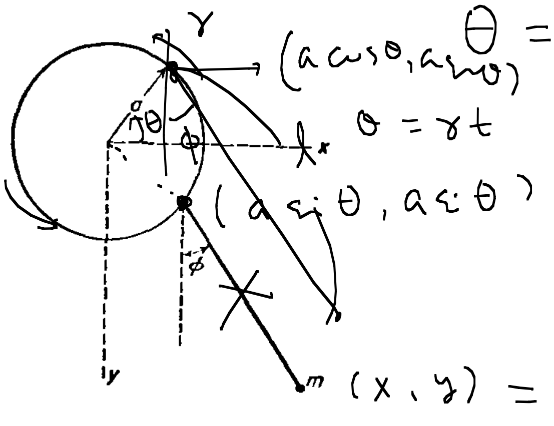
$$m_2 l^2 \ddot{\phi} + m_2 l \ddot{x} \cos\phi + m_2 g l \sin\phi = 0$$

$$\dot{x} = \frac{C_0 - m_2 l \dot{\phi} \cos\phi}{m_1 + m_2} \rightarrow \ddot{x} = -\frac{m_2}{m_1 + m_2} l \overset{\cdot}{(\dot{\phi} \cos\phi)}$$

[Prob 3]

$\theta \rightarrow$ ~~dyn. vari.~~

$$\theta = \gamma t$$



$$(x, y) = (a \cos \gamma t + l \sin\phi,$$
$$a \sin \gamma t - l \cos\phi)$$

$$\dot{x} = -a\gamma\sin\gamma t + l\dot{\phi}\cos\phi$$

$$\dot{y} = a\gamma\cos\gamma t + l\dot{\phi}\sin\phi$$

$$T = \frac{1}{2}m\left(a^2\gamma^2 + l^2\dot{\phi}^2 + 2a\gamma l\dot{\phi}\sin(\phi-\gamma t)\right)$$

$$V = -mg\left(a\sin\gamma t - l\cos\phi\right)$$

$$L = \frac{1}{2}m\left(\cancel{a^2\gamma^2} + l^2\dot{\phi}^2 + 2a\gamma l\dot{\phi}\sin(\phi-\gamma t)\right)$$
$$+mg\left(\cancel{a\sin\gamma t} - l\cos\phi\right)$$

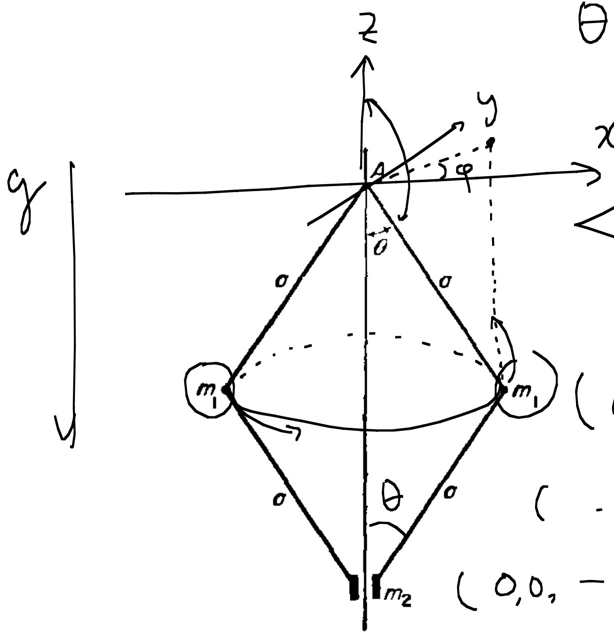
$$\frac{\partial L}{\partial\phi} = ma\gamma l\dot{\phi}\cos(\phi-\gamma t) + mgl\sin\phi$$

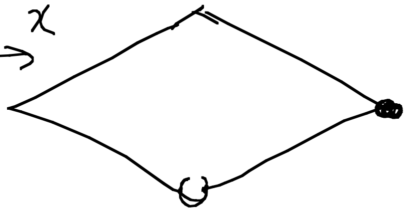
$$\frac{\partial L}{\partial\dot{\phi}} = ml^2\dot{\phi} + ma\gamma l\sin(\phi-\gamma t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} = ml^2\ddot{\phi} + ma\gamma l(\dot{\phi}-\gamma)\cos(\phi-\gamma t)$$

$$\hat{\theta} = \pi - \theta$$

## Prob4





$(a\sin\theta\cos\varphi, a\sin\theta\sin\varphi, -a\cos\theta)$

"

$( \cdot \quad \cdot \quad \cdot \quad \cdot )$

$(0, 0, -2a\cos\theta)$

$(0, 0, 2a\dot{\theta}\sin\theta)$

$$L = \frac{\overset{2\times}{m_1}}{2}a^2\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right) + 2m_2a^2\dot{\theta}^2\sin^2\theta$$
$$+2m_1ga\cos\theta + 2m_2ag\cos\theta$$

$$\frac{\partial L}{\partial \varphi} = 0 \longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = 0 \longrightarrow \frac{\partial L}{\partial \dot{\varphi}} = \text{const} = \ell$$

$$\|$$

$$2 m_1 a^2 \dot{\varphi} \sin^2\theta$$

$$\therefore \quad \dot{\varphi} = \frac{\ell}{2 m_1 a^2 \sin^2\theta}$$

$$\frac{\partial L}{\partial \theta} = 2 m_1 a^2 \dot{\varphi}^2 \sin\theta \cos\theta + 4 m_2 a^2 \dot{\theta}^2 \sin\theta \cos\theta$$

$$- 2(m_1 + m_2) g a \sin\theta \qquad \|$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(2 m_1 a^2 \dot{\theta} + 2 m_2 a^2 \dot{\theta} \sin^2\theta\right)$$

# Chap2. Conservation laws.

closed system : E , total momentm.

$$L = L(q_i, \dot{q}_i, t)_{i=1\cdots,s}$$

if $\frac{\partial L}{\partial q_k} = 0 \mapsto \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0 \rightarrow \frac{\partial L}{\partial \dot{q}_k} = \text{const}$

---

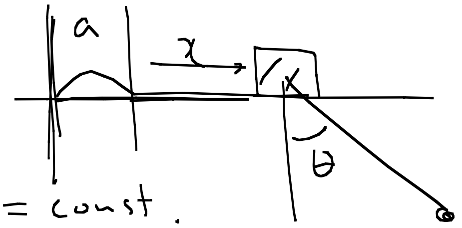
$\frac{\partial L}{\partial \dot{q}_i}$ = generalized momentum = $P_i$

$p_1, \cdots, P_s \xmapsto{\text{among these}} P_k$ is conserved

if $\frac{\partial L}{\partial q_k} = 0$

why $\frac{\partial L}{\partial x} = 0$ ?

$\hookrightarrow \frac{\partial L}{\partial \dot{x}} = P_x = \text{const}.$



$x \rightarrow x+a$ is a symmetry

$\updownarrow$

L is invariant under the transform.

§6. Energy

E is conserved if

$t \rightarrow t+a$ is sym.

$$L = L(q, \dot{q}, t)$$

$$\longmapsto L(q(t'), \underset{\parallel}{\dot{q}}, t' = t + a)$$

$$t' = t + a$$

$$\dot{q} = \frac{dq}{dt} = \boxed{\frac{dt'}{dt}}\frac{dq}{dt'} = \dot{q}$$

$$\therefore \text{ if } \quad L = L(q_i, \dot{q}_i, \cancel{t})$$

$\rightarrow$ L is invariat under $t \rightarrow t + a$

$$\frac{\partial f}{\partial t} \neq \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q}\dot{q}$$

$$f(q, t,$$

(ex) $f = x^2(t) + t$

$$\underbrace{\frac{\partial f}{\partial t}} = 1, \quad \underbrace{\frac{df}{dt}} = 2x\dot{x} + 1$$

---

no explicit dependence on t

$\updownarrow$

L is $L(q_i, \dot{q}_i, \cancel{t})$

$\swarrow$ $t \rightarrow t + a$ symmetry

$$\frac{\partial L}{\partial t} = 0 \iff \text{Enery is conserved}$$

---

$$\frac{dL}{dt} = \underbrace{\frac{\partial L}{\partial t}}_{\overset{\parallel}{0}} + \sum_{i=1}^{s} \dot{q}_i \underbrace{\frac{\partial L}{\partial q_i}}_{\uparrow} + \sum_{i=1}^{s} \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i}$$

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \checkmark$$

$$\frac{dL}{dt} = \sum_{i=1}^{s} \dot{q}_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \sum_{i=1}^{s} \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i}$$

$$= \frac{d}{dt} \sum_{i=1}^{s} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}$$

$$\frac{d}{dt}\left( \sum_{i=1}^{s}\left(\dot{q}_i \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i}\right) - L \right) = 0$$

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

$$\Rightarrow \underbrace{\sum_{i=1}^{s} \dot{q}_i p_i - L}_{\parallel} = \text{constant}$$

$$H = \text{Energy}$$

(ex)

$$L = \frac{m}{2}\dot{x}^2 - V(x) \rightarrow p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$m\dot{x}^2 - L = \frac{m}{2}\dot{x}^2 + V = E \checkmark$$

---

$$L = \frac{1}{2}\sum_{i,j=1}^{s} a_{ij}(q_1,\cdots q_s)\, \dot{q}_i \dot{q}_j - V(q_1 \cdots q_s)$$

$$a_{ij} = a_{ji}$$

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{1}{2}\sum_{i,j} a_{ij}\left(\delta_{ik}\dot{q}_j + \delta_{jk}\dot{q}_i\right)$$

$$= \frac{1}{2} \sum_j a_{kj} \dot{q}_j + \frac{1}{2} \sum_i a_{ik} \dot{q}_i$$

$$\left( \sum_{n=1}^{s} f_n \delta_{nk} = f_k \right) \qquad = \sum_i a_{ik} \dot{q}_i$$

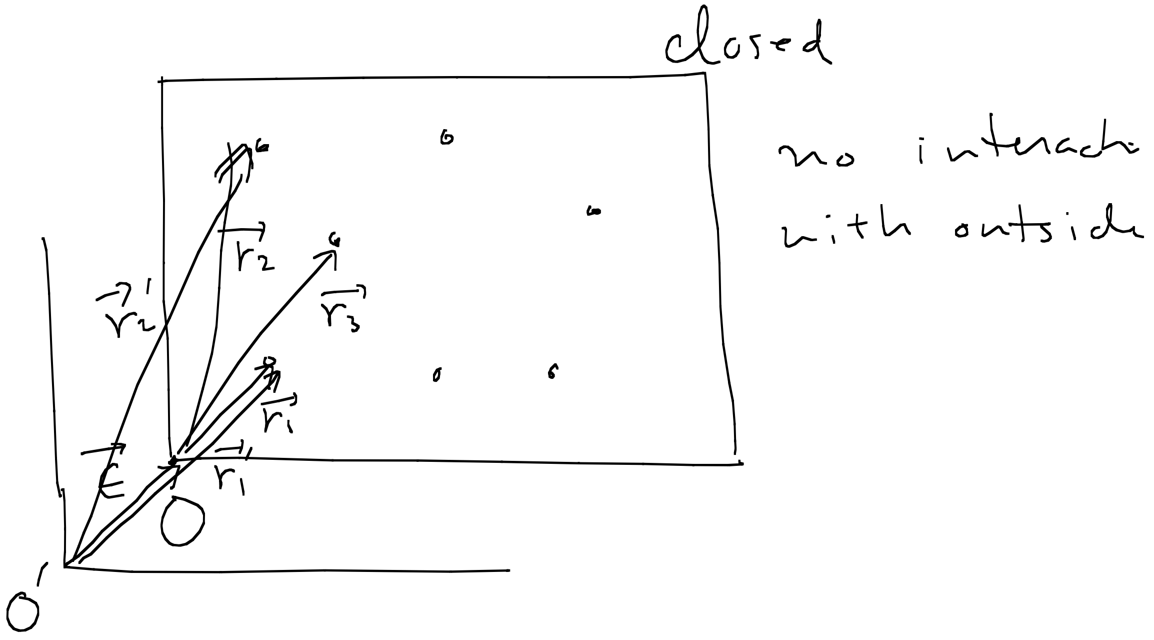
$$\sum_k \dot{q}_k p_k - L = \underbrace{\sum_k \dot{q}_k \left( \sum_i a_{ik} \dot{q}_i \right)} - L$$

$$= \sum_{ij} a_{ij} \dot{q}_i \dot{q}_j - L$$

$$\frac{1}{2} \sum_{ij} a_{ij} \dot{q}_i \dot{q}_j - V$$

$$= \underbrace{\frac{1}{2} \sum_{i,j} a_{ij} \dot{q}_i \dot{q}_j} + V = E$$

§7. momentum.



$$\vec{r}_a' = \vec{r}_a + \vec{\epsilon}$$

$\vec{r}_a \longrightarrow \vec{r}_a + \vec{\epsilon}$ is symmetry

for $a = 1, \cdots, s$

L is invariant.

$$\underbrace{L(\vec{r}_a+\vec{\epsilon})} - L(\cancel{\vec{r}_a}) = \delta L = 0$$

$$\delta L = L(\cancel{\vec{r}_a}) - \cancel{L(\vec{r}_a)} + \underbrace{\sum_a \vec{\epsilon}\cdot\frac{\partial L}{\partial \vec{r}_a}}_{\vec{\epsilon}\cdot\sum_a \frac{\partial L}{\partial \vec{r}_a}}$$

$$L(\cancel{\vec{r}_a}) + \sum_a \vec{\epsilon}\cdot\frac{\partial L}{\partial \vec{r}_a}$$

$$f(x+\epsilon) = f(x) + \epsilon f'(x)$$

for arbitrary $\vec{\epsilon}$

$$\therefore \quad \sum_a \frac{\partial L}{\partial \vec{r}_a} = 0 \quad \longleftrightarrow \quad \boxed{\sum_a \vec{F}_a = 0}$$

$$L = T(\dot{\vec{r}}_a) - V(\vec{r}_a)$$

$$\frac{\partial L}{\partial \vec{r}_a} = -\vec{\nabla}_a V(\vec{r}_a) = \vec{F}_a$$

---

$$\frac{\partial L}{\partial \vec{r}_a} = \frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot{\vec{r}}_a}} = \frac{d}{dt}\vec{p}_a$$

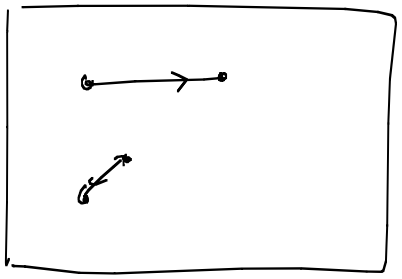
$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$\therefore \quad \sum_a \frac{d}{dt}\vec{p}_a = \frac{d}{dt}\sum_a \vec{p}_a = 0$$

$$\Rightarrow \quad \sum_a \vec{p}_a = \vec{P} = \text{constant}$$

Special case : no $\underbrace{V = 0}$

$$L = \sum_{a=1}^{s} T(\dot{\vec{r}}_a)$$



$$\vec{r}_a \longrightarrow \vec{r}_a + \vec{\epsilon}_a$$

$$\longrightarrow \quad \frac{\partial L}{\partial \dot{\vec{r}}_a} = \text{const} \qquad \frac{\partial L}{\partial \vec{r}_a} = 0 \quad a = 1, \cdots, s$$

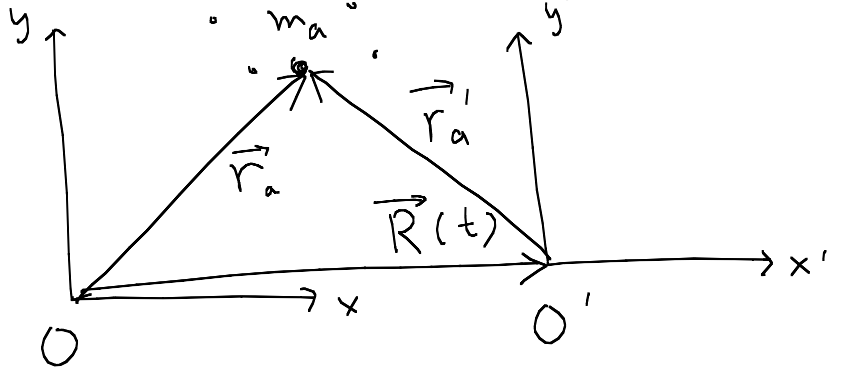
$$\longrightarrow \vec{p}_a = \text{const}$$

H.W. problem on page 16.

## §8. COM

frames : origin & axes

move relatively



$$\vec{r}_a = \vec{R} + \vec{r}_a'$$

$$\vec{v}_a = \frac{d\vec{r}_a}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{r}_a'}{dt} = \vec{V} + \vec{v}_a'$$

$$\vec{P} = \sum_a m_a \vec{v}_a = \sum_a m_a(\vec{V} + \vec{v}_a') = \overbrace{\sum_a m_a \vec{v}_a'}^{\vec{p}'} + \vec{V} \underbrace{\sum_a m_a}_{\mu}$$

choose moving frame O' such that $\vec{p}' = 0$

$$\vec V = \frac{\vec P}{\mu} = \frac{\sum m_a \boxed{\vec v_a}}{\mu} \leftarrow \frac{d\vec r_a}{dt} = \frac{d}{dt}\left(\frac{\sum_a m_a \vec r_a}{\mu}\right)$$

$O'$ : COM frame

$$\frac{d\vec R}{dt} \equiv \vec V = \frac{d}{dt}\left(\frac{\sum_a m_a \vec r_a}{\mu}\right)$$

$$\vec R = \frac{\sum_a m_a \vec r_a}{\mu}$$

$\uparrow$

COM

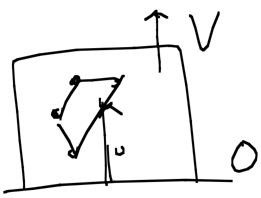
$\vec v_a = \vec V + \vec v_a'$

Kinetic energy

$$T = \sum_a \frac{1}{2} m_a \vec v_a^{\,2} = \sum_a \frac{1}{2} m_a \underbrace{(\vec V + \vec v_a')^2}_{\vec V^2 + \vec v_a'^{\,2} + 2\vec V\cdot\vec v_a'}$$

$$= \frac{1}{2}\mu V^2 + T' + \vec V\cdot \underbrace{\sum_a m_a \vec v_a'}_{0} \quad \because O' = COM$$

$$T = T' + \frac{1}{2}\mu V^2$$



potential egy $\begin{cases} \text{~~external~~ } U' \neq U \\ \underline{\underline{\text{internal}}} : |\vec r_a' - \vec r_b'| = |\vec r_a - \vec r_b| \\ \quad \therefore U' = U \end{cases}$

$$E = E' + \frac{1}{2}\mu V^2$$

(Prob) $L = T - U \qquad L' = T' - U'$

$$L = T' + \frac{1}{2}\mu V^2 - U' = L' + \frac{1}{2}\mu V^2 + \underset{\mu \vec V\cdot \underset{\uparrow \atop \frac{d\vec R'}{dt}}{\underbrace{\vec p'}}}{\overset{}{0}}$$

$$S = \int L\,dt = S' + \underbrace{\frac{1}{2}\mu V^2 t + \mu \vec V\cdot\vec R'}$$

$\therefore$ same e o m.

## §9. Ang. Mom.

$$L = L(q, \dot{q}, t)$$

$$t \to t + \varepsilon \to \frac{\partial L}{\partial t} = 0 \to \frac{dL}{dt} = \dot{q}\underbrace{\frac{\partial L}{\partial q}}_{}^{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}} + \ddot{q}\frac{\partial L}{\partial \dot{q}}$$

$$= \frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}}\right)$$

$$\therefore H = -L + \dot{q}\frac{\partial L}{\partial \dot{q}} \to \frac{dH}{dt} = 0$$

$$q \to q + \delta q \to \delta S = 0 \quad ; \text{ symmetry}$$

$$S = \int L\, dt \qquad \delta S = \int \left[L(q+\delta q, \dot{q}+\delta\dot{q}, t) - L(q, \dot{q}, t)\right] dt$$

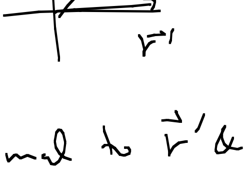
$$\delta q \underbrace{\frac{\partial L}{\partial q}}_{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}} + \delta\dot{q}\frac{\partial L}{\partial \dot{q}} = \frac{d}{dt}\left(\delta q \frac{\partial L}{\partial \dot{q}}\right) = 0 \leftrightarrow \delta S = 0$$

$$\Rightarrow \delta q \frac{\partial L}{\partial \dot{q}} = \text{conserved}$$

## Nöther theorem

- rotation invariance $\quad L = L(\vec{r}, \dot{\vec{r}}, t)$

$$\delta\vec{r} = \vec{r}' - \vec{r}$$



$\delta\vec{\phi}$ ; $|\delta\vec{\phi}| = \delta\phi$

direction : normal to $\vec{r}'$ & $\vec{r}$

$$\boxed{\delta\vec{r} = \delta\vec{\phi} \times \vec{r}} \to |\delta\vec{r}| = \delta\phi\, r \quad ✓$$

↑ no time dependence

$$\delta\vec{v} = \delta\vec{\phi} \times \vec{v}$$

$$L = L(\vec{r}_a, \vec{V}_a, t)$$

$$\delta L = \sum_a \left( \frac{\partial L}{\partial \vec{r}_a} \cdot \underbrace{\delta \vec{r}_a}_{\delta\vec{\phi} \times \vec{r}_a} + \frac{\partial L}{\partial \vec{V}_a} \cdot \underbrace{\delta \vec{V}_a}_{\delta\vec{\phi} \times \vec{V}_a} \right) = 0$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A})$$

$$= \delta\vec{\phi} \cdot \sum_a \underbrace{\left( \vec{r}_a \times \underbrace{\frac{\partial L}{\partial \vec{r}_a}}_{\frac{d}{dt}\frac{\partial L}{\partial \vec{V}_a}} + \vec{V}_a \times \frac{\partial L}{\partial \vec{V}_a} \right)}_{\frac{d}{dt}\left(\vec{r}_a \times \frac{\partial L}{\partial \vec{V}_a}\right)}$$

$$= \delta\vec{\phi} \cdot \underbrace{\frac{d}{dt}\left( \sum_a \vec{r}_a \times \frac{\partial L}{\partial \vec{V}_a} \right)}_{\overset{\shortparallel}{0}} = 0$$

for arbitrary $\delta\vec{\phi}$

$$\therefore \quad \sum_a \vec{r}_a \times \underbrace{\frac{\partial L}{\partial \vec{V}_a}}_{\vec{p}_a} = \text{conserved}.$$

$$L = \frac{1}{2}\sum_a m_a \vec{V}_a^2 - U(\vec{r}_a)$$

$$\frac{\partial L}{\partial \vec{V}_a} = m_a \vec{V}_a = \vec{p}_a$$

$\vec{r}_a \times \vec{p}_a = \vec{m}_a$ : angular momentum

$$\sum_a \vec{m}_a = \text{conserved} = \sum_a \vec{r}_a \times \vec{p}_a = \vec{M}$$

$E, \vec{P}, \vec{M}$ conserved.

COM frame ; (cf) $\vec{P} = \mu \vec{V}$ , $\vec{P}' = 0$

$$\vec{M} = \sum_a \vec{r}_a \times \vec{p}_a = \sum_a (\vec{r}_a{}' + \vec{R}) \times \underbrace{\vec{p}_a}_{m_a \vec{v}_a}$$

$$m_a \vec{v}_a \shortparallel m_a(\vec{V} + \vec{v}_a{}') \shortparallel m_a\vec{V} + \vec{p}_a{}'$$

$$= \sum_a (\vec{r}_a{}' + \vec{R}) \times (m_a \vec{V} + \vec{p}_a{}')$$

$$= \underbrace{\sum_a \vec{r}_a{}' \times \vec{p}_a{}'}_{\vec{M}' :\ \text{total A.M. in COM}} + \underbrace{\mu \vec{R} \times \vec{V}}_{\vec{R} \times \vec{P}} + \underbrace{\left(\sum_a m_a \vec{r}_a{}'\right)}_{\overset{\shortparallel}{0}} \times \vec{V} + \vec{R} \times \underbrace{\sum_a \vec{p}_a{}'}_{0}$$

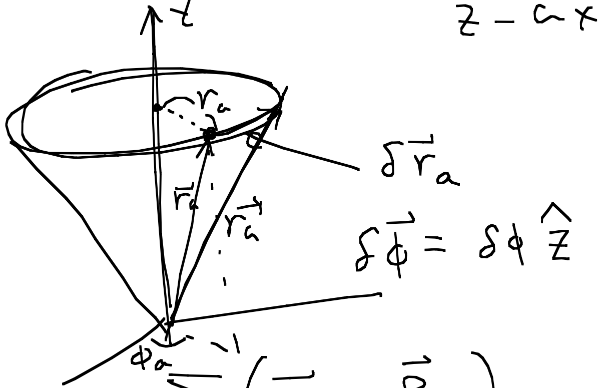
$$\therefore \quad \vec{M} = \vec{M}' + \vec{R} \times \vec{P}$$

$$\frac{d\vec{M}}{dt} = 0 = \frac{d\vec{M}'}{dt} \quad \because \quad \underset{\overset{\shortparallel}{\frac{\vec{P}}{\mu}}}{\vec{V}} \times \vec{P} + \vec{R} \times \underbrace{\dot{\vec{P}}}_{\overset{\shortparallel}{0}} = 0$$

---

rotational inv. about specific axis

z-axis : $\phi$



$$M^z_a = \sum_a \frac{\partial L}{\partial \dot{\phi}_a} = \sum_a (\vec{r}_a \times \underset{\overset{\shortparallel}{m_a \vec{v}_a}}{\vec{p}_a})_z$$

$$= \sum_a m_a (x_a \dot{y}_a - y_a \dot{x}_a)$$

Cylindrical coordinate

$x_a = r_a \cos\phi_a \quad y_a = r_a \sin\phi_a \ , z_a = z_a$

$$\left( L = \sum_a \frac{1}{2} m_a (\dot{r}_a^2 + r_a^2 \dot{\phi}_a^2 + \dot{z}_a^2) - U(r) \right.$$

no $\phi$ dependence

$$\frac{\delta L}{\delta \phi_a} = 0 \rightarrow \sum_a \frac{\partial L}{\partial \dot{\phi}_a} = \text{const}$$

$$\downarrow$$

$$\sum_a m_a r_a^2 \dot{\phi}_a = \text{conserved})$$

$$\dot{x}_a = \dot{r}_a \cos\phi_a - r_a \dot{\phi}_a \sin\phi_a$$

$$\dot{y}_a = \dot{r}_a \sin\phi_a + r_a \dot{\phi}_a \cos\phi_a$$

$$x_a \dot{y}_a - y_a \dot{x}_a = r_a \cos\phi_a \left( \dot{r}_a \cancel{\sin}\phi_a + r_a \dot{\phi}_a \cos\phi_a \right)$$

$$- r_a \sin\phi_a \left( \dot{r}_a \cancel{\cos}\phi_a - r_a \dot{\phi}_a \sin\phi_a \right)$$

$$= r_a^2 \dot{\phi}_a$$

$$M^z = \sum_a m_a r_a^2 \dot{\phi}_a$$

H.W. Probs on page 22.

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## §10. Mechanical similarity

$$\frac{\partial L}{\partial q_a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} = 0$$

$$L \to L' = L \times c$$

↑ const.

correlation length $= l \to \infty$

scale invariance

→ critical phenomena

( 임계현상 )

scale transf.

$$L = T - U$$

$$\vec{r}_a \to \vec{r}_a' = \alpha \vec{r}_a$$

↑ a-independent

scale covariance

$$U(\vec{r}_1 \cdots \vec{r}_n) \longrightarrow U(\alpha \vec{r}_1, \cdots, \alpha \vec{r}_n) = \alpha^{\textcircled{k}} U(\vec{r}_1, \cdots \vec{r}_n)$$

(ex) H.O. $U = \sum_a \frac{1}{2} K \vec{r}_a^{\,2} \longrightarrow U = \alpha^{\textcircled{2}} \sum_a \frac{1}{2} K \vec{r}_a^{\,2}$

$$U = \sum_a \frac{c}{|\vec{r}_a|} \longrightarrow k = -1$$

introduce : scale trasf in time

$$t \to \beta t$$

$$T = \sum_a \frac{1}{2} m_a \left(\frac{d\vec{r}_a}{dt}\right)^2 \longrightarrow \sum_a \frac{1}{2} m_a \left(\frac{d(\alpha \vec{r}_a)}{d(\beta t)}\right)^2$$

$$= \frac{\alpha^2}{\beta^2} \sum_a \frac{1}{2} m_a \dot{\vec{r}}_a^{\,2}$$

$$L = T - U \longrightarrow L' = T' - U' = \frac{\alpha^2}{\beta^2} T - \alpha^k U$$

$$= c\, L$$

$$\underline{\underline{\frac{\alpha^2}{\beta^2} = \alpha^k}} \longrightarrow \beta^2 = \alpha^{2-k} \longrightarrow \beta = \alpha^{1-\frac{k}{2}}$$

$$\vec{r}_a \to \alpha \vec{r}_a,\ t \to \underbrace{\alpha^{1-\frac{k}{2}} t}_{t'}$$

$$l' = \alpha l \longrightarrow \alpha = \frac{l'}{l} \Rightarrow \frac{t'}{t} = \underbrace{\left(\frac{l'}{l}\right)^{1-\frac{k}{2}}}$$

↑ Spatial length

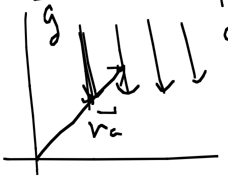
$$\frac{v'}{v} = \frac{l'/t'}{l/t} = \frac{l'}{l} \cdot \underbrace{\frac{t}{t'}}_{\left(\frac{l'}{l}\right)^{-1+\frac{k}{2}}} = \left(\frac{l'}{l}\right)^{\frac{k}{2}}$$

$$\frac{E'}{E} = \frac{v'^2}{v^2} = \left(\frac{l'}{l}\right)^k, \quad \frac{M'}{M} = \frac{l'v'}{lv} = \left(\frac{l'}{l}\right)^{1+\frac{k}{2}}$$

$k=2$ (H.O.) → $\frac{t'}{t} = 1$ ← scale invariant

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

$k=1$ $\quad U = c|\vec{r}_a|$



c

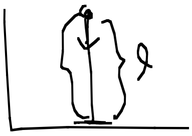
uniform

$$U = \vec{g} \cdot \vec{r}_a$$

$k=-1$

$\left(\frac{t'}{t}\right)^2=\left(\frac{l'}{l}\right)^3$ ← kepler 3rd law

$$\frac{t'}{t}=\sqrt{\frac{l'}{l}}$$



$l=\frac{1}{2}gt^2$

$t=\sqrt{\frac{2l}{g}}$

$\frac{t'}{t}=\sqrt{\frac{l'}{l}}$

---

$x+it=z$ , $x-it=\bar{z}$

$$\begin{pmatrix} t' \\ x' \end{pmatrix}=\begin{pmatrix} ch\beta & sh\beta \\ -sh\beta & ch\beta \end{pmatrix}\begin{pmatrix} t \\ x \end{pmatrix}$$

$t'=\gamma(x+vt)$ Lorentz

$x'=\gamma(x-vt)$

$\gamma=ch\beta$, $v\gamma=sh\beta$

$x\rightarrow\alpha x$

↳ $z\rightarrow f(z)$ , $\bar{z}\rightarrow f(\bar{z})$ : conformal transf.

→ ∞ 차원 symmetry

rotation $\vec{L}$ → generator → $[L_i,L_j]=i\epsilon_{ijk}L_k$

$O(3)$ ← $L_x, L_y, L_z$

algebra

↳ Virasoro algebra ; $L_n$ $n=-\infty,\cdots,\infty$

$[L_n,L_m]=(n-m)L_{n+m}+\cdots$

---

Virial theorem

$$\langle T\rangle=\langle U\rangle$$

$$\bar{f}=\frac{1}{\tau}\int_0^\tau f(t)\,dt$$

↑ time average

$T=\frac{1}{2}\sum_a \vec{v}_a\cdot\vec{p}_a \rightarrow 2T=\frac{d}{dt}\left(\sum_a \vec{p}_a\cdot\vec{r}_a\right)-\sum_a \vec{r}_a\cdot\dot{\vec{p}}_a$

$2\overline{T}=-\overline{\sum_a \vec{r}_a\cdot\dot{\vec{p}}_a}$

$\overline{\frac{dg}{dt}}=\frac{1}{\tau}\int_0^\tau \frac{dg}{dt}\,dt$

$=\frac{1}{\tau}\left[g(\tau)-g(0)\right]$

$=0$

$\dot{\vec{p}}_a=m_a\ddot{\vec{r}}_a=\vec{F}_a=-\vec{\nabla}_a U$

$=-\frac{\partial U}{\partial \vec{r}_a}$

$$2\overline{T}=\overline{\sum_a \vec{r}_a\cdot\frac{\partial U}{\partial \vec{r}_a}}=k\overline{\sum_a U(\vec{r}_a)}=k\overline{U}$$

$$U(\vec r) = c|\vec r|^k \rightarrow \frac{\partial U}{\partial \vec r} = \frac{\partial U}{\partial |\vec r|}\frac{\partial |\vec r|}{\partial \vec r} = \frac{\partial U}{\partial r}\frac{\vec r}{r}$$

$$|\vec r| = \sqrt{x^2+y^2+z^2} \qquad \frac{\partial |\vec r|}{\partial \vec r} = \vec\nabla(|\vec r|) = \frac{\cancel{2}\vec r}{\cancel{2} r} = \frac{\vec r}{r}$$

$$\frac{\partial U}{\partial r} = kC|\vec r|^{k-1} \longrightarrow \vec r \cdot \frac{\partial U}{\partial \vec r} = k\, C|\vec r|^{k-2} \underbrace{\vec r \cdot \vec r}_{r^2}$$

$$\underline{\underline{=}}\, k \underbrace{c|\vec r|^k}_{U}$$

$$2\bar T = \underline{\underline{k}}\, \bar U$$

$$\bar T + \bar U = \bar E = \left(\frac{k}{2}+1\right)\bar U \longrightarrow \bar U = \frac{\boxed{\bar E}^{\,=E}}{\frac{k}{2}+1}$$

$$\bar T\left(1+\frac{2}{k}\right) \longrightarrow \bar T = \frac{E}{1+\frac{2}{k}}$$

$\underline{k=2}$ $\quad \bar T = \bar U = \frac{E}{2}$ ← equipartition theorem.

$\underline{k=-1}$ $\quad \bar U = 2E$ ✓ Hydrogen atom

$\bar T = -E \rightarrow E < 0$

H.W.

Probs on 24

# Chap 3.

§11. one dimension → $q$

$$L = \frac{1}{2} a(q) \dot{q}^2 - U(q)$$

$$\frac{\partial L}{\partial q} = \frac{1}{2} a' \dot{q}^2 - U' = \frac{d}{dt}(a \dot{q})$$

simplest case : $q = x$ $\quad L = \frac{1}{2} m \dot{x}^2 - U(x)$

$$\dot{x} \times \left( \underline{m \ddot{x} = - U'(x)} \right) \Rightarrow m \dot{x} \ddot{x} = - \dot{x} U'(x)$$

$$\downarrow$$

$$\frac{d}{dt}\left(\frac{1}{2} m \dot{x}^2 + U\right) = 0 \quad \longleftarrow \quad \frac{m}{2} \frac{d}{dt}(\dot{x}^2) = - \frac{d}{dt} U(x)$$

$$\therefore \underline{\frac{1}{2} m \dot{x}^2 + U = E} \leftarrow \text{const} \qquad *$$

$$\rightarrow \quad \frac{dx}{dt} = \sqrt{\frac{2}{m}(E - U(x))}$$

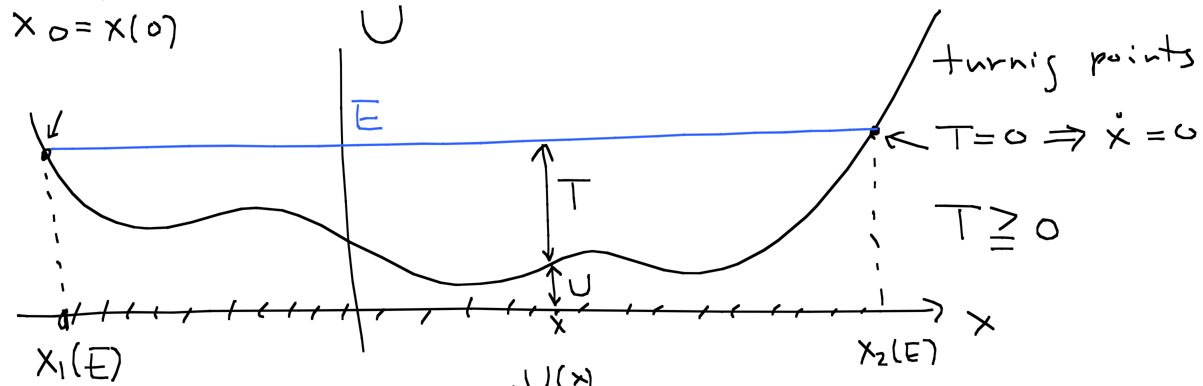
$\longrightarrow$ +

$\longleftarrow$ −

$t \to -t$

time inversion sym.

$$\int_{x_0 = x(0)}^{x} \frac{1}{\sqrt{\frac{2}{m}(E - U(x))}} dx = \int_0^t dt = t$$



stability :

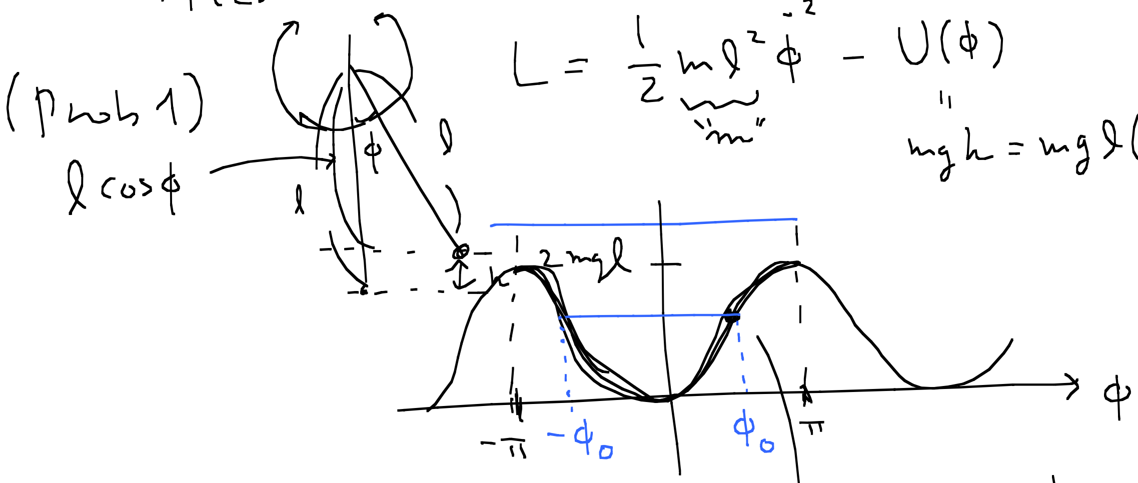
U: bounded from below

→ no periodic motion.

$$\sqrt{\frac{m}{2}}\int_{x_1(E)}^{x_2(E)}\frac{dx}{\sqrt{E-U(x)}}=\frac{T(E)}{2}$$ ← period.

$$L=\frac{1}{2}\underbrace{ml^2}_{\text{"}m\text{"}}\dot{\phi}^2-U(\phi)$$

$$\overset{\shortparallel}{mgh}=mgl\underbrace{(1-\cos\phi)}_{2\sin^2\frac{\phi}{2}}$$



$$E=2mgl\sin^2\frac{\phi_0}{2}$$

$$0<E<2mgl$$

$$\frac{T}{2}=\sqrt{\frac{\cancel{m}l^{\cancel{2}}}{2}}\underbrace{\int_{-\phi_0}^{\phi_0}}_{2\int_0^{\phi_0}}\frac{d\phi}{\sqrt{2\cancel{m}g\cancel{l}}\sqrt{\sin^2\frac{\phi_0}{2}-\sin^2\frac{\phi}{2}}}$$

$$4\sqrt{\frac{l}{g}}K\left(\sin\frac{\phi_0}{2}\right)$$

$$T=2\sqrt{\frac{l}{g}}\int_0^{\phi_0}\frac{d\phi}{\sqrt{\sin^2\frac{\phi_0}{2}-\sin^2\frac{\phi}{2}}}=2\sqrt{\frac{l}{g}}\frac{1}{\cancel{\sin\frac{\phi_0}{2}}}\int_0^{\frac{\pi}{2}}\frac{2\cancel{\sin\frac{\phi_0}{2}}\cancel{\cos\xi}\,d\xi}{\cancel{\sqrt{1-\sin^2\xi}}\sqrt{1-\sin^2\frac{\phi_0}{2}\sin^2\xi}}$$

Elliptic integrals ⇒ $\left\{\begin{array}{l}\text{complete integral of 1st kind}\\ \vdots\end{array}\right.$

$$\text{"}K\text{"}(k)\equiv\int_0^{\frac{\pi}{2}}\frac{d\xi}{\sqrt{1-k^2\sin^2\xi}}$$

sin, exp, ....

$$\sin\xi\equiv\frac{\sin\frac{\phi}{2}}{\sin\frac{\phi_0}{2}}\rightarrow\cos\xi\,d\xi=\frac{1}{\sin\frac{\phi_0}{2}}\left(\cos\frac{\phi}{2}\right)\frac{d\phi}{2}$$

$$d\phi=2\sin\frac{\phi_0}{2}\frac{\cos\xi\,d\xi}{\sqrt{1-\underbrace{\sin^2\frac{\phi_0}{2}\sin^2\xi}_{\sin^2\frac{\phi}{2}}}}$$

$K(m)$

$\phi_0 \ll 1 \qquad \sin\frac{\phi_0}{2} \ll 1 \qquad m \ll 1$

$$T = 4\sqrt{\frac{l}{g}}\left(\frac{\pi}{2} + \frac{\pi}{8}\sin^2\frac{\phi_0}{2} + \cdots\right).$$

---

H.W. $\qquad q \to q + \eta$

$$\frac{1}{2}\int\left(\underline{\frac{\partial^2 L}{\partial q^2}}\,\eta^2 + \underline{\frac{\partial^2 L}{\partial \dot q^2}}\,\dot\eta^2 + \underbrace{2\frac{\partial^2 L}{\partial q\,\partial \dot q}\,\eta\dot\eta}\right)dt$$

$$\frac{\partial^2 L}{\partial q\,\partial \dot q}\,\frac{d}{dt}(\eta^2)$$

$$= \frac{d}{dt}(\quad) - \eta^2\,\underline{\frac{d}{dt}\left(\frac{\partial^2 L}{\partial q\,\partial \dot q}\right)}$$

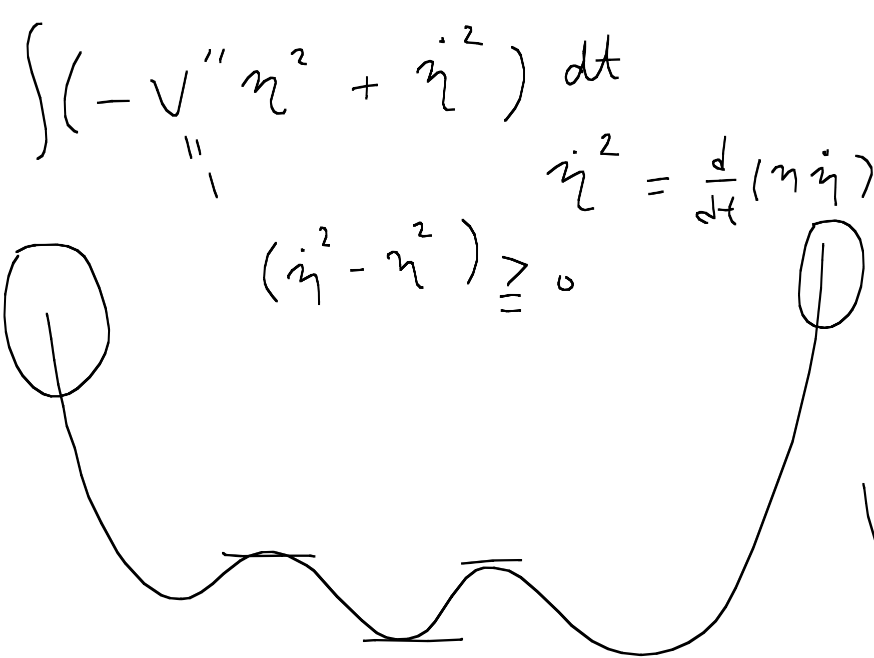
$$\frac{\partial^2 L}{\partial q^2} - \frac{d}{dt}\frac{\partial^2 L}{\partial q\,\partial \dot q} \leftarrow L = \frac{1}{2}\dot q^2 - V(q)$$

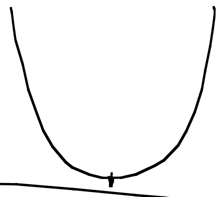
$$\left(\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot q}\right) \longrightarrow \frac{\partial^2 L}{\partial q^2} = -V''$$

$$= \frac{1}{2}\int\left(-\underset{\overset{\shortparallel}{1}}{V''}\,\eta^2 + \dot\eta^2\right)dt$$

$$\dot\eta^2 = \frac{d}{dt}(\eta\dot\eta) - \eta\ddot\eta$$

$$(\dot\eta^2 - \eta^2) \geqq 0$$



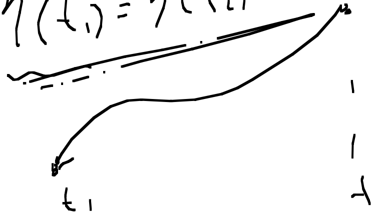


$f(x) = (x^2-1)^2, \quad x^2$

$f'(x) = 0 \longrightarrow x = 1, -1, 0$

$$\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} = 0}$$

$\eta(t_1) = \eta(t_2) = 0$



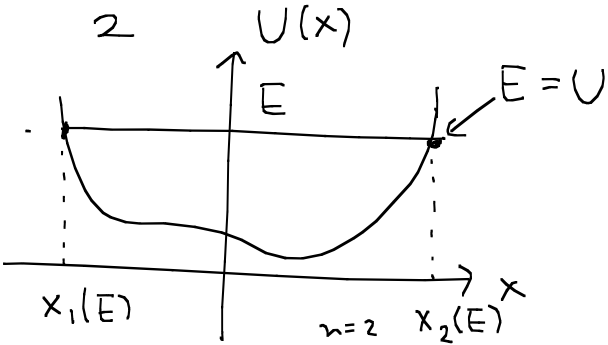
$\eta(t) = c\,t + d$
$= c(t - t_1)$

E-L Eq : unique

---

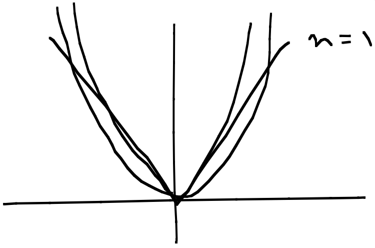
$$\sqrt{\frac{m}{2}} \int_{x_1(E)}^{x_2(E)} \frac{dx}{\sqrt{E - U(x)}} = \frac{T(E)}{2}$$

period.



Prob 2.

(a) $U = A|x|^n$



$$4\sqrt{\frac{m}{2}} \int_0^{x(E)} \frac{dx}{\sqrt{E - Ax^n}} = T$$

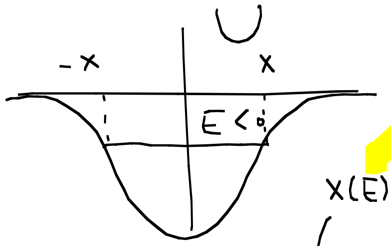
$$4\sqrt{\frac{m}{2}} \int_0^{x(E)} \frac{1}{\sqrt{E}} \frac{dx}{\sqrt{1 - \underbrace{\frac{Ax^n}{E}}_{y^n}}} = T$$

$y \equiv x\left(\frac{A}{E}\right)^{\frac{1}{n}}$

$dx = \left(\frac{E}{A}\right)^{\frac{1}{n}} dy$

$$T = 4\sqrt{\frac{m}{2E}} \left(\frac{E}{A}\right)^{\frac{1}{n}} \underbrace{\int_0^1 \frac{1}{\sqrt{1 - y^n}} dy}_{\sqrt{\pi}\,\frac{\Gamma(1+\frac{1}{n})}{\Gamma(\frac{1}{2}+\frac{1}{n})}} \propto E^{\frac{1}{n} - \frac{1}{2}}$$

(b) $U = -\frac{U_0}{\cosh^2 \alpha x}$ $\quad -U_0 < E < 0$



$$T = 4\sqrt{\frac{m}{2}} \int_0^{x(E)} \frac{dx}{\underbrace{\sqrt{E + \frac{U_0}{\cosh^2 \alpha x}}}} \qquad E = -|E|$$

$y = \sinh \alpha x$

$dy = \alpha \cosh \alpha x \, dx$

$$\frac{\cosh \alpha x \, dx}{\sqrt{U_0 + E \underbrace{\cosh^2 \alpha x}_{(1+\sinh^2 \alpha x)}}} = \frac{dy/\alpha}{\sqrt{\underbrace{\underbrace{U_0 + E + E y^2}_{U_0 - |E| - |E| y^2}}_{\vee \, 0}}}$$

$$= \frac{1}{\alpha\sqrt{U_0 - |E|}} \frac{dy}{\sqrt{\underset{\uparrow}{1} - \underbrace{\frac{|E|}{U_0 - |E|} y^2}}}$$

$$\rightarrow \sqrt{\frac{|E|}{U_0 - |E|}}\, y \equiv \sin z$$

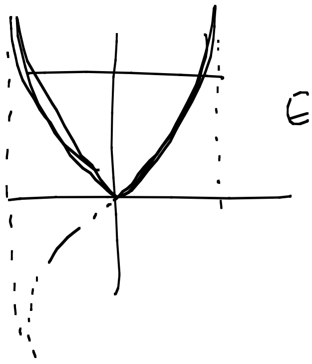
$$\downarrow$$

$$\boxed{\sqrt{\frac{|E|}{U_0 - |E|}}}\, dy = \cos z \, dz$$

$$T = 4\sqrt{\frac{m}{2}} \int_0^{Z(E)} \frac{1}{\alpha \cancel{\sqrt{U_0 - |E|}}} \sqrt{\frac{\cancel{U_0 - |E|}}{|E|}} \; \overset{\sin^2 z}{\frac{\cancel{\cos z}\, dz}{\cancel{\cos z}}}$$

$$= 4\sqrt{\frac{m}{2}} \frac{1}{\alpha\sqrt{|E|}} \underset{\frac{\pi}{2}}{\underset{\parallel}{Z(E)}} = \frac{\pi}{\alpha}\sqrt{\frac{2m}{|E|}}$$

(c) $U = U_0 \tan^2 \alpha x$



$$T = 4\sqrt{\frac{m}{2}} \int_0^{X(E)} \frac{dx}{\underbrace{\sqrt{E - U_0 \tan^2 \alpha x}}}$$

$$\frac{dy}{\alpha}$$

$$\parallel$$

$$\frac{\cos \alpha x \, dx}{\sqrt{E \underbrace{\cos^2 \alpha x}_{1 - \sin^2 \alpha x} - U_0 \sin^2 \alpha x}} = \frac{\boxed{\cos \alpha x \, dy}}{\sqrt{E - (E + U_0)\sin^2 \alpha x}}$$

$\sin \alpha x \equiv y$

$dy = \alpha \cos \alpha x \, dx$

$$= 4\sqrt{\frac{m}{2}} \int_0^{y(E)} \frac{dy/\alpha}{\sqrt{E - (E + U_0) y^2}}$$

$$= 4\sqrt{\frac{m}{2}}\,\frac{1}{\alpha\sqrt{E}}\int\frac{dy}{\sqrt{1-\underbrace{\cancel{\frac{E+U_0}{E}y^2}}_{\sin^2 z}}}$$

$$y=\sqrt{\frac{E}{E+U_0}}\sin z$$

$$dy=\sqrt{\;''\;}\;\cancel{\cos z}\,dz$$

$$= 4\sqrt{\frac{m}{2}}\,\frac{1}{\alpha\cancel{\sqrt{E}}}\sqrt{\frac{\cancel{E}}{E+U_0}}\,\frac{\pi}{2} = \frac{\pi}{\alpha}\sqrt{\frac{2m}{E+U_0}}\;//$$

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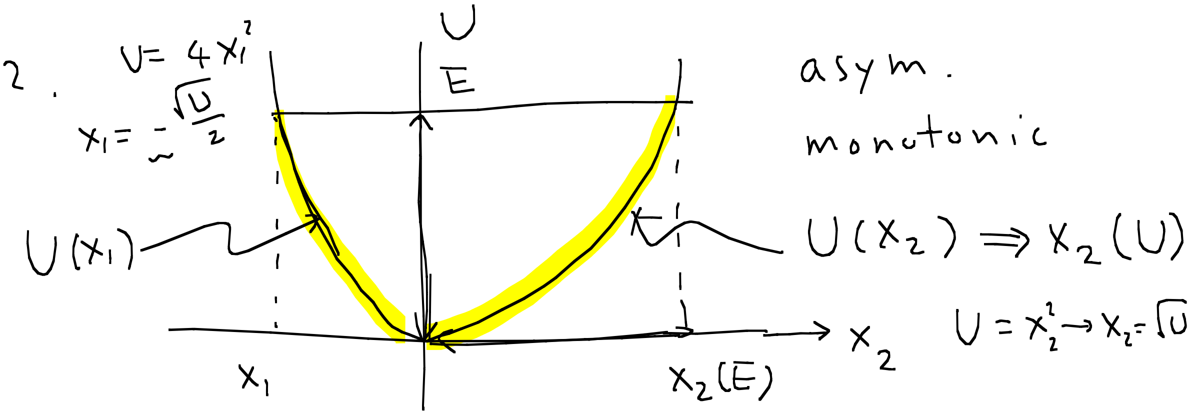
§12.



Inverse problem : period $\longrightarrow U$

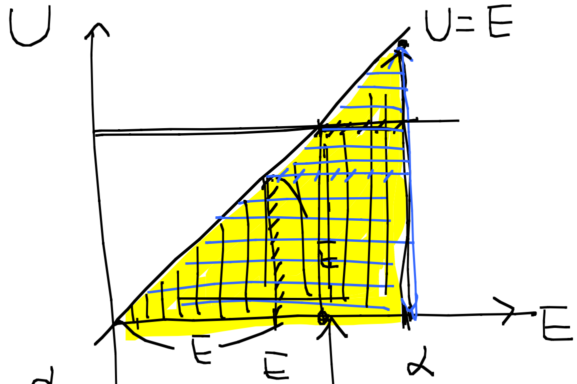
$$T(E)=\sqrt{2m}\int_{x_1(E)}^{x_2(E)}\frac{dx}{\sqrt{E-U}}$$

$$=\sqrt{2m}\left(\int_{x_1}^{0}\frac{dx}{\sqrt{E-U}}+\underbrace{\int_{0}^{x_2}\frac{dx}{\sqrt{E-U}}}\right)$$

$$\underbrace{\int_E^0}_{-\int_0^E}{}^{''}\;\frac{\frac{dx_1}{dU}\,dU}{\sqrt{E-U}} \qquad \underbrace{\int_0^E}{}^{''}\;\frac{\frac{dx_2}{dU}\,dU}{\sqrt{E-U}}$$

$$=\sqrt{2m}\underbrace{\int_0^E}\left(\frac{dx_2}{dU}-\frac{dx_1}{dU}\right)\frac{dU}{\sqrt{E-U}}$$

$$\underbrace{\int_0^\alpha\frac{\boxed{T(E)}\,dE}{\sqrt{\alpha-E}}}=\sqrt{2m}\underbrace{\int_0^\alpha dE\int_0^E dU}\underbrace{\left(\frac{dx_2}{dU}-\frac{dx_1}{dU}\right)\frac{1}{\sqrt{\alpha-E}\sqrt{E-U}}}_{f(E,U)}$$



$$\underbrace{\int_0^\alpha dU \int_U^\alpha dE}\ f(E,U)$$

$$= \sqrt{2m}\int_0^\alpha dU \int_U^\alpha dE \underbrace{\left(\frac{dx_2}{dU}-\frac{dx_1}{dU}\right)}_{\text{function of only } U} \frac{1}{\sqrt{\alpha-E}}\,\frac{1}{\sqrt{E-U}}$$

$$= \sqrt{2m}\ \underbrace{\int_0^\alpha dU\left(\frac{dx_2}{dU}-\frac{dx_1}{dU}\right)}\ \underbrace{\int_U^\alpha dE\, \frac{1}{\sqrt{\alpha-E}}\,\frac{1}{\sqrt{E-U}}}_{\pi}$$

$$= \sqrt{2m}\ \pi\ \left( x_2(U)\Big|_0^\alpha - x_1\Big|_0^\alpha \right)$$

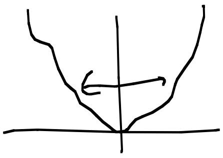
$$= \pi\sqrt{2m}\ \left(x_2(\alpha) - x_1(\alpha)\right) \quad \text{for any } \alpha$$

$$\Rightarrow \underbrace{x_2(U)}_{\Uparrow} - \underbrace{x_1(U)}_{} = \frac{1}{\pi\sqrt{2m}}\int_0^U \frac{T(E)\,dE}{\sqrt{U-E}}$$

potential

$\Uparrow$ if $\underbrace{T(E)}$ is given

if sym.



$x_1 = -x_2$

$$x_2(U) = \frac{1}{2\pi\sqrt{2m}}\int_0^U \frac{T(E)\,dE}{\sqrt{U-E}}$$

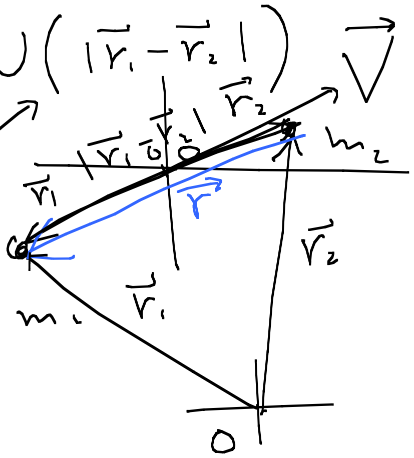
(ex) $= U^2 \rightarrow U = \sqrt{x}$

## §13. reduced mass (two-body)

$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(|\vec{r}_1 - \vec{r}_2|)$$

$$\underline{\vec{r} = \vec{r}_1 - \vec{r}_2}$$

$U(r)$



$$\text{CM}: \vec{R} \equiv \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \underline{= 0}$$

$$\vec{P} = (m_1 + m_2)\vec{V} = \text{const}.$$

$$m_1 \vec{r} = m_1 \vec{r}_1 - m_1 \vec{r}_2$$

$$-) \quad 0 = m_1 \vec{r}_1 + m_2 \vec{r}_2$$

$$m_1 \vec{r} = -(m_1 + m_2)\vec{r}_2 \rightarrow \vec{r}_2 = -\frac{m_1}{m_1 + m_2} \vec{r}$$

$$m_2 \vec{r} = m_2 \vec{r}_1 - m_2 \vec{r}_2$$

$$+) \quad 0 = m_1 \vec{r}_1 + m_2 \vec{r}_2$$

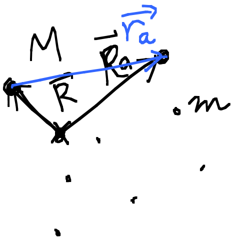
$$m_2 \vec{r} = (m_1 + m_2)\vec{r}_1 \rightarrow \vec{r}_1 = \frac{m_2}{m_1 + m_2} \vec{r}$$

$$L = \underbrace{\frac{1}{2} m_1 \left( \frac{m_2}{m_1 + m_2} \dot{\vec{r}} \right)^2 + \frac{1}{2} m_2 \left( -\frac{m_1}{m_1 + m_2} \dot{\vec{r}} \right)^2} - U(r)$$

$$= \frac{1}{2} \underbrace{\frac{m_1 m_2}{m_1 + m_2}}_{m} \dot{\vec{r}}^{\,2} - U(r)$$

"reduced mass"

(prob)



$$0 = M\vec{R} + \sum_{a=1}^{n} \underbrace{\vec{R}_a}_{\vec{R} + \vec{r}_a} m$$

$$'' \quad \overbrace{(M + nm)}^{\mu} \vec{R} + m \sum_a \vec{r}_a$$

$$\vec{R} + \vec{r}_a = \vec{R}_a$$

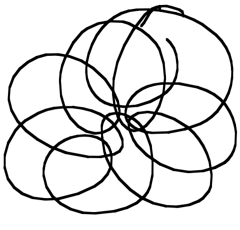
$$\boxed{\vec{R} = -\frac{m}{\mu} \sum_a \vec{r}_a}$$

$$M \gg m \rightarrow \frac{m}{\mu} \ll 1$$

# §14. Central field

KAM Theorem.

chaotic



3 d.o.f

2 body problem ⟶ 1 body

6 d.o.f

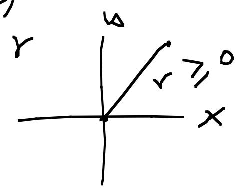
$$U = U(\vec{r}_1 - \vec{r}_2)$$

$$\downarrow$$

$$U(\overbrace{\underbrace{|\vec{r}_1 - \vec{r}_2|}_{|\vec{r}| = r})}^{2 \text{ d.o.f.}}$$

cylindrical

$$\vec{r} = (\underset{\sim}{r}, \phi, z)$$



$$\dot{\vec{r}}^2 = \dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2$$

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\phi}^2 + \underset{\sim}{\dot{z}^2}) - U(r)$$

$$\frac{\partial L}{\partial z} = 0 \rightarrow \dot{z} = \cancel{\text{const}}$$

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\phi}^2) - \underset{\sim}{U(r)}$$

$$\underset{\sim}{r, \phi}$$

$$\downarrow$$

1 d.o.f.

$$\frac{\partial L}{\partial \phi} = 0 \longrightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0 \longrightarrow \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} = M$$

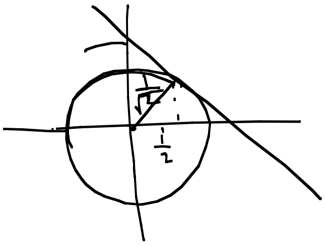
$$\Downarrow$$

$$\underset{\sim}{\dot{\phi} = \frac{M}{mr^2}}$$

constant.

use Lagrange's Undetermined multiplier.

$f(x,y) = x^2+y^2$ $\qquad$ $x+y=1 = g$



$$F = f + \lambda g = x^2+y^2+\lambda(x+y-1)$$

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0 \longrightarrow 2x + x = 2y+\lambda = 0$$

$$\rightarrow x = y = -\frac{\lambda}{2} = \frac{1}{2}$$

$$\lambda = -1$$

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r) + \lambda\left(\dot{\phi} - \frac{M}{mr^2}\right)$$

$$\frac{\partial L}{\partial \phi} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0 \rightarrow \frac{\partial L}{\partial \dot{\phi}} = \underbrace{\lambda + mr^2\overset{\frac{M}{mr^2}}{\dot{\phi}} = M}$$

$$\frac{\partial L}{\partial r} = mr\dot{\phi}^2 - U'(r) + \frac{2M\lambda}{\underbrace{mr^3}} = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}$$

$$\Rightarrow m\ddot{r} = mr\left(\frac{M}{mr^2}\right)^2 \overset{0}{-} U'(r)$$

$$= \frac{M^2}{mr^3} - U'(r)$$

$$= -\frac{d}{dr}\left(\underbrace{U + \frac{M^2}{2mr^2}}_{U_{eff}(r)}\right)$$

$$\therefore m\ddot{r} = -\frac{d}{dr}U_{eff}(r) \qquad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$E = \sum_a p_a \dot{q}_a - L$$

$$= m\dot r^2 + mr^2\dot\phi^2 - \left[\underbrace{\frac12 m\dot r^2 + \frac12 mr^2\dot\phi^2} - U\right]$$

$$\left(\dot\phi = \frac{M}{mr^2}\right)$$

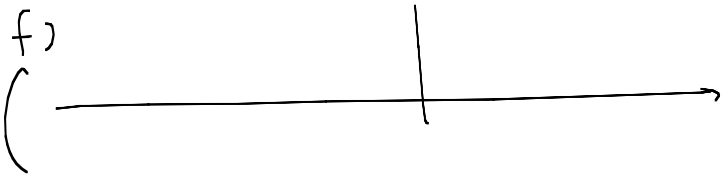
$$E = \underbrace{\frac12 m\dot r^2}_{\text{"effective" Kinti}} + \frac12 mr^2\dot\phi^2 + U$$

$$+ \underbrace{\frac12\frac{M^2}{mr^2} + U}_{\text{effective pot. enrgy}}$$

1 d.o.f. "$\underline{\underline{r}}$" only.

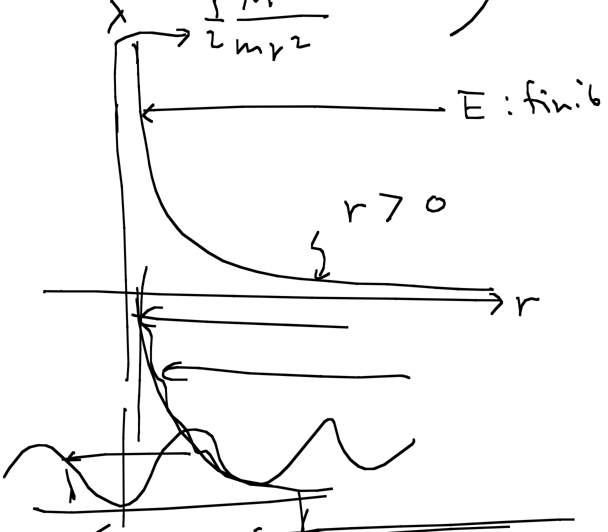
$$U \longrightarrow U_{\text{eff}} = \underset{r}{U} + \underline{\underline{\frac12\frac{M^2}{mr^2}}}$$

(cf)



$-\infty < x < \infty$

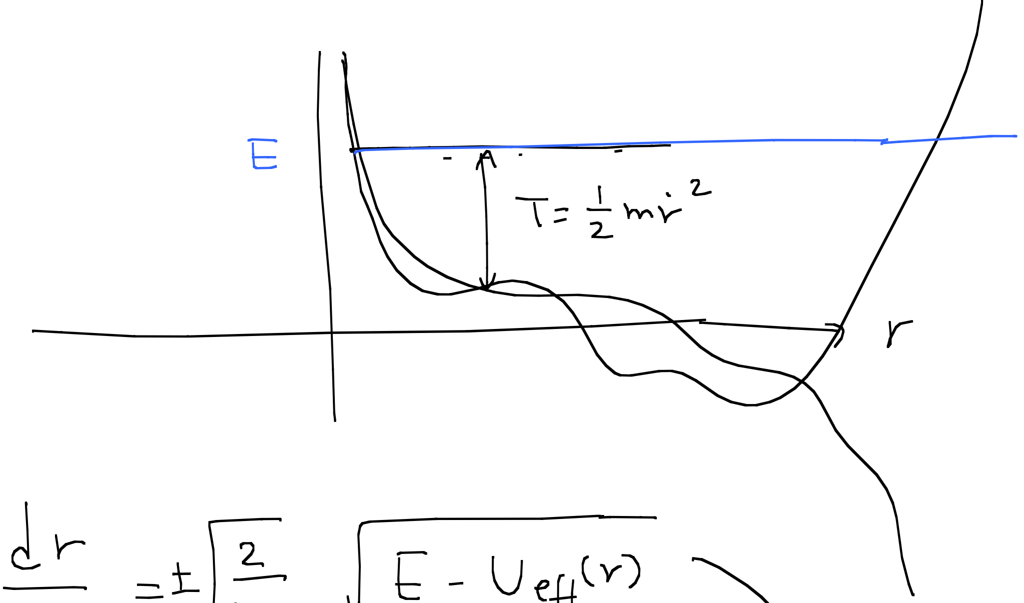
$\Rightarrow$ $\boxed{r \geq 0}$



(ex) $U = \sin kr$

~~(exampl) $U = -\frac{k}{r^3}$~~ $k>0$

so, effective 1D problem



$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{m}}\sqrt{E - U_{eff}(r)}$$

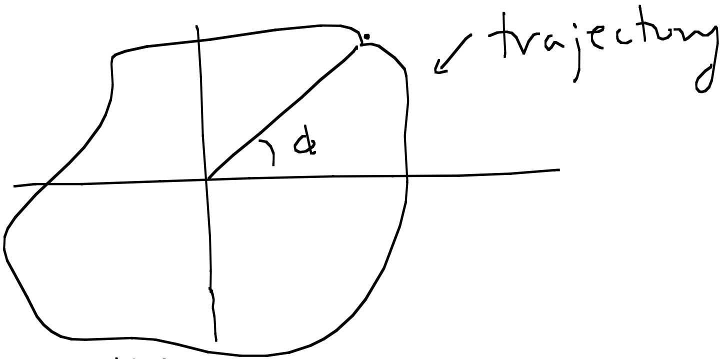
$$\boxed{t = \int_0^t dt = \pm\int_{r_0=r(0)}^{r}\sqrt{\frac{2}{m}}\frac{dr}{\sqrt{E - U_{eff}(r)}}}$$

$$\underbrace{\frac{d\phi}{dt} = \frac{M}{mr^2}}_{} \longrightarrow d\phi = \frac{M}{mr^2}\,dt$$

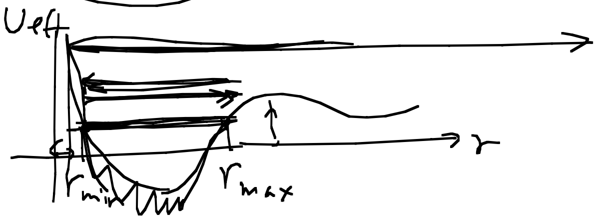
$$= \pm\frac{M}{mr^2}\sqrt{\frac{m}{2}}\frac{dr}{\sqrt{E - U_{eff}(r)}}$$

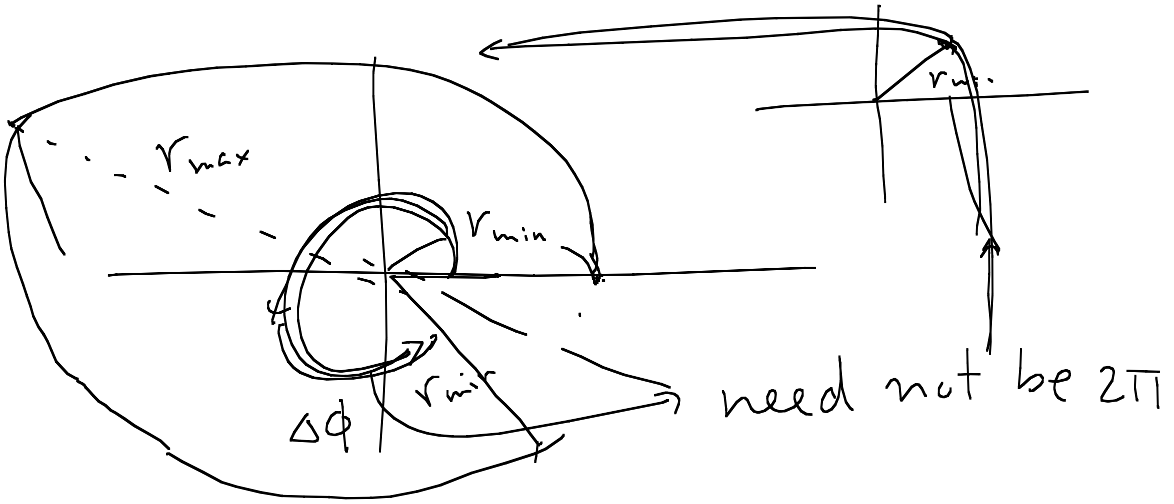
$$\therefore \boxed{\phi = \int_0^\phi d\phi = \pm\frac{M}{\sqrt{2m}}\int_{r_0}^{r}\frac{dr}{r^2\sqrt{E - U_{eff}(r)}}}$$

$$\phi = f(r) \longrightarrow r = f^{-1}(\phi)$$



If $U_{eff}$ is bounded





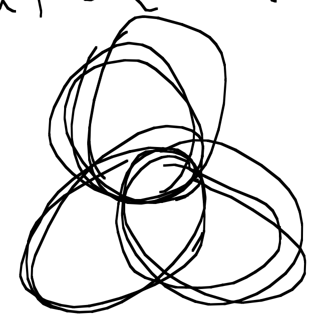
$$\therefore \Delta\phi = \int_0^{\Delta\phi} d\phi = 2\frac{M}{\sqrt{2m}}\int_{r_{min}}^{r_{max}} \frac{dr}{r^2\sqrt{E-\underbrace{U_{eff}(r)}_{U+\frac{M^2}{2mr^2}}}}$$

$$\neq 2\pi$$

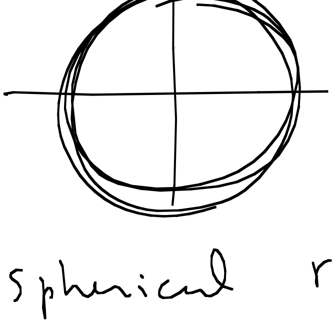
$$= \begin{cases} 2\pi \left(\boxed{\frac{m}{n}}\right) \leftarrow \text{rational} & \longrightarrow \text{closed} \\ 2\pi \cdot \text{irrational} & \longrightarrow \text{not closed} \end{cases}$$

$\Delta\phi = 2\pi m$ after $n$ revolution

→ closed



prob 1.



spherical $r = l, \theta, \phi$

$$L = \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - \underbrace{U(\theta)}_{mgl(1-\cos\theta)}$$

$$\frac{\partial L}{\partial \dot\phi} \equiv M_z = m l^2 \sin^2\theta\, \dot\phi \longrightarrow \dot\phi = \frac{M_z}{m l^2 \sin^2\theta} = \frac{d\phi}{dt}$$

$$E = \frac{1}{2} m l^2 \left(\dot\theta^2 + \dot\phi^2 \sin^2\theta\right) + mgl(1-\cos\theta)$$

$$\Rightarrow \underbrace{\frac{1}{2} m l^2 \dot\theta^2}_{\text{"effective K.E)}} + \underbrace{\frac{1}{2} m l^2 \left(\frac{M_z}{m l^2 \sin^2\theta}\right)^2 \sin^2\theta - mgl\cos\theta + \cancel{mgl}}_{U_{eff}(\theta)}$$

$$\frac{d\theta}{dt} = \pm\sqrt{\frac{2}{m l^2}}\sqrt{E - \frac{M_z^2}{2 m l^2 \sin^2\theta} + mgl\cos\theta}$$

$$\int dt = t = \int \frac{d\theta}{(\quad)}, \quad dt \cdot \frac{M_z}{m l^2 \sin^2\theta} = d\phi$$

$$\phi = \int d\phi = \int \cdots\, d\theta.$$

$$t = \int \frac{\sin\theta\, d\theta}{\sqrt{\frac{2E\sin^2\theta}{ml^2} - \frac{\cancel{2}M_z^2}{(ml^2)^2 \cancel{\sin^2\theta}} + \frac{2g}{l}\cos\theta \underbrace{\sin^2\theta}_{1-\cos^2\theta}}}$$

$1-\cos^2\theta$

$$= \int \frac{\sin\theta\, d\theta \longrightarrow -dx}{\sqrt{\underbrace{\left(\frac{2E}{ml^2} - \frac{M_z^2}{(ml^2)^2}\right)}_{a} + \underbrace{\frac{2g}{l}}_{b}\cos\theta - \underbrace{\frac{2E}{ml^2}}_{c}\cos^2\theta - \underbrace{\frac{2g}{l}}_{d}\cos^3\theta}}$$

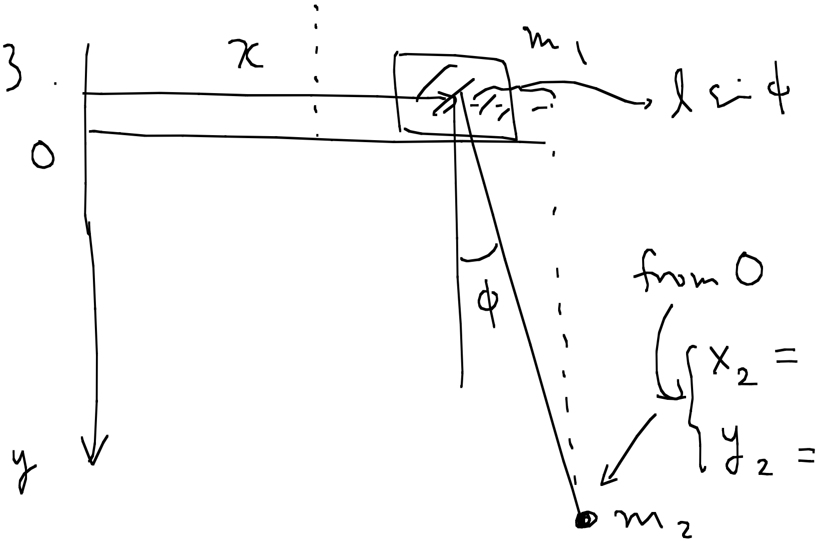
$\cos\theta \equiv x \qquad dx = -\sin\theta\, d\theta$

$$\int \frac{dx}{\sqrt{\underbrace{a+bx+cx^2+dx^3}_{d(x-\alpha)(x-\beta)(x-\gamma)}}}$$

$$= F\left(\sin^{-1}\left(\sqrt{\frac{\gamma-x}{\gamma-\beta}}\right), \left(\frac{\beta-\gamma}{\alpha-\gamma}\right)\right)(x-\gamma)\sqrt{\frac{x-\alpha}{\gamma-\alpha}}\sqrt{\frac{x-\beta}{\gamma-\beta}}\sqrt{\frac{\beta-\gamma}{x-\gamma}} \times \frac{1}{\sqrt{a+bx+cx^2+dx^3}}\Bigg)$$

$$F(\theta_0,m) = \int_0^{\theta_0} \frac{d\theta}{\sqrt{1-m\sin^2\theta}} \qquad E(\theta_0,m) = \int_0^{\theta_0} d\theta \sqrt{1-m\sin^2\theta}$$

Prob3.



$$-\frac{m_2 l \sin\phi}{m_1+m_2}$$

from O

$$\begin{cases} x_2 = x + l\sin\phi \\ y_2 = l\cos\phi \end{cases}$$

$$L = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2\left(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi\right) + m_2 g l\cos\phi$$

$$\frac{\partial L}{\partial x} = 0 \longrightarrow \frac{\partial L}{\partial \dot{x}} \equiv P_x = (m_1+m_2)\dot{x} + m_2 l\dot{\phi}\cos\phi$$

$$= \frac{d}{dt}\Big[\underbrace{(m_1+m_2)x + m_2 l\sin\phi}_{\underbrace{m_1 x + m_2(x + l\sin\phi)}_{x\text{-coord of CM}}}\Big]$$

In CM frame: of x-axis $= 0$ $\longrightarrow P_x = 0 \longrightarrow x = -\frac{m_2 l\sin\phi}{m_1+m_2}$

$$\dot{x} = -\frac{m_2 l}{m_1+m_2}\dot{\phi}\sin\phi$$

$$E = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2\left(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi\right) - m_2 g l\cos\phi$$

$$= \frac{1}{2}m_2 l^2\dot{\phi}^2\left(1-\frac{m_2}{m_1+m_2}\cos^2\phi\right) - m_2 g l\cos\phi$$

$$\frac{d\phi}{dt} = \dot{\phi} = \pm\sqrt{\frac{E+m_2 g l\cos\phi}{\frac{1}{2}m_2 l^2\left(1-\frac{m_2}{m_1+m_2}\cos^2\phi\right)}}$$

$$\int dt = \int\frac{d\phi}{\sqrt{\frac{E+m_2 g l\cos\phi}{\boxed{\frac{1}{2}m_2 l^2}\left(1-\frac{m_2}{m_1+m_2}\cos^2\phi\right)}}}$$

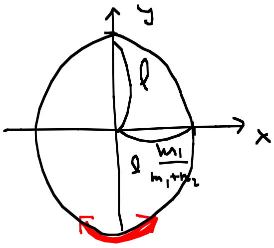
$$t = l\sqrt{\frac{1}{2}\frac{m_2}{m_1+m_2}}\int\frac{\sqrt{m_1+m_2\sin^2\phi}}{\sqrt{E+m_2 g l\cos\phi}}d\phi$$

---

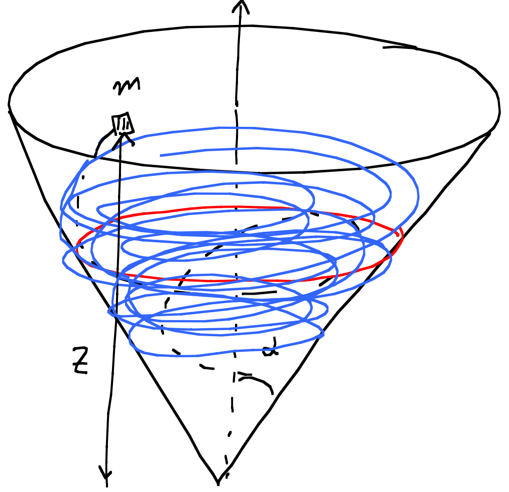
$$-\frac{m_2 l\sin\phi}{m_1+m_2}$$

$$\begin{cases} x_2 = x + l\sin\phi = l\sin\phi\left(\frac{m_1}{m_1+m_2}\right) \\ y_2 = l\cos\phi \end{cases}$$

$$\therefore \left(\frac{x_2}{\underset{< l}{\boxed{l\left(\frac{m_1}{m_1+m_2}\right)}}}\right)^2 + \left(\frac{y_2}{l}\right)^2 = 1$$



Prob2.



spherical ∴ $\theta = \alpha$

$\dot{\theta} = 0$

$r, \phi \rightarrow$ variable

$$z = r \cos\alpha$$

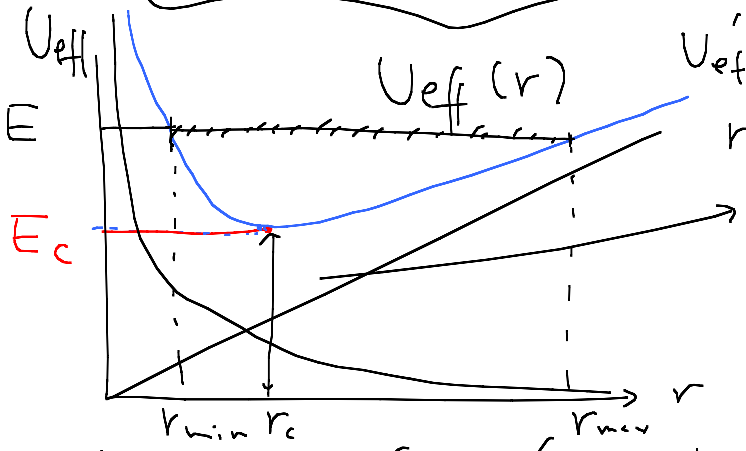
$$L = \frac{1}{2} m \left[ \dot{r}^2 + r^2 \dot{\phi}^2 \sin^2\alpha ) \right] - mgr \cos\alpha$$

$$\frac{\partial L}{\partial \phi} = 0 \rightarrow \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2\alpha \, \dot{\phi} = M_z$$

$$\dot{\phi} = \frac{M_z}{m \sin^2\alpha \, r^2}$$

$$E = \frac{1}{2} m \left[ \dot{r}^2 + r^2 \dot{\phi}^2 \sin^2\alpha ) \right] + mgr \cos\alpha$$

$$= \frac{1}{2} m \dot{r}^2 + \underbrace{\frac{M_z^2}{2m \sin^2\alpha \, r^2} + mgr \cos\alpha}$$



$$U_{eff}' = 0$$

$$r_c' = \left( \frac{M_z^2}{m^2 \sin^2\alpha \, g \cos\alpha} \right)^{\frac{1}{3}}$$

$$\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)} = \frac{dr}{dt} \xrightarrow{t=} \int dt = \int \frac{dr}{\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}}$$

$$= \int \frac{r \, dr}{\sqrt{\frac{2}{m}\left( -\frac{M_z^2}{2m \sin^2\alpha} + E r^2 - mg \cos\alpha \, r^3 \right)}} = E(\cdots)$$

## §15. Kepler.

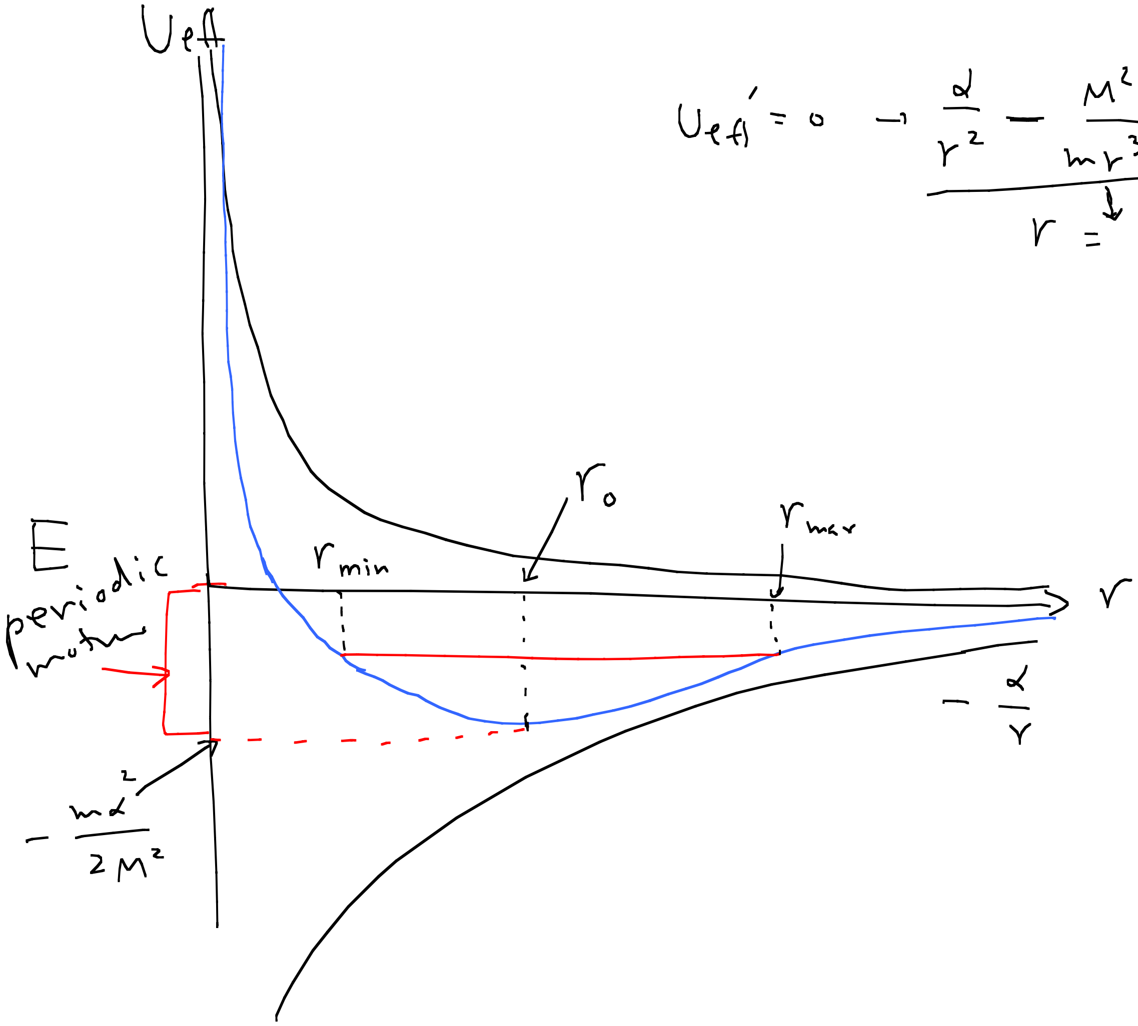
$U = -\frac{\alpha}{r}$ $\qquad \alpha = G M_\odot m$

$U_{eff} = -\frac{\alpha}{r} + \frac{M^2}{2mr^2}$ $\qquad U_{eff,min} = -\frac{m\alpha^2}{2M^2}$

$U_{eff}' = 0 \to \frac{\alpha}{r^2} - \frac{M^2}{mr^3} = 0$

$\downarrow$

$r = \frac{M^2}{\underset{\underset{GM_\odot m}{\uparrow}}{m\alpha}}$

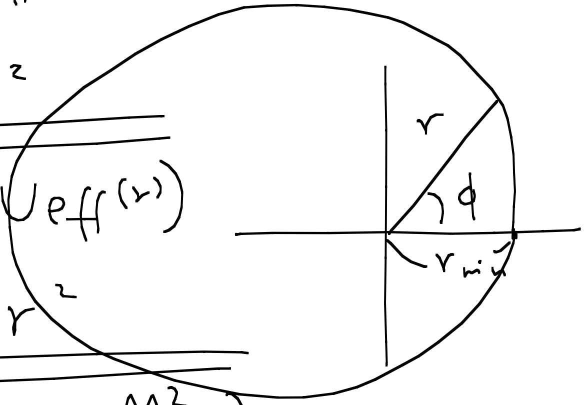


① $-\frac{m\alpha^2}{2M^2} < E < 0$

$E = \frac{1}{2} m\dot{r}^2 + U_{eff}(r)$, $\quad M = mr^2\dot{\phi}$

$$\phi = \int \frac{M\, dr/r^2}{\sqrt{2m(E - U_{eff}(r))}}$$

$$\tilde{\phi} = \int_{r_{min} \equiv r_1}^{r} \frac{M\, dr/r^2}{\sqrt{2m\left(E + \frac{\alpha}{r} - \frac{M^2}{2mr^2}\right)}}$$



$$\sqrt{2mE + \frac{2m\alpha}{r} - \frac{M^2}{r^2}} = \sqrt{2mE + 2m\alpha u - M^2u^2}$$

$$\frac{1}{r} \equiv u \qquad du = -\frac{1}{r^2}\,dr$$

$$= \int_{u}^{u_1 = \frac{1}{r_1}} \frac{\cancel{M}\,du}{\cancel{M}\sqrt{\underbrace{\frac{2mE}{M^2} + \frac{m^2\alpha^2}{M^4}}_{\equiv A^2} - \Big(\underbrace{u - \frac{m\alpha}{M^2}}_{u'}\Big)^2}} \qquad \longrightarrow du' = du$$

$$= \int_{u'}^{u_1' = u_1 - \frac{m\alpha}{M^2}} \frac{du'}{\sqrt{A^2 - u'^2}}$$

$$u' \equiv A\cos\phi$$

$$du' = \underset{\sim}{-}A\sin\phi\,d\phi$$

$$\tilde{\phi} = \int_{\phi_1}^{\phi} \frac{A\cancel{\sin\phi}\,d\phi}{\sqrt{\cancel{A^2 - A^2\cos^2\phi}}} = \phi - \phi_1$$

$$A\cos\phi_1 = \frac{1}{r_1} - \frac{m\alpha}{M^2}$$

$$\phi = \tilde{\phi} + \phi_1$$

$$u' = A\cos(\tilde{\phi} + \phi_1) = \overset{\frac{1}{r}}{u} - \frac{m\alpha}{M^2}$$

$$\therefore\ r = \left[\underbrace{\frac{m\alpha}{M^2}}_{\frac{1}{p}} + \underbrace{\sqrt{\frac{m^2\alpha^2}{M^4} + \frac{2mE}{M^2}}}_{\frac{e}{p}}\cos(\underbrace{\tilde{\phi} + \phi_1}_{\equiv\phi})\right]^{-1}$$

$r_{min} \longrightarrow \phi = 0$ $\qquad r_{min} = \dfrac{1}{\dfrac{m\alpha}{M^2} + \sqrt{\dfrac{m^2\alpha^2}{M^4} + \dfrac{2mE}{M^2}}}$

(perigee)

‖

근일점 perihelion

$r_{max} \longrightarrow \phi = \pi$ $\qquad r_{max} = \dfrac{1}{\dfrac{m\alpha}{M^2} - \sqrt{\dfrac{m^2\alpha^2}{M^4} + \dfrac{2mE}{M^2}}}$

(apogee)

‖

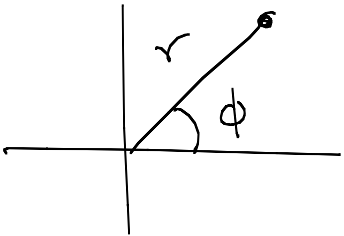
원일점 aphelion $\qquad -\dfrac{m\alpha^2}{2M^2} < E < 0$

$1 > \text{eccentricity } e > 0$

$$\frac{1}{p} \equiv \frac{m\alpha}{M^2} \; ; \; \frac{e}{p} \equiv \sqrt{\frac{m^2\alpha^2}{M^4} + \frac{2mE}{M^2}} \; ; \quad r_{max} = \frac{p}{1-e} \; , \; r_{min} = \frac{p}{1+e}$$

$$r = \left[ \frac{1}{p} + \frac{e}{p} \cos\phi \right]^{-1}$$

$$\boxed{1 + e\cos\phi = \frac{p}{r}}$$



$$x = r\cos\phi \qquad y = r\sin\phi$$

$$\underset{\sqrt{x^2+y^2}}{\overset{\downarrow}{r}} + e\,\underbrace{r\cos\phi}_{x} = p$$

$$\left(\sqrt{x^2+y^2}\right)^2 = (p - ex)^2$$

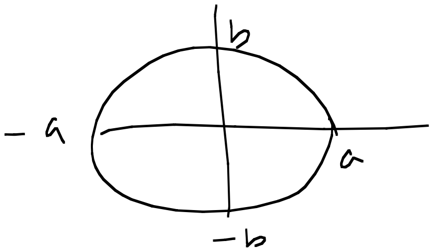
$$x^2 + y^2 = p^2 - 2pex + e^2x^2$$

$$\underbrace{(1-e^2)x^2 + 2pex} + y^2 = p^2$$

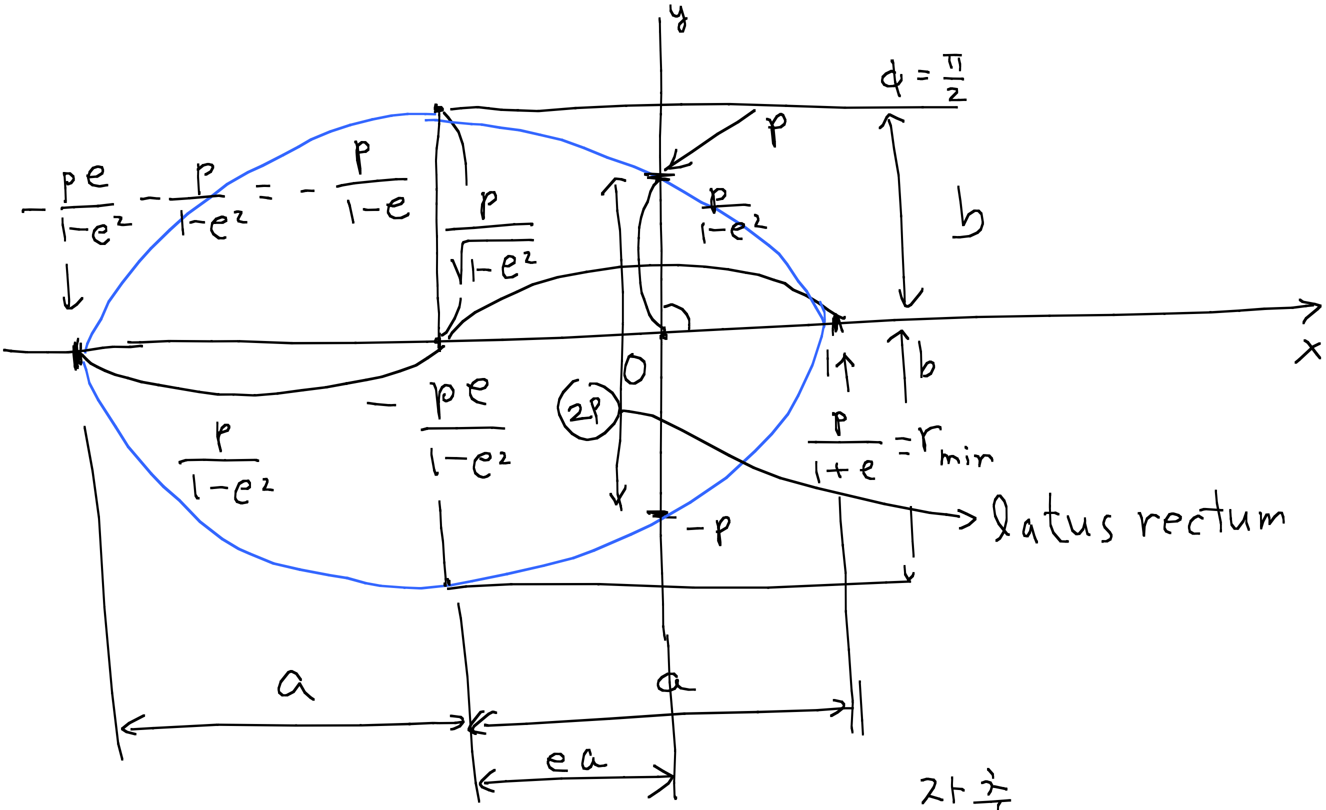
$$(1-e^2)\left(x + \frac{pe}{1-e^2}\right)^2 + y^2 = \frac{p^2e^2}{1-e^2} + p^2$$

$$= \frac{p^2}{1-e^2}$$

$$\frac{\left(x + \frac{pe}{1-e^2}\right)^2}{\frac{p^2}{(1-e^2)^2}} + \frac{y^2}{\boxed{\frac{p^2}{1-e^2}}} = 1$$

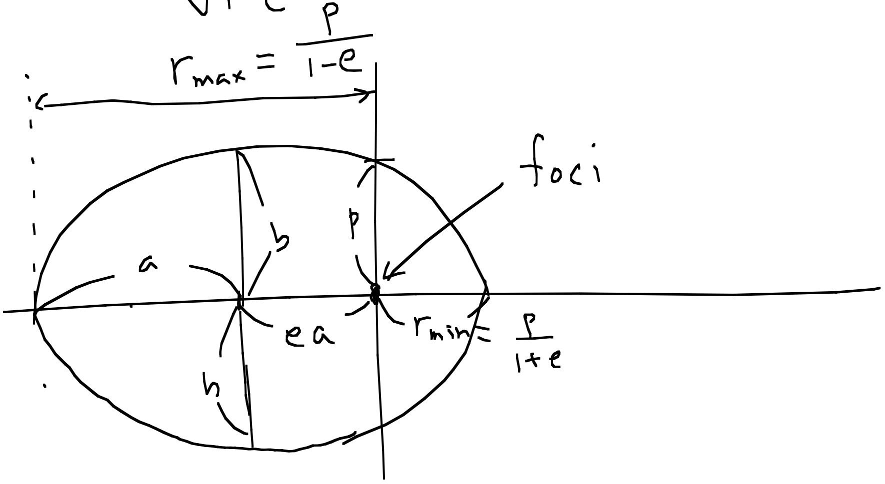


$$\leftarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



$a = \frac{p}{1-e^2}$ 장축 major axis

$b = \frac{p}{\sqrt{1-e^2}}$ 단축 minor axis



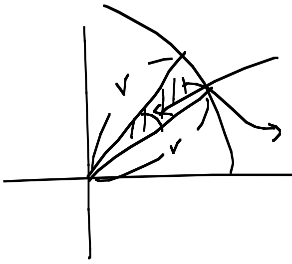
$$p = \frac{M^2}{m\alpha} \qquad e = \sqrt{1+\frac{2EM^2}{m\alpha^2}} \rightarrow 1-e^2 = \frac{2M^2|E|}{m\alpha^2}$$

$$a = \frac{\cancel{M^2}}{\cancel{m\alpha}} \frac{\cancel{m}\alpha^{\cancel{2}}}{2\cancel{M^2}|E|} = \frac{\alpha}{2|E|}, \qquad b = \sqrt{\frac{M^2}{2m|E|}}$$

$\alpha = GM_{\odot}m$ → $b$ = indep of $G$

$M \equiv m r^2 \dot{\phi}$

- 1st law: ellipse



$\to dA = \frac{1}{2} r \, r \, d\phi$

$= \frac{1}{2} r^2 d\phi$

- $\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi} = \frac{M}{2m}$ = constant ; K's 2nd law

$$A = \frac{M}{2m} T = \pi a b =$$

$$= \pi \frac{\alpha}{2|E|} \sqrt{\frac{M^2}{2m|E|}}$$

$$\frac{\cancel{M^2}}{(2m)^2} T^2 = \pi^2 \frac{\alpha^2 \cancel{M^2}}{8|E|^3} \qquad \frac{1}{|E|} = \frac{2a}{\alpha}$$

$$T^2 = \frac{\pi^2 m^2 \alpha^2}{2|E|^3} = \frac{\pi^2 m^2 \cancel{\alpha^2}}{2} \cdot \frac{8a^3}{\alpha^3}$$

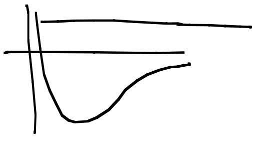
$$T^2 = \frac{4\pi^2 m^2}{\alpha} a^3$$

- K's 3rd law "Harmonic law"

$$T^2 \propto a^3$$

---

② $E > 0 : e > 1$

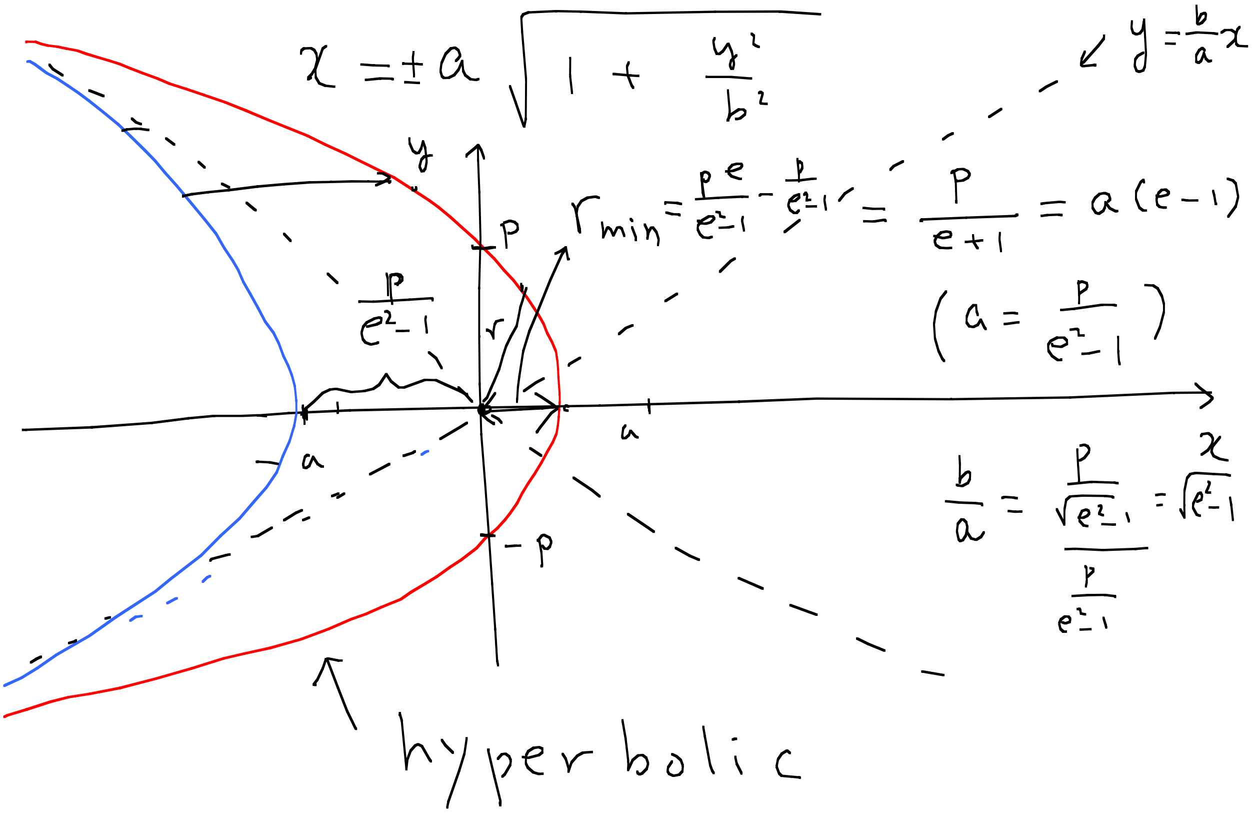


$$\frac{\left(x - \frac{pe}{e^2-1}\right)^2}{\frac{p^2}{(1-e^2)^2}} - \frac{y^2}{\frac{p^2}{e^2-1}} = 1$$

↑ ↑

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\left( \begin{array}{c} e^2 - 1 > 1 \\ < 1 \\ \cancel{a > b} \\ \cancel{a < b} \end{array} \right)$$



$$x = \pm a\sqrt{1+\frac{y^2}{b^2}}$$

$$y = \frac{b}{a}x$$

$$r_{min} = \frac{pe}{e^2-1} - \frac{p}{e^2-1} = \frac{p}{e+1} = a(e-1)$$

$$\left(a = \frac{p}{e^2-1}\right)$$

$$\frac{b}{a} = \frac{\frac{p}{\sqrt{e^2-1}}}{\frac{p}{e^2-1}} = \sqrt{e^2-1}$$

April 21 → no class

April 28 → mid term exam

$$t = \int_0^t dt = \pm\int_{r_0=r(0)}^{r} \sqrt{\frac{2}{m}} \frac{dr}{\sqrt{E-U_{eff}(r)}}$$

$$U_{eff} = -\frac{\alpha}{r} + \frac{M^2}{2mr^2}$$

$$= \sqrt{\frac{m}{2|E|}} \int \frac{r\,dr}{\sqrt{\underbrace{-r^2 + \frac{\alpha}{|E|}r - \frac{M^2}{2m|E|}}_{-\left(r-\frac{\alpha}{2|E|}\right)^2 + \underbrace{\frac{\alpha^2}{4E^2} - \frac{M^2}{2m|E|}}_{}}}} = \sqrt{\frac{m}{2|E|}} \int \frac{r\,dr}{\sqrt{a^2e^2-(r-a)^2}}$$

$$= \underbrace{\left(\frac{\alpha}{2E}\right)^2}_{a^2} \underbrace{\left(1-\frac{2EM^2}{2m\alpha^2}\right)}_{e^2}$$

$$p = \frac{M^2}{m\alpha}$$

$$e = \sqrt{1+\frac{2EM^2}{m\alpha^2}}$$

$$\rightarrow e^2 - 1 = \frac{2EM^2}{m\alpha^2}$$

$\left( a = \frac{p}{e^2-1} = \frac{M^{\cancel{2}}}{m\cancel{\alpha}} \; \frac{m\cancel{\alpha}^2}{2E\cancel{M^2}} = \frac{\alpha}{2E} \rightarrow \right.$

$$\underline{r - a = ae(-\cos\xi)} \rightarrow dr = ae\sin\xi\, d\xi$$

$$r = a(1 - e\cos\xi)$$

$$t = \sqrt{\frac{m}{2E}} \int \frac{a(1-e\cos\xi)}{\cancel{ae}\ \cancel{\sin\xi}}\ \cancel{ae}\ \cancel{\sin\xi}\ d\xi \qquad e<1$$

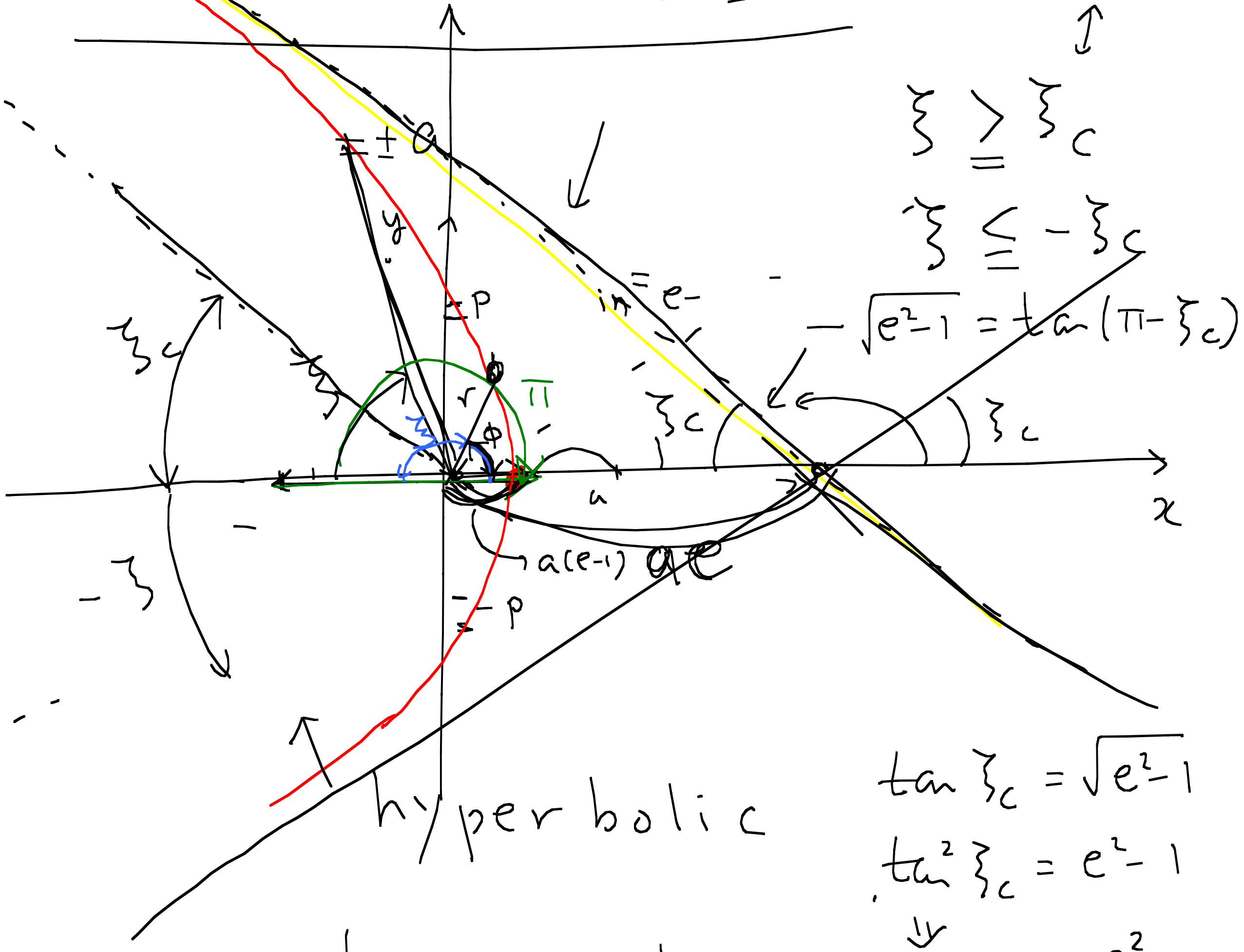
$$= a\sqrt{\frac{m}{2E}}\left(\xi - e\sin\xi\right) + c$$

let

$t = 0 \quad \xi = 0 \quad \rightarrow \quad c = 0$.

$\hookrightarrow r(t=0) = a(1-e)$

$$\underline{r = a(1 - e\cos\xi) \geq 0}$$



$$\tan\xi_c = \sqrt{e^2-1}$$

$$\tan^2\xi_c = e^2-1$$

$$\Downarrow$$

$$\cos\xi_c = \frac{1}{e}, \Leftarrow \quad \frac{1}{\cos^2\xi_c} = 1 + \tan^2\xi_c = e^2$$

$$\underline{r - a = ae(\cos\phi)} \longrightarrow dr = -ae\sin\phi\, d\phi$$

$$r = a(1+e\cos\phi)$$

$$t = \pm\sqrt{\frac{m}{2E}}\int\frac{a(1+e\cos\phi)\,(-ea\cancel{\sin}\phi\, d\phi)}{ae\cancel{\sin}\phi}$$

$$= \mp a\sqrt{\frac{m}{2E}}\,(\phi + e\sin\phi) + c.$$

$\uparrow$

$$t = 0 \qquad \phi = 0 \longrightarrow c = 0$$

$$t = a\sqrt{\frac{m}{2E}}\,(1 + e\sin\phi)$$

$$r = a(1 + e\cos\phi), \quad \frac{p}{r} = 1 + e\cos\phi$$

$$r\cos\phi = x, \quad r\sin\phi = y$$

$$p = r + ex = a(\overbrace{1 + e\cos\phi)}^{r} + ex$$

$$x = a(\cos\xi - e), \quad y = a\sqrt{1-e^2}\sin\xi$$

---

$E>0$ (hyperbolic) → $e>1$ $\qquad r = a(e\cosh\xi - 1)$

$$t = \sqrt{\frac{m}{2|E|}}\int\frac{r\,dr}{\sqrt{(r+a)^2-(ae)^2}} \nwarrow \times \qquad \begin{aligned} r + a &= ae\cosh\xi \\ dr &= ae\sinh\xi\, d\xi \end{aligned}$$

$$= \sqrt{\frac{m}{2E}}\int\frac{a(-1+e\cosh\xi)\,ae\cancel{\sinh}\xi\, d\xi}{ae\cancel{\sinh}\xi}$$

$$= a\sqrt{\frac{m}{2E}}\,(-\xi + e\sinh\xi)$$

*

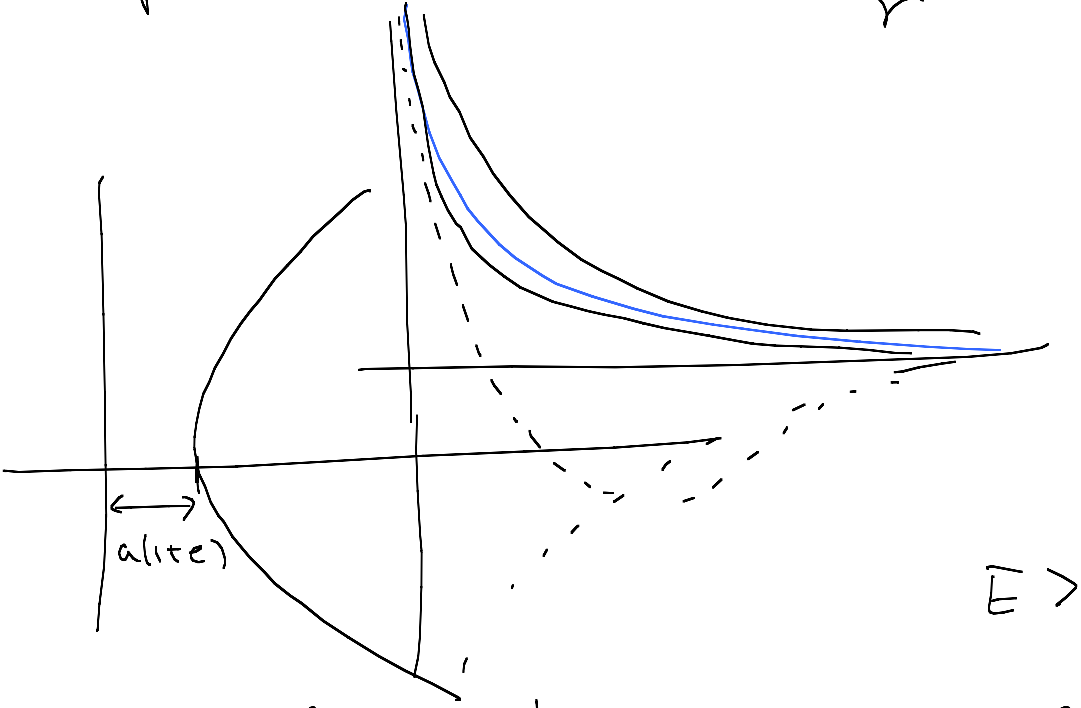
$$t = \int_0^t dt = \pm \int_{r_0 = r(0)} \sqrt{\frac{2}{m}} \frac{dr}{\sqrt{E - U_{eff}(r)}}$$

$$U_{eff} = -\frac{\alpha}{r} + \frac{M^2}{2mr^2}$$

$$= \sqrt{\frac{m}{2E}} \int \frac{r\,dr}{\sqrt{\underbrace{r^2 + \frac{\alpha r}{E} - \frac{M^2}{2mE}}_{(r+a)^2 - (ae)^2}}}$$

---

repulsive $U(r) = \frac{\alpha}{r}$ $\alpha > 0$



$E > 0$

$$t = \sqrt{\frac{m}{2E}} \int \frac{r\,dr}{\sqrt{\underbrace{r^2 - \left(\frac{\alpha r}{E} + \frac{M^2}{2mE}\right)}_{\left(r - \frac{\alpha}{2E}\right)^2 + \frac{\alpha^2}{4E^2} - \frac{M^2}{2mE}}}}$$

$$(r-a)^2 - a^2e^2$$

$$= \sqrt{\frac{m}{2E}} \int \frac{\overset{a(1+e\cosh\xi)}{}\,\cancel{ae\sinh\xi}\,d\xi}{\cancel{ae\sinh\xi}}$$

$$= a\sqrt{\frac{m}{2E}} (\xi + e\sinh\xi)$$

$r = a(1+e\cosh\xi)$

$\rightarrow r - a = ae\cosh\xi$

$dr = ae\sinh\xi$

# Chap 4. Collision

$$2 \text{ particle} \longrightarrow \left\{\begin{array}{l} CM \\ LAB \end{array}\right\} 1 \text{ d.o.f.}$$

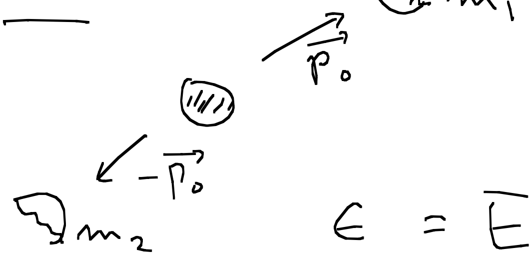
CM : $\vec{P}_1 + \vec{P}_2 = 0$

LAB : $\vec{P}_2 = 0$

$$\vec{v}_{10} = \overset{\vec{v}_0}{\overset{\shortparallel}{\frac{\vec{p}_0}{m_1}}}, \quad \vec{v}_{20} = -\frac{\vec{p}_0}{m_2} = -\frac{m_1}{m_2}\vec{v}_0$$

Disintegration

CM: at rest.



$$E_i = E_{1i} + \frac{p_0^2}{2m_1} + E_{2i} + \frac{p_0^2}{2m_2}$$

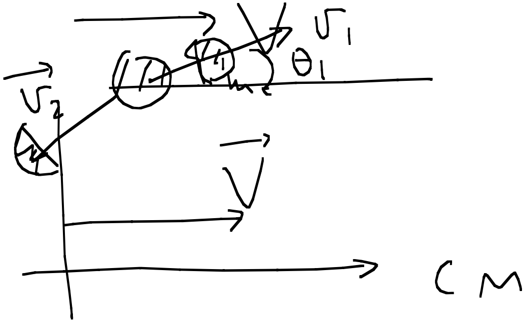
$\uparrow$ internal

$$\epsilon = E_i - E_{1i} - E_{2i} = \frac{p_0^2}{2}\underbrace{\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}_{\frac{1}{m}} = \frac{p_0^2}{2m}$$

$\epsilon > 0$ to be disintegrated

---

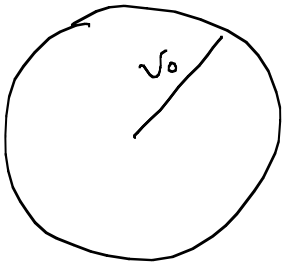
"LAB" frame

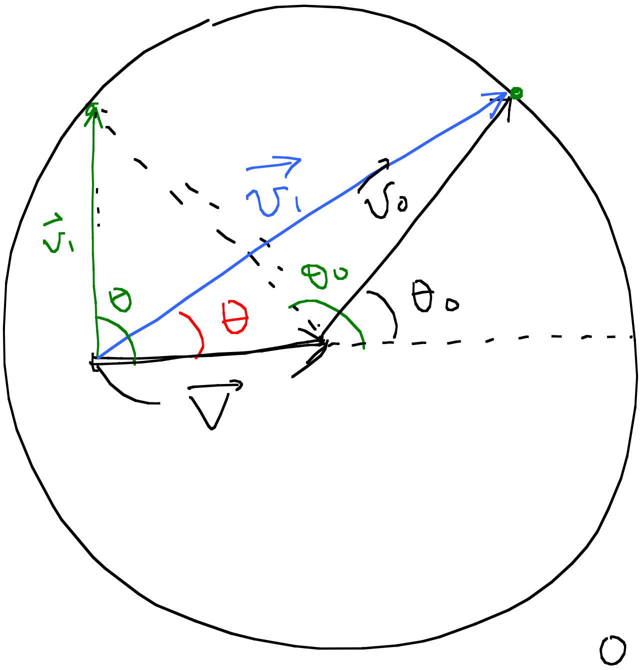


$$\underbrace{\vec{v}_1 - \vec{V}} = \vec{v}_{10} = \vec{v}_0$$

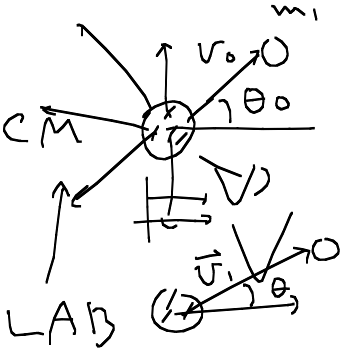
$$\vec{v}_2 - \vec{V} = \vec{v}_{20} = -\frac{m_1}{m_2}\vec{v}_0$$

$$v_1^2 + V^2 - 2v_1V\cos\theta_1 = v_0^2$$

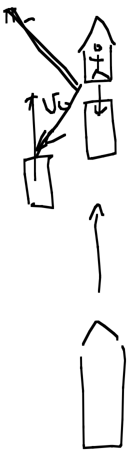




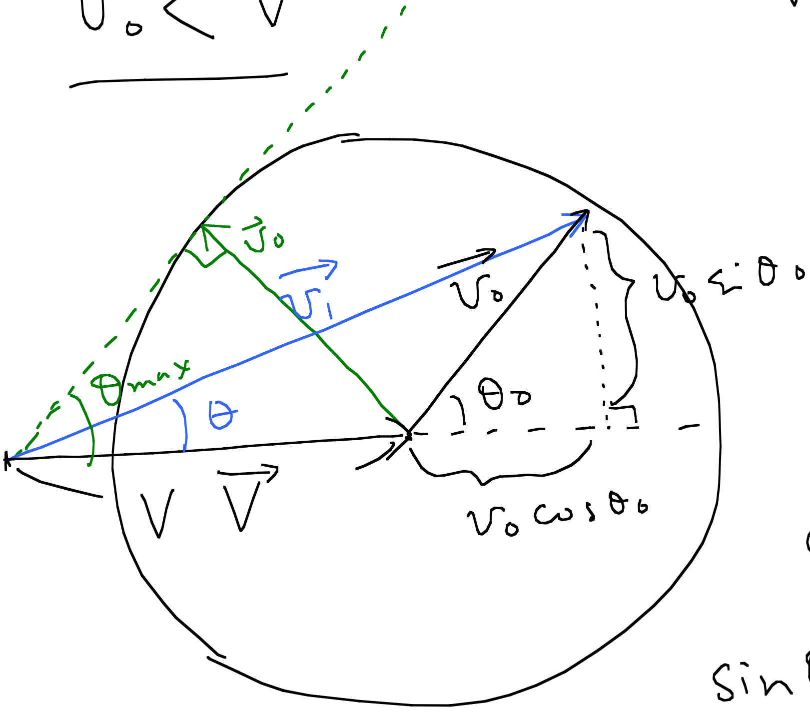
$v_0 > V$



$0 \le \theta \le 2\pi$



$v_0 < V$



$0 \le \theta_0 \le 2\pi$

$0 \le \theta \le \theta_{max}$

$$\sin\theta_{max} = \frac{v_0}{V}$$

$$\boxed{\tan\theta = \frac{v_0 \sin\theta_0}{V + v_0\cos\theta_0}}$$

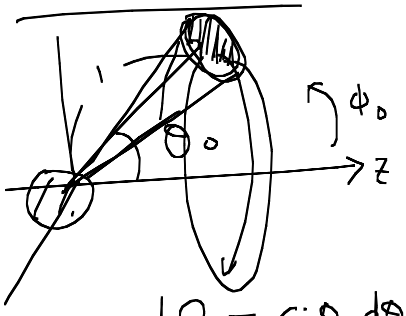
$$(V + v_0\cos\theta_0)^2 = (\cot\theta\ v_0)^2\left(\sqrt{1-\cos^2\theta_0}\right)^2$$

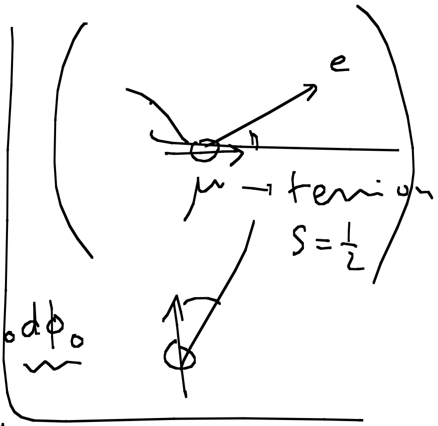
$$V^2 + 2Vv_0\cos\theta_0 + v_0^2\cos^2\theta_0 = \cot^2\theta\, v_0^2(1-\cos^2\theta_0)$$

$$\cos\theta_0 = -\frac{V}{v_0}\sin^2\theta \pm \cos\theta\sqrt{1-\frac{V^2}{v_0^2}\sin^2\theta}$$

$$\sin^2\theta \leq \frac{v_0^2}{V^2}$$

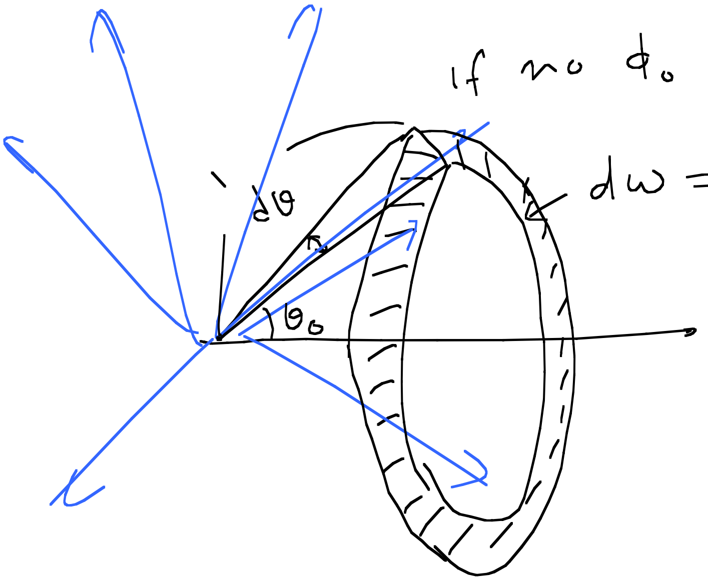
CM frame : <u>random</u> disintegration





$$d\Omega = \sin\theta_0 d\theta_0 \underbrace{d\phi_0}$$

if no $\phi_0$ dependence
(∵ rotational sym around z axis)



$$dw = 2\pi \sin\theta_0 d\theta_0$$

$$dp = \frac{dw}{4\pi} = \frac{1}{2}\underbrace{\sin\theta_0 d\theta_0}_{d(\cos\theta_0)}$$

<u>LAB</u>

$$\vec{v}_1 = \vec{v}_0 + \vec{V}$$

$$v_1^2 = \underbrace{v_0^2}_{\text{constant}} + V^2 + 2v_0 V\cos\theta_0$$

$$d(v_1^2) = 2v_0 V\, d\cos\theta_0$$

$$dp = \frac{1}{2}d(\cos\theta_0) = \frac{1}{4v_0 V\cdot\frac{1}{2}m_1}\, d\left(\overbrace{v_1^2\frac{1}{2}m_1}^{dT}\right)$$

$$= \frac{1}{2m_1 v_0 V}\, dT$$

$$T = \frac{1}{2}m_1 v_1^2 = \frac{1}{2}m_1\left(v_0^2+V^2+2v_0 V\cos\theta_0\right)$$

$\theta_0 = +\pi$ $\qquad T = \frac{1}{2} m_1 (v_0 - V)^2 = T_{min}$

$\theta_0 = 0$ $\qquad T = \frac{1}{2} m_1 (v_0 + V)^2 = T_{max}$

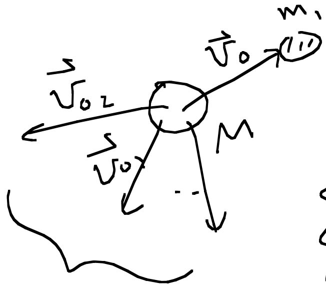
$$\int dp = \frac{1}{2 m_1 v_0 V} \int_{T_{min}}^{T_{max}} dT = 1$$

---

more than 2 particles

in CM: $\qquad E_i = E_{1i} + \underbrace{\frac{p_0^2}{2m_1}}_{T_{10}} + \sum_{a=2}^{N} \left( E_{ai} + \frac{\vec{p}_{0a}^{\,2}}{2m_a} \right)$

$$T_{10} = \Big( \overbrace{E_i - E_{1i} - E_i'}^{\epsilon} \Big) - \sum_{a=2}^{N} \frac{\vec{p}_{0a}^{\,2}}{2m_a} \qquad E_i' = \sum_a E_{ai}$$



$$\vec{p}_0 = - \sum_{a=2}^{N} \vec{p}_{0a}$$

$$\sum_a \vec{p}_{0a} = \sum_a m_a \vec{v}_{0a}$$

for maximum $\vec{v}_0^{\,2}$, $\quad \vec{v}_{0a} \equiv \vec{v}_0' \Rightarrow \vec{p}_0 = - \underbrace{\Big(\sum_a m_a\Big)}_{M - m_1} \vec{v}_0'$

$$\therefore \; T_{10}^{max} = \frac{\vec{p}_0^{\,2}}{2m_1} = \epsilon - \sum_a \frac{1}{2} m_a v_0'^2$$

$$= \epsilon - \frac{1}{2} (M - m_1) \left( \frac{1}{M - m_1} \right)^2 p_0^2$$

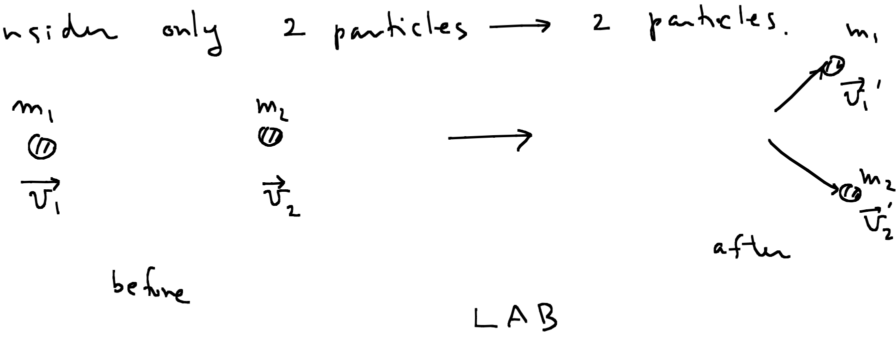
$$= \epsilon - \frac{m_1}{M - m_1} T_{10}^{max}$$

$$\Rightarrow T_{10}^{max} = \frac{M - m_1}{M} \epsilon \quad //$$

# §17. Elastic Collision

Kinetic energy is preserved during collision.

we will consider only 2 particles → 2 particles.



$\vec p_1 + \vec p_2 = \vec p_1' + \vec p_2'$

CM : frame moving with CM velocity in LAB frame

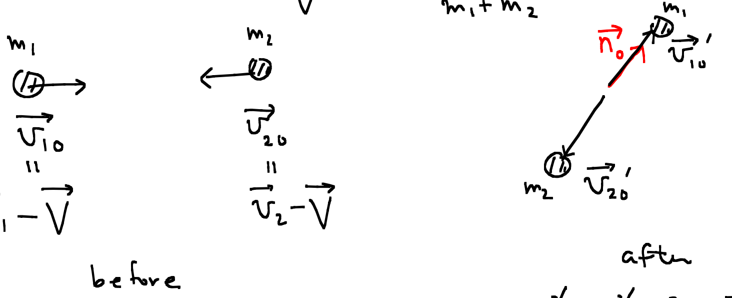
$$\vec V = \frac{\overbrace{m_1 \vec v_1}^{\vec p_1} + \overbrace{m_2 \vec v_2}^{\vec p_2}}{m_1 + m_2}$$ invariant during collision

$\vec v_{10} = \vec v_1 - \vec V$

$= \frac{m_2}{m_1+m_2}(\underbrace{\vec v_1 - \vec v_2}_{\equiv \vec v})$

$\vec v_{20} = \vec v_2 - \vec V$

$= \frac{m_1}{m_1+m_2}(\underbrace{\vec v_2 - \vec v_1}_{-\vec v})$



$\vec p_{10}' + \vec p_{20}' = 0$

CM $\vec v_{10}' = \vec v_1' - \vec V$ , $\vec v_{20}' = \vec v_2' - \vec V$

$p_{10} = \frac{m_2}{m_1+m_2} v$ , $|\vec v| = |\vec v_1 - \vec v_2|$ $\quad \vec p_{10} + \vec p_{20} = 0$

Elastic : CM: $E_{b0} = \frac{1}{2} m_1 v_{10}^2 + \frac{1}{2} m_2 v_{20}^2 = \frac{p_{10}^2}{2m_1} + \frac{p_{20}^2}{2m_2} = \frac{p_{10}^2}{2m}$ ∵ $p_{10} = p_{20}$

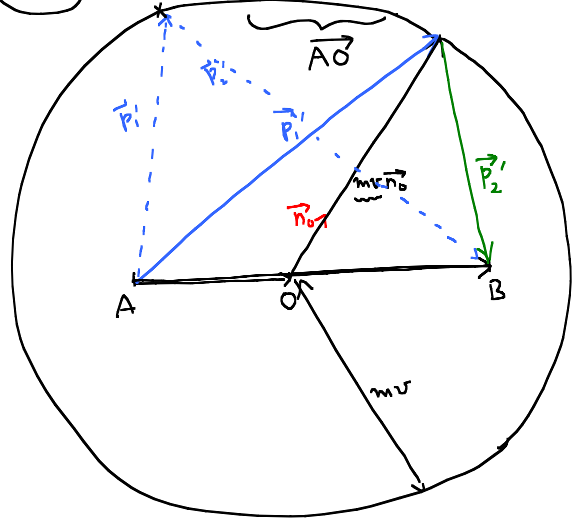
$= E_{a0} = \frac{p_{10}'^2}{2m_1} + \frac{p_{20}'^2}{2m_2} = \frac{p_{10}'^2}{2m}$ ∴ $\boxed{p_{10} = p_{10}'} = \frac{m_2}{m_1+m_2} v$ $(= p_{20} = p_{20}')$

∴ $\vec p_{10}' = m_1 \overbrace{\frac{m_2}{m_1+m_2} v}^{v_{10}} \vec n_0 = -\vec p_{20}' = -m_2 \vec v_{20}' = -m_2(\vec v_2' - \vec V)$

$= -\vec p_2' + m_2 \vec V$

$= m_1 \vec v_{10}' = m_1(\vec v_1' - \vec V) = \vec p_1' - m_1 \frac{\vec p_1 + \vec p_2}{m_1+m_2}$

∴ $\vec p_1' = \overset{m}{\boxed{\frac{m_1 m_2}{m_1+m_2}}} v \vec n_0 + \underbrace{m_1 \frac{\vec p_1 + \vec p_2}{m_1+m_2}}_{\vec{AO}}$ ; $\vec p_2' = -\overset{m}{\boxed{\frac{m_1 m_2}{m_1+m_2}}} v \vec n_0 + \underbrace{m_2 \frac{\vec p_1 + \vec p_2}{m_1+m_2}}_{\vec{OB}}$



$\vec{AO} \propto \vec{OB}$

$mv = \frac{m_1 m_2}{m_1+m_2} |\vec v_1 - \vec v_2|$

$|\vec{AO}| = \frac{m_1}{m_1+m_2}(m_1 \vec v_1 + m_2 \vec v_2)$

$\Downarrow$ $\vec v_2 = 0 \to v = v_1$

$mv = \frac{m_1 m_2}{m_1+m_2} v_1$

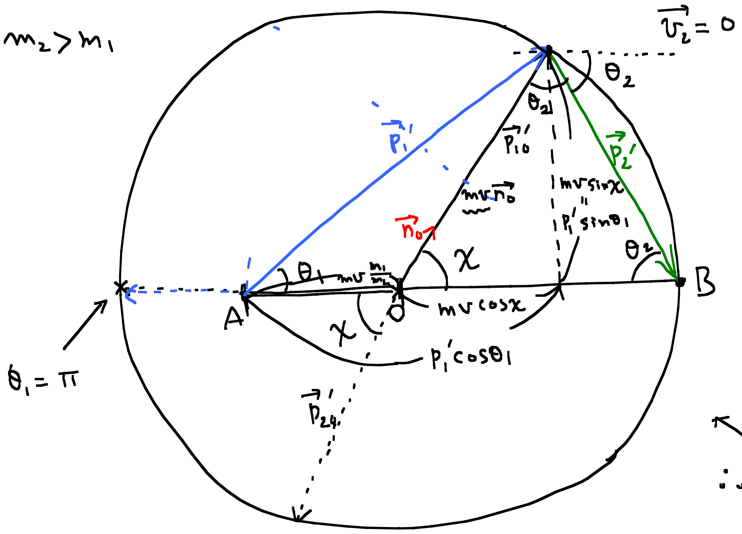
$|\vec{AO}| = \frac{m_1 m_1}{m_1+m_2} v_1 = m \frac{m_1}{m_2} v$

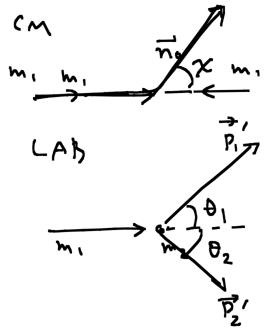
$|\vec{OB}| = \frac{m_1 m_2}{m_1+m_2} v_1 = mv$

Now, LAB frame collision usually fixes target. : $\vec v_2 = 0$

fixed target experiment

$m_2 > m_1$



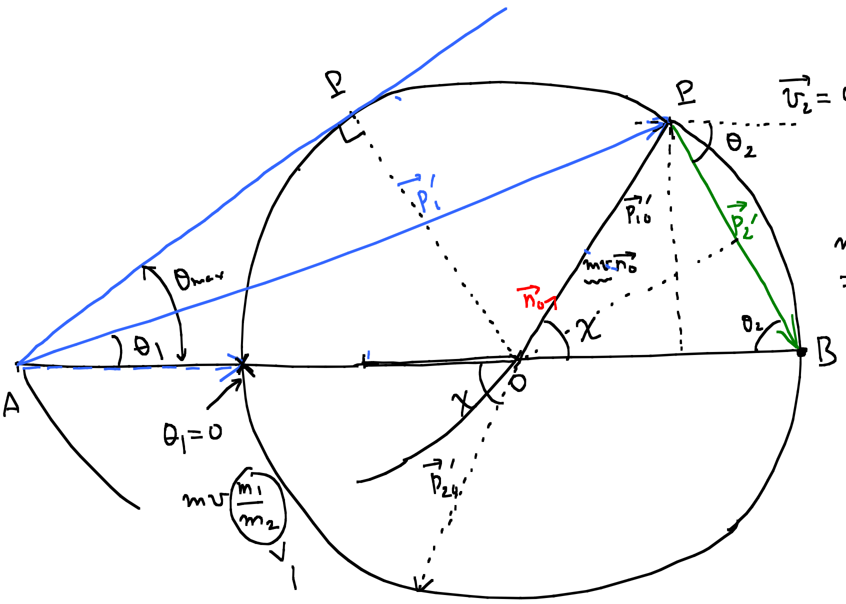


$$mv\sin\chi = p_1{}'\sin\theta_1$$

$$mv\frac{m_1}{m_2} + mv\cos\chi = p_1{}'\cos\theta_1$$

$$\therefore \boxed{\tan\theta_1 = \frac{\sin\chi}{\frac{m_1}{m_2} + \cos\chi}}$$

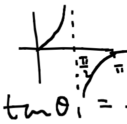
$$\boxed{\theta_2 = \frac{\pi - \chi}{2}}$$



$\underline{\underline{m_2 < m_1}}$

$$0 \leqq \theta_1 \leqq \theta_{max} < \pi$$

$$\sin\theta_{max} = \frac{mv}{AO} = \frac{m_2}{m_1} < 1$$



$\theta_{max} = ?$

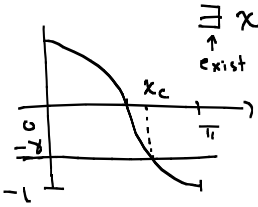
$$\tan\theta_1 = \frac{\sin\chi}{\textcircled{\gamma} + \cos\chi}$$

$$\boxed{\tan\theta_1 = \frac{\sin\chi}{\frac{m_1}{m_2} + \cos\chi}}$$

if $m_1 < m_2$ $\frac{m_1}{m_2} \equiv \gamma < 1$

if $m_1 > m_2$ $\gamma > 1$

$\underset{\substack{\uparrow \\ \text{exist}}}{\exists} \chi_c \underset{\substack{\uparrow \\ \text{suchthat}}}{\ni:} \gamma + \cos\chi_c = 0$



$$mv\sin\chi = p_1{}'\sin\theta_1$$

$$mv\underbrace{\frac{m_1}{m_2}}_{\gamma} + mv\cos\chi = p_1{}'\cos\theta_1 \rightarrow p_1{}'^2 = (mv)^2\left(\sin^2\chi + (\gamma + \cos\chi)^2\right)$$

$$= (mv)^2\left(1 + \gamma^2 + 2\gamma\cos\chi\right)$$

$$\therefore p_1{}' = m_1v_1{}' = \underbrace{mv}_{\substack{\| \\ \frac{m_1m_2}{m_1+m_2}v_1}}\sqrt{1 + \gamma^2 + 2\gamma\cos\chi}$$

$$p_2{}' = 2\,mv\cos\theta_2 = 2\overset{\|}{mv}\sin\frac{\chi}{2}$$

$$\theta_2 = \frac{\pi - \chi}{2}$$

Special cases

① $\chi = \pi$ → $m_1 < m_2$ : $\underset{\substack{\| \\ m_1\vec{v}_1{}'}}{\vec{p}_1{}'} = -\frac{\vec{v}_1}{\cancel{v_1}}\, \underset{\substack{\| \\ \frac{m_1m_2}{m_1+m_2}}}{m}\cancel{v_1}\underbrace{\left(1 - \frac{m_1}{m_2}\right)}_{\frac{m_2-m_1}{m_2}} \longrightarrow \vec{v}_1{}' = -\frac{m_2-m_1}{m_1+m_2}\vec{v}_1$ $(m_2 > m_1)$

$\|$

$m_1 > m_2$ : $m_1\vec{v}_1{}' = \frac{\vec{v}_1}{v_1}\, mv\left(\frac{m_1}{m_2} - 1\right) \longrightarrow \vec{v}_1{}' = \frac{m_1 - m_2}{m_1 + m_2}\vec{v}_1$ $(m_1 > m_2)$

$$m_2\vec{v}_2{}' + m_1\vec{v}_1{}' = m_1\vec{v}_1 \longrightarrow \vec{v}_2{}' = \frac{m_1}{m_2}(\vec{v}_1 - \vec{v}_1{}') = \frac{m_1}{m_2}\left(\vec{v}_1 - \frac{m_1 - m_2}{m_1 + m_2}\vec{v}_1\right)$$

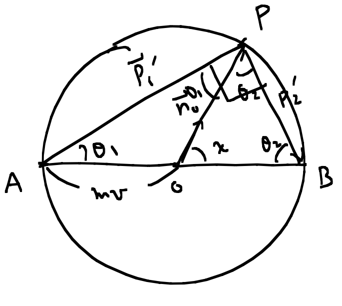
$$= \frac{2m_1}{m_1 + m_2}\vec{v}_1$$

$$E'_{2,max} = \frac{1}{2} m_2 v'^2_{2,max} \leftarrow$$ ↑ maximum speed of $m_2$

$$= \frac{1}{2} m_2 \left(\frac{2m_1}{m_1+m_2}\right)^2 v_1^2$$

$$= \frac{4 m_1 m_2}{(m_1+m_2)^2} \underbrace{\frac{m_1 v_1^2}{2}}_{E_1}$$

② $m_1 = m_2$



$$\theta_1 + \theta_2 = \frac{\pi}{2}$$

$$\theta_2 = \frac{\pi - \chi}{2} \rightarrow \theta_1 = \frac{\chi}{2}$$

$$m_1 v_1' = 2 m v_1 \cos\frac{\chi}{2} = m_1 v_1 \cos\frac{\chi}{2}$$

$$m = \frac{m_1 m_1}{m_1+m_1} = \frac{m_1}{2}$$

$$\therefore v_1' = v_1 \cos\frac{\chi}{2}$$

$$m_2 v_2' = 2 \frac{m_1}{2} v_1 \sin\frac{\chi}{2} \rightarrow v_2' = v_1 \sin\frac{\chi}{2}$$

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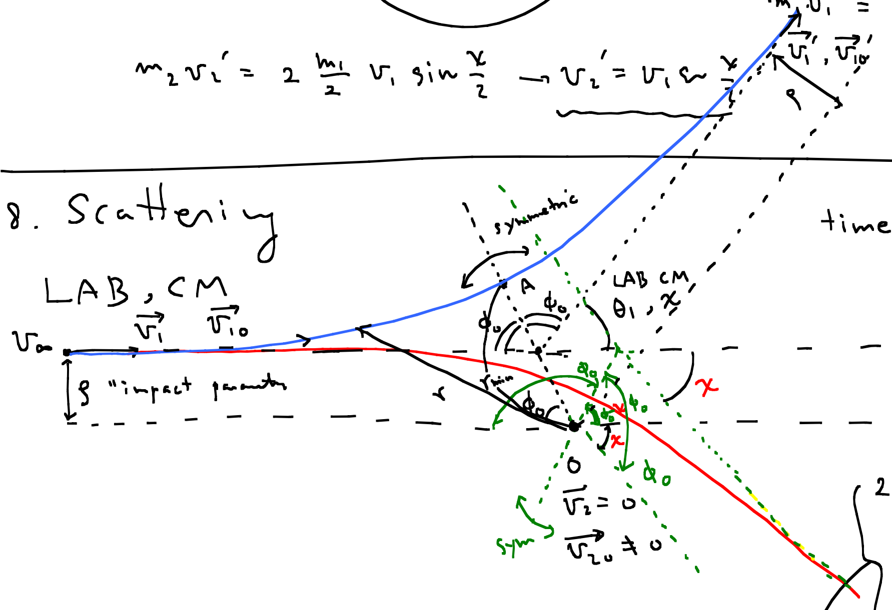
## §18. Scattering

time reversal sym.



$$2\phi_0 + \chi = \pi$$

$$\chi = \pi - 2\phi_0, \quad 0 \leq \phi_0 \leq \frac{\pi}{2}$$

$$\phi_0 + \phi_0 - \chi = \pi$$

$$\chi = 2\phi_0 - \pi, \quad \frac{\pi}{2} \leq \phi_0 \leq \pi$$

$$\chi = |\pi - 2\phi_0|$$

in CM frame : → 1 particle problem

with $\vec{r} = \vec{r}_1 - \vec{r}_2$

$$U \rightarrow U + \frac{M^2}{2mr^2} = U_{eff}$$

$$M = mr^2\dot{\phi} \rightarrow \frac{d\phi}{dt} = \frac{M}{mr^2}$$

$$E = \frac{1}{2} m\dot{r}^2 + U_{eff}(r)$$

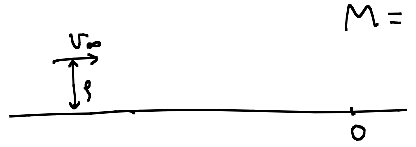
$$\frac{dr}{dt} = \sqrt{\frac{2}{m}(E - U_{eff})}$$

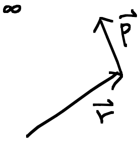
$$\int d\phi = \int \frac{M}{mr^2} \underset{\frac{dr}{\sqrt{\frac{2}{m}(E-U_{eff})}}}{\underset{\shortparallel}{dt}}$$

$$\phi_0 = \int_0^{\phi_0} d\phi = \int_{r_{min}}^{\infty} \frac{M/r^2}{\sqrt{2m(E - \underset{\uparrow}{U} - \frac{M^2}{2mr^2})}} dr$$

$$E = \frac{1}{2} m v_\infty^2$$

$$M = \rho m v_\infty$$





$$\vec{M} = \vec{r} \times \vec{p}$$

$$M = rp\sin\theta$$

$$= \rho p$$

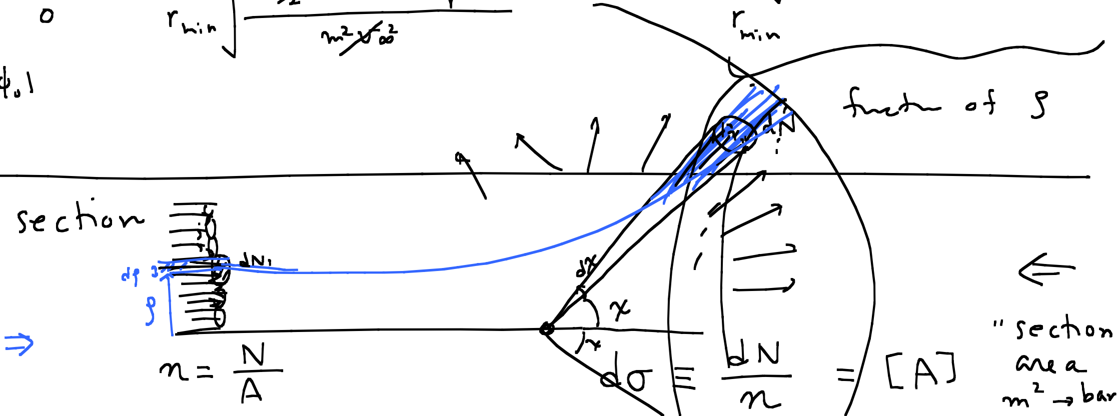
$$\underset{\uparrow}{\phi_0} = \int_0^{\phi_0} d\phi = \int_{r_{min}}^{\infty} \frac{\rho \quad /r^2}{\sqrt{\frac{2m \cancel{\frac{1}{2} m v_\infty^2} - \cancel{2m} U - \frac{\rho^2 m^2 v_\infty^2}{r^2}}{m^2 \cancel{v_\infty^2}}}} dr = \int_{r_{min}}^{\infty} \frac{\rho/r^2}{\sqrt{1 - \frac{2}{mv_\infty^2} U - \frac{\rho^2}{r^2}}} dr$$

$$\chi = |\pi - 2\phi_0|$$

function of $\rho$

---

cross section

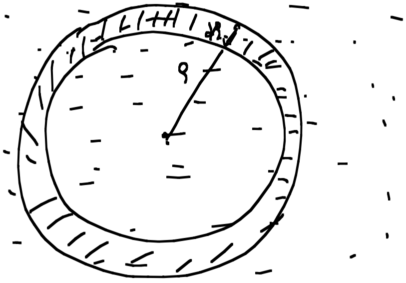


$$n = \frac{N}{A}$$

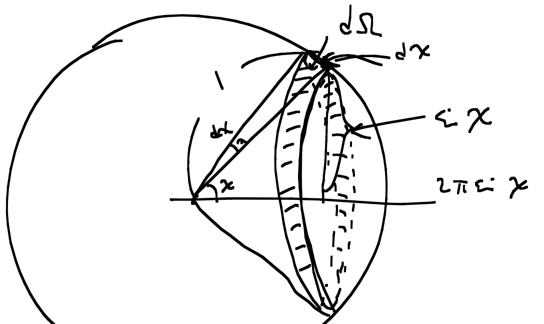
$$d\sigma \equiv \frac{dN}{n} = [A]$$

"section" area $m^2 \rightarrow$ barn

picobam



$$\frac{d\sigma}{d\chi} = \frac{1}{n}\frac{dN}{d\chi}$$



$$dN_1 = n\, 2\pi \rho\, d\rho = dN$$

$$d\sigma = \frac{dN_1}{n} = \frac{\cancel{n}\, 2\pi\rho\, d\rho}{\cancel{n}}$$

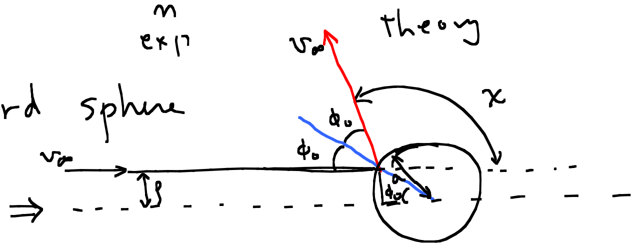
$$= 2\pi\rho\, d\rho$$

$$\frac{d\sigma}{d\Omega} = 2\pi\rho\left|\frac{d\rho}{d\Omega}\right| = \frac{\cancel{2\pi}\rho}{\cancel{2\pi}\sin\chi}\left|\frac{d\rho}{d\chi}\right| \leftarrow d\Omega = 2\pi\sin\chi\, d\chi$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\rho}{\sin\chi}\left|\frac{d\rho}{d\chi}\right|}$$

$\uparrow$ $\frac{dN}{n}$ exp $\qquad$ $\uparrow$ theory

**Prob1.** Hard sphere



$\rho = a\sin\phi_0$ , $\rho \le a$

$$\begin{cases}\chi = \pi - 2\phi_0 , & \rho \le a \\ \chi = 0 , & \rho > a\end{cases}$$

$$\rho = a\sin\phi_0 = a\sin\frac{\pi-\chi}{2} = a\cos\frac{\chi}{2}$$

$$\downarrow$$

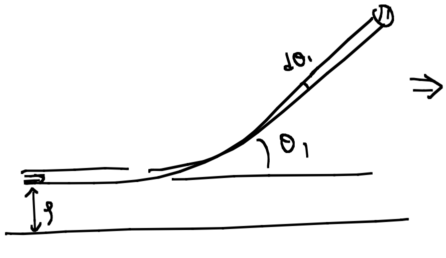
$$\frac{d\rho}{d\chi} = -\frac{a}{2}\sin\frac{\chi}{2}$$

CM.

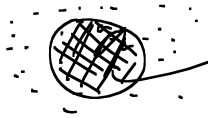
$$\frac{d\sigma}{d\Omega} = \frac{a\cos\frac{\chi}{2}}{\underset{``2\sin\frac{\chi}{2}\cos\frac{\chi}{2}}{\sin\chi}}\frac{a}{2}\sin\frac{\chi}{2}$$

$$= \frac{a^2}{4} \longrightarrow \sigma = \int\underbrace{\frac{d\sigma}{d\Omega}}\underset{\underset{2\pi\sin\chi d\chi}{\shortparallel}}{d\Omega} = \frac{a^2}{4}\underbrace{\int d\Omega}_{4\pi} = \pi a^2$$

LAB.



⇒



$$d\sigma = \frac{dN}{n} \quad : \text{same for LAB or CM}$$

$$\frac{d\sigma}{d\Omega_1} = \frac{\rho}{\sin\theta_1}\left|\frac{d\rho}{d\theta_1}\right| = \frac{\rho\sin\chi}{\sin\theta_1\sin\chi}\left|\frac{d\rho}{d\chi}\right|\left|\frac{d\chi}{d\theta_1}\right|$$

$$= \frac{\sin\chi}{\sin\theta_1}\left|\frac{d\chi}{d\theta_1}\right|\left(\frac{d\sigma}{d\Omega}\right)_{CM}$$

$$\left(\frac{d\sigma}{d\Omega_1}\right)_{LAB} = \frac{\sin\chi}{\sin\theta_1}\left|\frac{d\chi}{d\theta_1}\right|\left(\frac{d\sigma}{d\Omega}\right)_{CM.}$$

$$\uparrow$$

$$\tan\theta_1 = \frac{\sin\chi}{\frac{m_1}{m_2} + \cos\chi} \longrightarrow \theta_1(\chi)$$

**Prob2.**

$$\epsilon = E_2' = \frac{1}{2}m_2 v_2'^2 = \frac{1}{2}m_2\left(v_1\sin\frac{\chi}{2}\right)^2 = \epsilon_{max}\sin^2\frac{\chi}{2}$$

$$\frac{d\epsilon}{d\chi} = \frac{2}{2}\epsilon_{max}\sin\frac{\chi}{2}\cos\frac{\chi}{2} = \frac{\sin\chi}{2}\epsilon_{max}$$

$$\frac{d\sigma}{d\chi} = \frac{1}{2}\pi a^2\sin\chi \rightarrow \frac{d\sigma}{d\epsilon} = \frac{d\sigma}{d\chi}\frac{d\chi}{d\epsilon}$$

$$= \frac{1}{2}\pi a^2 \cancel{\sin\chi}\frac{2}{\cancel{\sin\chi}\,\epsilon_{max}} = \frac{\pi a^2}{\epsilon_{max}} = \frac{d\sigma}{d\epsilon}$$

$$\downarrow$$

$$\sigma = \int_0^{\epsilon_{max}}\frac{\pi a^2}{\epsilon_{max}}\,d\epsilon = \pi a^2$$

Prob3. $U(\alpha\vec{r}) = \alpha^k U(\vec{r})$ $\quad \underbrace{\frac{v'}{v} = \left(\frac{l'}{l}\right)^{\frac{k}{2}}}$

$$\boxed{U(r) \sim \frac{1}{r^n}}$$

$U(\alpha r) \sim \frac{1}{\alpha^n r^n} \rightarrow \underline{k=-n}$ ; impact parameter $\rho$

$v_\infty$

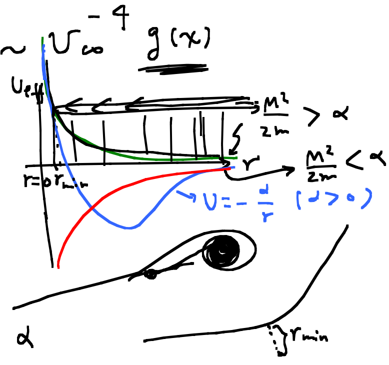
$v_\infty \propto \rho^{\frac{k}{2}} = \rho^{-\frac{n}{2}} \rightarrow \underline{\rho = v_\infty^{-\frac{2}{n}} f(x)}$

$$\frac{d\sigma}{d\Omega} = \frac{\rho}{\sin x}\left|\frac{d\rho}{dx}\right| = \frac{v_\infty^{-\frac{2}{n}} f(x)}{\sin x} v_\infty^{-\frac{2}{n}} f'(x) = \underline{v_\infty}^{-\frac{4}{n}} g(x)$$

$n=1$ (Rutherford) $\quad \frac{d\sigma}{d\Omega} \sim v_\infty^{-4} \underline{g(x)}$

Prob4. $U = -\frac{\alpha}{r^2}$ $\quad U_{eff} = U + \underbrace{\frac{M^2}{2mr^2}}_{(M \neq 0)}$

cross section of particles which reach at $r=0$. $\quad \left((cf) \; -\frac{\alpha}{r}\right)$

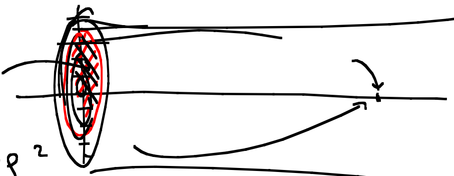


$$U_{eff} = \frac{\frac{M^2}{2m} - \alpha}{r^2}$$

$\therefore$ scattering happens only when $\frac{M^2}{2m} \leq \alpha$
when particle reaches at $r=0$

$M = m\rho v_\infty$

$m\rho^2 v_\infty^2 \leq 2\alpha \longrightarrow \rho \leq \frac{1}{v_\infty}\sqrt{\frac{2\alpha}{m}} \equiv \rho_{max}$

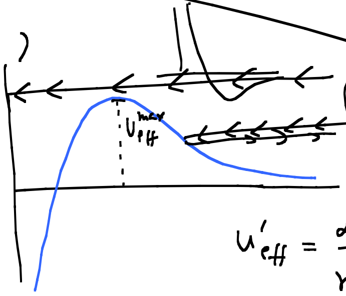


$N_1 = n \cdot \pi \rho_{max}^2$

$\sigma = \frac{N_1}{n} = \pi \rho_{max}^2$

Prob5. $U = -\frac{\alpha}{r^n}$ $\quad (n=1 \; ; \rightarrow \sigma = 0)$

$n>2$ ;



$\left(-\frac{\alpha}{r^n}\right) + \left(\frac{M^2}{2mr^2}\right)$
$\parallel$
$U_{eff}$

$E \geq U_{eff}^{max}$

$U'_{eff} = \frac{\alpha n}{r^{n+1}} - \frac{M^2}{mr^3} = 0$

$r^{n-2} = \frac{\alpha n m}{M^2} \equiv \beta$

$r = (\beta)^{\frac{1}{n-2}}$

$$U_{eff}^{max} = -\alpha \beta^{-\frac{n}{n-2}} + \frac{M^2}{2m}\beta^{-\frac{2}{n-2}}$$

$$= \frac{M^2}{2m}\beta^{-\frac{2}{n-2}}\underbrace{\left(1 - \frac{2\cancel{m}\cancel{\alpha}}{\cancel{M^2}}\frac{1}{\beta}\right)}_{\left(1-\frac{2}{n}\right) > 0} \quad \frac{\cancel{M^2}}{\cancel{\alpha}\cancel{m}n}$$

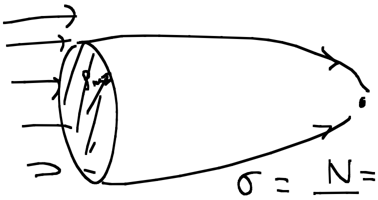
$$= \left(1-\frac{2}{n}\right)\frac{M^2}{2m}\left(\frac{M^2}{nm\alpha}\right)^{\frac{2}{n-2}}$$

$M = m\rho v_\infty$

$$= \underbrace{\left(1-\frac{2}{n}\right)\frac{1}{2}(mv_\infty^2)^{\underbrace{1+\frac{2}{n-2}}_{\frac{n}{n-2}}}}_{C} \boxed{(\rho^2)^{\overbrace{1+\frac{2}{n-2}}^{\frac{n}{n-2}}}} \boxed{\left(\frac{1}{n\alpha}\right)^{\frac{2}{n-2}}}$$

$$U_{eff}^{max} = \underset{\sim}{C} \cdot \rho^{\frac{2n}{n-2}} \leq \frac{1}{2}mv_\infty^2$$

$$0 \leq \rho \leq \rho_{max} = \left(\frac{mv_\infty^2}{2C}\right)^{\frac{n-2}{2n}}$$



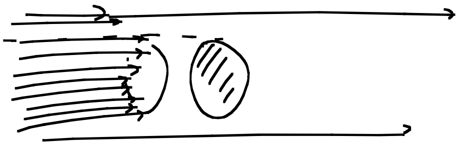
$$\sigma = \frac{N}{n} = \frac{\cancel{n}\pi\rho_{max}^2}{\cancel{n}} = \pi\rho_{max}^2$$

Prob 6.

$U(r) = -\frac{\alpha}{r}$ $\quad r > R$

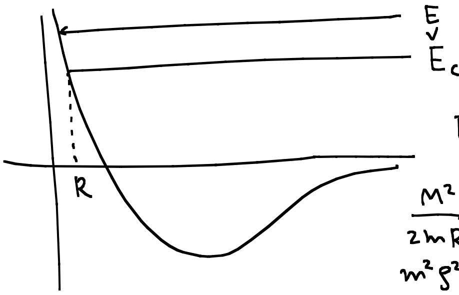
$\alpha = \gamma m_1 m_2$ $\quad G_N$

$m_1$



$U_{eff}(r) = -\frac{\alpha}{r} + \frac{M^2}{2mr^2}$ $\quad m = \frac{m_2 m_1}{m_1+m_2} \approx m_1$

$m_2 \gg m_1$



$E_c = U_{eff}(R)$

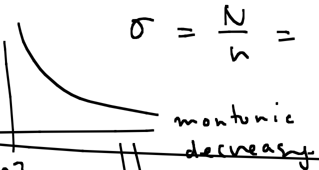
$$\frac{M^2}{2mR^2} - \frac{\alpha}{R} = E_c$$

$$\rho = \rho_{max} \leftarrow \frac{m^2\rho^2 v_\infty^2}{2mR^2} - \frac{\alpha}{R} = \frac{1}{2}mv_\infty^2$$

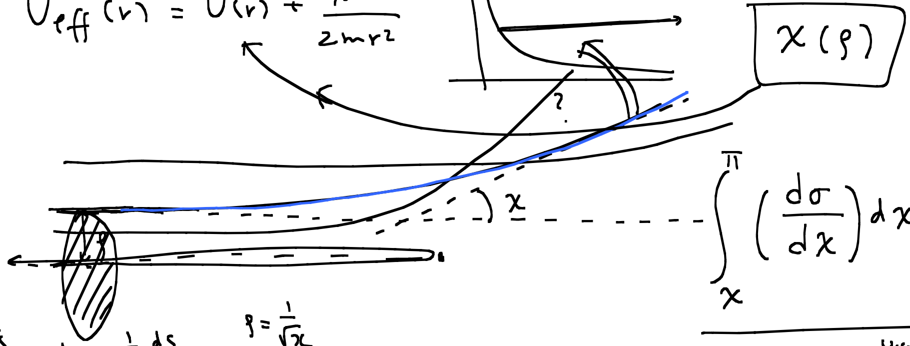
$$\sigma = \frac{N}{n} = \pi\rho_{max}^2 = \frac{2\pi R^2}{mv_\infty^2}\left(\frac{1}{2}mv_\infty^2 + \frac{\alpha}{R}\right)$$

$$= \pi R^2\left(1 + \frac{2\gamma \cancel{m_1} m_2}{\cancel{m}Rv_\infty^2}\right)_{\text{''}}$$

Prob 7. $U(r)$ = repulsive



$$U_{eff}(r) = U(r) + \frac{M^2}{2mr^2}$$



$$\int_\chi^\pi \left(\frac{d\sigma}{d\chi}\right)d\chi = \frac{N(\text{▨})}{n}$$
$$= \pi\rho^2$$

$r = \frac{1}{s} \rightarrow dr = -\frac{1}{s^2}ds$ $\quad \rho = \frac{1}{\sqrt{x}}$

$$\overset{\uparrow}{s} = \frac{1}{r}, \quad x = \frac{1}{\rho^2}, \quad w = \sqrt{1 - \frac{U}{E}} \rightarrow \frac{U}{E} = 1 - w^2$$

$E = \frac{1}{2}mv_\infty^2$

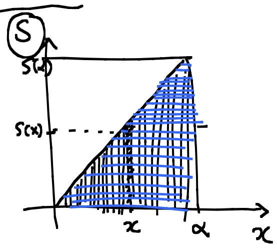
$$\underbrace{\frac{\pi - \chi(x)}{2}} = \phi_0 = \int_{r_{min}}^\infty \frac{\rho/r^2 \, dr}{\sqrt{\underbrace{1 - \frac{\rho^2}{r^2} - \frac{U}{E}}}} \leftarrow \sqrt{E - U_{eff}(r)}$$

$\chi = \pi - 2\phi_0$

$\phi_0 = \frac{\pi - \chi(\rho)}{2}$

$$= \int_0^{s_0} \frac{\overset{\shortparallel}{\frac{1}{\sqrt{x}}} \cancel{s^2} \frac{1}{\cancel{s^2}} ds}{\sqrt{\underbrace{1 - \frac{s^2}{x} - (1 - w^2)}_{w^2 - \frac{s^2}{x}}}} = \int_0^{s_0} \frac{ds}{\sqrt{xw^2 - s^2}}$$

$\underbrace{xw^2 - s^2 = 0}$ when $s = s_0$



$$\int_0^\alpha dx \frac{1}{\sqrt{\alpha - x}} \cdot \frac{\pi - \chi(x)}{2} = \int_0^\alpha dx \int_0^{s(x)} \frac{ds}{\sqrt{\alpha - x}\sqrt{xw^2 - s^2}}$$

$$\frac{\pi}{2}\underbrace{\int \frac{1}{\sqrt{\alpha - x}}dx}_{-2\sqrt{\alpha - x}\Big|_0^\alpha = 2\sqrt{\alpha}} = \int_0^{s(\alpha)} \frac{ds}{w}\int_{x(s)}^\alpha dx \frac{1}{\sqrt{\alpha - x}\sqrt{x - \frac{s^2}{w^2}}}$$

$$\pi\sqrt{\alpha} - \int_0^\alpha dx \frac{\chi(x)}{2\sqrt{\alpha - x}} = \pi\int_0^{s(\alpha)} \frac{ds}{w}$$

$$\int_a^b \frac{dx}{\sqrt{b - x}\sqrt{\underbrace{x - a}_{y}}} = \int_0^{b-a} \frac{dy}{\underset{\underset{z}{\shortparallel}}{\boxed{\sqrt{y}}}\sqrt{\underbrace{(b-a)}_{\alpha^2} - y}}$$

$$\shortparallel$$

$$-\sqrt{\alpha - x}\,\chi(x)\Big|_0^\alpha + \int_0^\alpha \sqrt{\alpha - x}\,\chi'(x)\,dx = \int_0^\alpha \frac{2\cancel{z}dz}{\cancel{z}\sqrt{\alpha^2 - z^2}} = \int_0^{\frac{\pi}{2}} \frac{2\cancel{\alpha}\cancel{\cos\theta}\,d\theta}{\cancel{\alpha}\cancel{\cos\theta}} = \pi$$

$\chi(\overset{\infty}{0}) = \chi(\rho = \infty) = 0$

$x = y + a$, $dz = \frac{dy}{2\sqrt{y}}$

$dy = 2z\,dz$

$z = \alpha\sin\theta$

$dz = \alpha\cos\theta\,d\theta$

$$\pi\sqrt{\alpha} - \int_0^\alpha \sqrt{\alpha - x}\,\chi'(x)\,dx = \pi\int_0^{s(\alpha)} \frac{ds}{w} \qquad (B)$$

differal of $d\alpha$

$$\pi\sqrt{\alpha + d\alpha} - \int_0^{\alpha + d\alpha} \underbrace{\sqrt{\alpha + d\alpha - x}}\,\chi'(x)\,dx = \pi\int_0^{s(\alpha + d\alpha)} \frac{ds}{w} \qquad (A)$$

$\sqrt{\alpha + d\alpha} \overset{\shortparallel}{=} \sqrt{\alpha}\left(1 + \frac{d\alpha}{2\alpha}\right)$

$(\alpha - x)\left(1 + \frac{d\alpha}{\alpha - x}\right)$

$A - B$

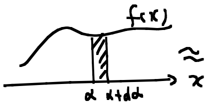
$$\overset{\shortparallel}{\sqrt{\alpha - x}}\left(1 + \underbrace{\frac{d\alpha}{2(\alpha - x)}}\right) = \sqrt{\alpha - x} + \frac{d\alpha}{2\sqrt{\alpha - x}}$$

$$\frac{\pi}{2}\frac{d\alpha}{\sqrt{\alpha}} - \underbrace{\left[\int_0^{\alpha+d\alpha}\sqrt{\alpha-x}\,\chi'(x)\,dx - \int_0^{\alpha}\sqrt{\alpha-x}\,\chi'(x)\,dx\right]}_{} - d\alpha\int_0^{\alpha}\frac{\chi'(x)}{2\sqrt{(\alpha-x)}}dx$$

$$\int_{\alpha}^{\alpha+d\alpha}\underbrace{\sqrt{\alpha-x}\ \chi'(x)}\,dx \approx d\alpha\ \sqrt{\alpha-x}\ \chi'(x)\Big|_{x=\alpha}=0$$

$$S(\alpha+d\alpha)=S(\alpha)+d\alpha\,S'(\alpha)$$

$$= \pi\int_0^{S(\alpha+d\alpha)}\frac{ds}{w} - \pi\int_0^{S(\alpha)}\frac{ds}{w}$$



$$\approx d\alpha\,f(\alpha)$$

$$= \pi\int_{S(\alpha)}^{S(\alpha)+\underbrace{d\alpha\,S'(\alpha)}}\frac{1}{w}\,ds \approx \pi\frac{\cancel{d\alpha}\,\boxed{S'(\alpha)}\overset{=\frac{ds}{\cancel{d\alpha}}}{}}{w} = \pi\frac{ds}{w}$$

$$\frac{\pi}{2}\frac{d\alpha}{\sqrt{\alpha}} - d\alpha\int_0^{\alpha}\frac{\chi'(x)}{2\sqrt{(\alpha-x)}}dx = \pi\frac{ds}{w}$$

$$\alpha \equiv \frac{S^2}{W^2} \longrightarrow d\alpha = 2\frac{S}{W}d\left(\frac{S}{W}\right)$$

$$\sqrt{\alpha}=\frac{S}{W}$$

$$\pi\, d\left(\frac{S}{W}\right) - 2\frac{S}{W}d\left(\frac{S}{W}\right)\int_0^{\frac{S^2}{W^2}}\frac{\chi'(x)}{\cancel{2}\sqrt{\frac{S^2}{W^2}-x}}dx = \cancel{\frac{\pi ds}{w}}$$

$$\pi\left(\overset{\downarrow}{\cancel{\frac{ds}{w}}} - \frac{s\,dw}{w^2}\right) - \cancel{2}\frac{S}{W}d\left(\frac{S}{W}\right)\int\ \text{"}$$

$$d\left(\frac{S}{W}\right) = \frac{ds}{w} - \frac{s\,dw}{w^2}$$

$$\cancel{-}\,\pi\,\cancel{\frac{S}{W}}\underbrace{\frac{dw}{w}}_{d(\log w)} \cancel{-}\,\cancel{\frac{S}{W}}\,d\left(\frac{S}{W}\right)\int_0^{\frac{S^2}{W^2}}\frac{\chi'(x)}{\sqrt{\phantom{x}}}dx = 0$$

$$\boxed{\pi\ d\log w = -d\left(\frac{S}{W}\right)\underbrace{\int_0^{\frac{S^2}{W^2}}\frac{\chi'(x)}{\sqrt{\frac{S^2}{W^2}-x}}dx}}$$

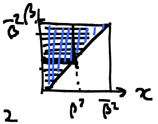
$$\pi\log w = \pi\int_1^{\log w}d(\log w) = -\int_0^{\frac{S}{W}}\left[\ \ \right]d\left(\frac{S}{W}\right) \ -\ \boxed{w = e^{-\frac{1}{\pi}\underbrace{\int_0^{\frac{S^2}{W^2}}\left[\ \ \right]d\left(\frac{S}{W}\right)}}}$$

$$w \equiv \sqrt{1-\frac{U}{E}} \rightarrow w=1 \text{ when } U\overset{(\infty)}{=}0 \leftarrow S=\frac{1}{r}\overset{r\to\infty}{=}0$$

$$d\beta^2 = 2\beta\,d\beta$$

$$\int_0^{\frac{S}{W}}\left[\int_0^{\frac{S^2}{W^2}}\frac{\chi'(x)}{\sqrt{\underset{\overset{\shortparallel}{\beta^2}}{\boxed{\frac{S^2}{W^2}}}-x}}dx\right]d\underset{\overset{\shortparallel}{\beta}}{\left(\frac{S}{W}\right)} = \int_0^{\bar{\beta}^2}\frac{d(\beta^2)}{\boxed{2}\beta}\int_0^{\beta^2}\frac{\chi'(x)}{\sqrt{\beta^2-x}}dx$$

$$=$$



$$= \int_0^{\bar{\beta}^2}dx\,\chi'(x)\int_x^{\bar{\beta}^2}\frac{1}{\beta\sqrt{\beta^2-x}}d\beta^2$$

$$\int_x^{\bar{\beta}^2}\frac{d\beta^2}{2\sqrt{\beta^2}\ \sqrt{\beta^2-x}}$$

$$d\beta = \sqrt{x}\ \sinh y\,dy$$

$$\beta \equiv \sqrt{x}\ \cosh y$$

$$\beta^2 = x\cosh^2 y$$

$$\sqrt{\beta^2-x} = \sqrt{x}\ \sqrt{\sinh^2 y}$$

$$= \int\frac{2\cancel{\beta}\,d\beta}{2\cancel{\beta}\sqrt{\beta^2-x}}$$

$$\beta^2 = \bar{\beta}^2 = x\cosh^2\bar{y}$$

$$\bar{y} = \cosh^{-1}\left(\frac{\bar{\beta}}{\sqrt{x}}\right)$$

$$= \int_0^{\cosh^{-1}\left(\frac{\bar{\beta}}{\sqrt{x}}\right)}\frac{2\cancel{\sqrt{x}}\,\cancel{\sinh y}\,dy}{2\cancel{\sqrt{x}}\,\cancel{\sinh y}}$$

$$= \cosh^{-1}\left(\frac{\bar{\beta}}{\sqrt{x}}\right)$$

$$= \int_0^{\bar{\beta}^2}dx\ \cosh^{-1}\left(\frac{\bar{\beta}}{\sqrt{x}}\right)\frac{d\chi}{dx}$$

$$x = \frac{1}{\rho^2} \quad dx = -\frac{2}{\rho^3}d\rho$$

$$= \int_{\infty}^{\frac{1}{\bar{\beta}}}\cancel{-\frac{2}{\rho^3}}\ \cosh^{-1}\left(\frac{\rho}{rw}\right)\frac{d\chi}{d\rho}\cancel{\left(-\frac{\rho^3}{2}\right)}d\rho = \int_{\infty}^{rw}\cosh^{-1}\left(\frac{\rho}{rw}\right)\frac{d\chi}{d\rho}d\rho$$

$$\bar{\beta}^2 = \frac{S^2}{W^2} = \frac{1}{r^2W^2} \longrightarrow \frac{1}{\bar{\beta}} = rw$$

$$w = e^{-\frac{1}{\pi}\underbrace{\int_{\infty}^{rw}}_{\ominus\int_{rw}^{\infty}}\cosh^{-1}\left(\frac{\rho}{rw}\right)\frac{d\chi}{d\rho}\,d\rho}$$

$$= e^{\underbrace{+\frac{1}{\pi}\int_{rw}^{\infty} \cosh^{-1}\left(\frac{\rho}{rw}\right) \frac{dx}{d\rho}\, d\rho}}$$

$$\rightarrow \underbrace{\cosh^{-1}\left(\frac{\rho}{rw}\right) x(\rho)\Big|_{rw}^{\infty}}_{=0} = 0$$

$$- \int \underbrace{\cosh^{-1}\left(\frac{\rho}{rw}\right)} x(\rho)\, d\rho$$

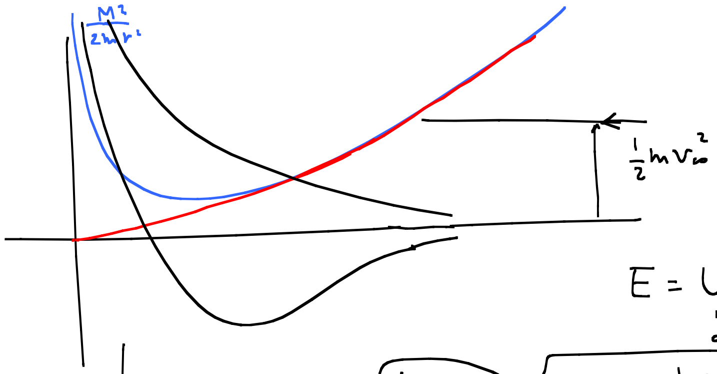
$$\frac{1}{rw}\frac{1}{\sqrt{\left(\frac{\rho}{rw}\right)^2 - 1}} = \frac{1}{\sqrt{\rho^2 - r^2w^2}}$$

$$\boxed{w = e^{-\frac{1}{\pi}\int_{rw}^{\infty} \frac{\boxed{x(\rho)}}{\sqrt{\rho^2 - r^2w^2}}\, d\rho}}$$

$$= f(rw) \Rightarrow w = g(r) = \sqrt{1 - \frac{U}{E}}$$

$$\Rightarrow U = U(r) \,//$$

$$\frac{1}{r} \qquad \frac{1}{r}e^{-\alpha r}$$



$$E = U + \frac{1}{2}mv^2$$
$$\overset{\shortparallel}{\infty}$$

1972

~~Coleman~~ Harvard

Politzer (Caltech)

Gross (Princeton)

Wilzek

$U \to 0$ $(r \to \infty)$ asymptotic free $(r \to 0)$

$r \ll r_c \sim 1\text{fm}$

$$U(r) \sim \frac{1}{1 \pm \underset{\frac{1}{137}}{\alpha} \underbrace{\log \frac{r_c}{r}}_{137}} \xrightarrow[r \to 0]{} 0$$

$$r = \underbrace{1\text{fm}}_{10^{-13}}\ \underbrace{e^{-137}}_{10^{-30}} \sim 10^{-43}\text{m}$$

"Landau pole"

## §19. Rutherford's Scattering

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{\rho/r^2}{\sqrt{1-\frac{\rho^2}{r^2}-\frac{2\alpha}{mv_\infty^2 r}}}\,dr$$

$U = \frac{\alpha}{r}$

$\alpha = \begin{cases} \frac{e^2}{4\pi\varepsilon_0} \\ -\gamma m_1 m_2 \end{cases}$

$r \rightarrow u = \frac{1}{r}$ $du = -\frac{1}{r^2}dr$

$$= \int_0^{u_{max}} \frac{du}{\sqrt{\left(\frac{1}{\rho^2} - u^2 - \gamma u\right)}} \quad \left(\gamma \equiv \frac{2\alpha}{\rho^2 m v_\infty^2}\right) = \int \frac{du}{\sqrt{\left(\frac{1}{\rho^2}+\frac{\gamma^2}{4}\right) - \left(u+\frac{\gamma}{2}\right)^2}} = \int_0^{\phi_0} \frac{A\cancel{\cos\theta d\theta}}{\sqrt{A^2 - \cancel{A^2\sin^2\theta}}}$$

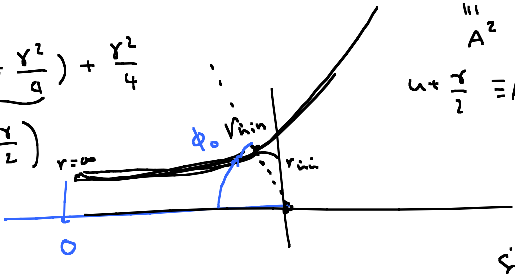
$$\frac{1}{\rho^2} - \underbrace{\left(u^2 + \gamma u + \frac{\gamma^2}{4}\right)}_{\left(u+\frac{\gamma}{2}\right)} + \frac{\gamma^2}{4}$$

$\left(\frac{1}{\rho^2}+\frac{\gamma^2}{4}\right) \overset{|||}{} A^2$

$= \arcsin\left(\frac{u+\frac{\gamma}{2}}{A}\right)\Big|_0^{u_{max}}$

$u_{max} = \frac{1}{r_{min}}$

$$\boxed{u_{max} + \frac{\gamma}{2} = A}$$



$u + \frac{\gamma}{2} \equiv A\sin\theta \rightarrow du = A\cos\theta\, d\theta$

$\downarrow$

$\phi_0 = \frac{\pi}{2} - \sin^{-1}\left(\frac{\gamma}{2A}\right)$

$\sin^{-1}\left(\frac{\gamma}{2A}\right) = \frac{\pi}{2} - \phi_0$

$\frac{\gamma}{2A} = \sin\left(\frac{\pi}{2} - \phi_0\right) = \cos\phi_0$

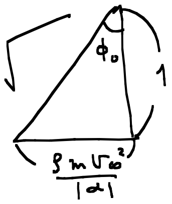
$$\phi_0 = \cos^{-1}\left(\frac{\gamma/2}{\sqrt{\frac{1}{\rho^2}+\left(\frac{\gamma}{2}\right)^2}}\right) \leftarrow$$

$\alpha > 0$

$$= \cos^{-1}\left(\frac{1}{\sqrt{1+\frac{4}{\rho^2\gamma^2}}}\right)$$

$\rho^2\gamma^2 = \frac{4\alpha^2}{\rho^2 m^2 v_\infty^4}$

$$\therefore \phi_0 = \cos^{-1}\left(\frac{1}{\sqrt{1+\frac{\rho^2 m^2 v_\infty^4}{\alpha^2}}}\right) ''$$



$\alpha > 0$

$\phi_0 = \frac{1}{2}(\pi - \chi)$

$\tan\phi_0 = \cot\frac{\chi}{2}$

$\rightarrow \tan\phi_0 = \left(\frac{|\alpha|}{\rho m v_\infty^2}\right)^{-1} \rightarrow \rho = \frac{\alpha\tan\phi_0}{mv_\infty^2}$

$$\boxed{\rho^2 = \frac{\alpha^2}{m^2 v_\infty^4}\cot^2\frac{\chi}{2}}$$

$\frac{\alpha^2}{m^2 v_\infty^4} = \beta^2$

$\rightarrow \rho = \beta\cot\frac{\chi}{2}$

$\frac{d\rho}{d\chi} = -\frac{\beta}{2}\frac{1}{\sin^2\frac{\chi}{2}}$

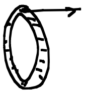
$\begin{cases} 2\phi_0 + \chi = \pi \\ \chi = \pi - 2\phi_0 \quad 0 \le \phi_0 < \frac{\pi}{2} \quad (\alpha > 0) \\ \phi_0 + \phi_0 - \chi = \pi \quad (\alpha<0) \end{cases}$

$\chi = 2\phi_0 - \pi$, $\frac{\pi}{2} \le \phi_0 \le \pi$

$\phi_0 = \frac{\chi+\pi}{2} \rightarrow \tan\phi_0 = -\cot\frac{\chi}{2}$

CM: $\frac{d\sigma}{d\Omega} = \frac{\rho}{(\sin\chi)}\left|\frac{d\rho}{d\chi}\right| = \frac{1}{2}\frac{\beta^2\cos\frac{\chi}{2}}{\underbrace{\sin\chi}\sin^3\frac{\chi}{2}} = \frac{\frac{\alpha^2}{m^2v_\infty^4}}{4\sin^4\frac{\chi}{2}}$

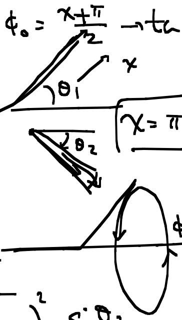
$2\sin\frac{\chi}{2}\cos\frac{\chi}{2}$



$$\boxed{d\sigma = 2\pi\rho\left|\frac{d\rho}{d\chi}\right|d\chi}$$

$d\sigma = 2\pi\beta^2 \frac{\cot\frac{\chi}{2}}{2\sin^2\frac{\chi}{2}}d\chi$

$= \pi\beta^2\frac{\cos\frac{\chi}{2}}{\sin^3\frac{\chi}{2}}d\chi$



$$\boxed{\chi = \pi - 2\theta_2}$$

$|d\chi| = 2|d\theta_2|$

$\phi \rightarrow 2\pi$

LAB: $d\sigma_2 = \pi\left(\frac{\alpha}{2mv_\infty^2}\right)^2 \frac{\cos(\;'' \;)}{\sin^3\left(\frac{\pi-2\theta_2}{2}\right)}d\chi$

$\overset{||}{\frac{\sin\theta_2}{\cos^3\theta_2}} \rightarrow d\sigma_2 = 2\pi\left(\frac{\alpha}{2mv_\infty^2}\right)^2\frac{\sin\theta_2}{\cos^3\theta_2}d\theta_2$

$d\Omega_2 \overset{||}{} 2\pi\sin\theta_2 d\theta_2$

$= \left(\frac{\alpha}{2mv_\infty^2}\right)^2\frac{d\Omega_2}{\cos^3\theta_2}$

H.W.

$d\sigma_1 = \pi\beta^2\frac{\cos\frac{\chi}{2}}{\sin^3\frac{\chi}{2}}d\chi$

$$\boxed{\tan\theta_1 = \frac{\sin\chi}{\frac{m_1}{m_2}+\cos\chi}} \rightarrow \chi \text{ in terms of } \theta_1$$

if Indistinguishable : $m_1 = m_2 \rightarrow \chi = 2\theta_1 \rightarrow d\sigma_1 = 2\pi\beta^2\frac{\cos\theta_1}{\sin^3\theta_1}d\theta_1$

$= \beta^2\frac{\cos\theta_1}{\sin^4\theta_1}d\Omega_1$

$\overset{\uparrow}{2\pi\sin\theta_1 d\theta_1}$

LAB
$\theta_1, \theta_2 \rightarrow \theta$

$\downarrow$

$$d\sigma = \left(\frac{\alpha}{2mv_\infty^2}\right)^2\left(\frac{1}{\sin^4\theta}+\frac{1}{\cos^4\theta}\right)\cos\theta\, d\Omega$$

---

in terms of $\epsilon = \frac{1}{2}m_2 v_2'^2 = \underbrace{\frac{2m^2}{m_2}v_\infty^2}_{\overset{|||}{\beta}}\sin^2\frac{\chi}{2} \rightarrow d\epsilon = \cdots \sin\frac{\chi}{2}\cos\frac{\chi}{2}\frac{1}{2}d\chi$

$\uparrow$
Energy of target

$\frac{d\epsilon}{\epsilon} = \frac{1}{2}\frac{\cos\frac{\chi}{2}}{\sin\frac{\chi}{2}}d\chi$

H.W: Problem 54, 55

$$d\sigma = 2\pi\rho^2 \frac{1}{\boxed{\sin^2\frac{\chi}{2}}} \frac{d\epsilon}{\epsilon} = 2\pi\rho^2 B \frac{d\epsilon}{\epsilon^2}$$

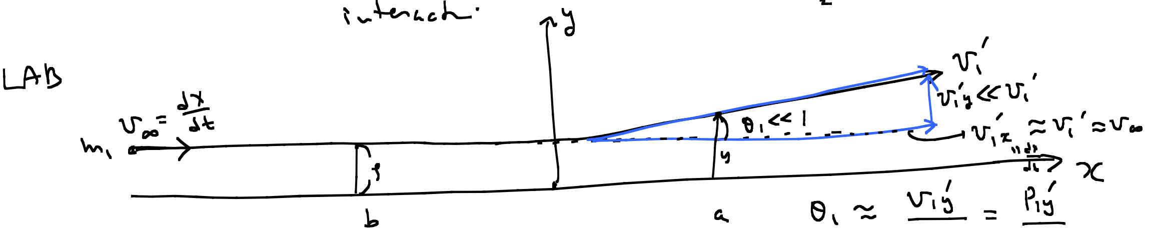
$$\sin^2\frac{\chi}{2} \rightarrow \frac{\epsilon}{B}$$

$$\frac{d\epsilon}{\epsilon} = 2\frac{\sin\cancel{\chi/2}}{\cancel{\cos\frac{\chi}{2}}}$$

$$d\sigma = \frac{2\pi\rho^2}{2}\frac{\cancel{\cos\frac{\chi}{2}}}{\sin^3\frac{\chi}{2}}\boxed{d\chi}$$

## §20. Small-angle ( $|\alpha| << 1$ )

$\uparrow$ interaction



$$\theta_1 \approx \frac{v'_{1y}}{v_1'} = \frac{P'_{1y}}{P_1' \approx P_1 = m_1 v_\infty}$$

$$\frac{d}{dt}P_y = F_y \longrightarrow P'_{1y} = \int_{-\infty}^{\infty} F_y\, dt \quad dt = \frac{dx}{v_\infty}$$

$$\downarrow$$

$$\Delta P_y = \int F_y\, dt$$

$$F_y = -\frac{y}{r}U'(r)$$

$$F_y = -\frac{\partial U(r)}{\partial y} \quad r = \sqrt{x^2+y^2}$$

$$= -U'(r)\boxed{\frac{\partial r}{\partial y}} = \frac{y}{r}$$

approx: $y \approx \rho$

$$\therefore \quad P'_{1y} = -\int_{-\infty}^{\infty} \frac{\rho}{\cancel{r}} U'(r) \frac{\boxed{dx}}{v_\infty} \rightarrow \frac{\cancel{r}}{\sqrt{r^2-\rho^2}}\, dr$$

$$r = \sqrt{x^2+y^2} \approx \sqrt{x^2+\rho^2} \rightarrow x^2 = r^2 - \rho^2$$

$$dr = \frac{x\, dx}{r} = \frac{\sqrt{r^2-\rho^2}}{r}dx$$

$$\boxed{\theta_1 = -\frac{\rho}{m_1 v_\infty^2} 2\int_\rho^\infty \frac{U'(r)}{\sqrt{r^2-\rho^2}}\, dr} \leftarrow$$

$x: -\infty \rightarrow \infty \quad r \rightarrow 2 \times \rho \rightarrow \infty$

$\leftarrow$ $\theta_1 << 1$

$$\downarrow$$

$$d\sigma = \left|\frac{d\rho}{d\theta_1}\right| \frac{\rho}{\boxed{\sin\theta_1}_{\approx \theta_1}} d\theta_1$$

Probl. $$\phi_0 = \int_{r_{min}}^{\infty} \frac{\rho/r^2}{\sqrt{1 - \frac{\rho^2}{r^2} - \frac{2U}{mv_\infty^2}}}\, dr \qquad U << 1$$

$$= -\frac{\partial}{\partial\rho}\int \sqrt{\underbrace{\underbrace{1-\frac{\rho^2}{r^2}}_{a} - \boxed{\frac{2U}{mv_\infty^2}}}_{\epsilon}}\, dr$$

$$\sqrt{a-\epsilon} = \sqrt{a}\sqrt{1-\frac{\epsilon}{a}}$$
$$= \sqrt{a}\left(1-\frac{\epsilon}{2a}\right)$$

Prob2. $$U = \frac{\alpha}{r^n} \quad (\alpha << 1)$$

$$U' = -\frac{\alpha n}{r^{n+1}}$$

$$\int_\rho^\infty \frac{U'}{\sqrt{r^2-\rho^2}}\, dr$$

$$= -\alpha n \underbrace{\int \frac{\frac{1}{r^{n+1}}}{\sqrt{r^2-\rho^2}}\, dr}$$

$$\frac{\rho^2}{r^2} = u$$

$\rightarrow$ $\Gamma$ - function

# Chap 5. small oscillation

## §21. one d.o.f. $x$

$$U(x) = \frac{1}{2} k x^2$$

$$m\ddot{x} + kx = 0 \longrightarrow \ddot{x} + \omega^2 x = 0 \qquad \omega^2 = \frac{k}{m}$$

$$E = \frac{1}{2} k a^2 = \frac{1}{2} m \omega^2 a^2$$

$$x = c_1 \cos\omega t + c_2 \sin\omega t$$

$$= \underset{\substack{\uparrow \\ \text{real} > 0}}{a} \cos(\omega t + \underset{\sim}{\alpha}) \rightarrow \mathrm{Re}\left(\underline{A e^{i\omega t}}\right) \qquad A = a e^{i\alpha}$$

H.W.: on 60, 61

## §22. forced.

$$m\ddot{x} + kx = F(t) \longrightarrow \ddot{x} + \omega^2 x = \frac{F}{m}$$

$$F(t) = f \cos(\underset{\uparrow}{\gamma} t + \beta) \rightarrow x \overset{\text{assume}}{=} \frac{f/m}{\omega^2 - \gamma^2} \cos(\gamma t + \beta)$$

$$(\omega^2 b - \gamma^2 b) \cos(\gamma t + \beta) = \frac{f}{m} \cos(\gamma t + \beta)$$

$$b = \frac{f/m}{\omega^2 - \gamma^2}$$

$$\boxed{x = a\cos(\omega t + \alpha) + \frac{f/m}{\omega^2 - \gamma^2} \cos(\gamma t + \beta)}$$

$$x = a\cos(\omega t + \alpha) + \frac{f/m}{\omega^2 - \gamma^2}\left(\cos(\gamma t + \beta) - \cos(\omega t + \beta)\right) + \frac{f/m}{\omega^2 - \gamma^2}\cos(\omega t + \beta)$$

$$a' \cos(\omega t + \alpha')$$

$$x = a' \cos(\omega t + \alpha') + \underbrace{\frac{f/m}{\omega^2 - \gamma^2}\left(\cos(\gamma t + \beta) - \underbrace{\cos(\omega t + \beta)}\right)}_{} \frac{\partial}{\partial \omega}$$

resonance ; $\gamma \rightarrow \omega$

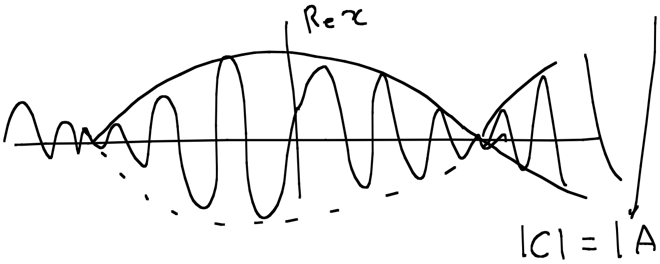
$$\underbrace{\frac{f/m}{2\omega} \textcircled{t}}_{\text{Amplitude}} \sin(\omega t + \beta)$$

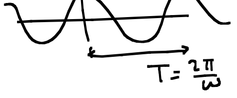
near resonance : $\gamma = \omega + \epsilon$

$\omega \gg \epsilon$

$$x = A e^{i\omega t} + B e^{i\overbrace{(\omega + \epsilon)}^{\gamma} t} = \underset{\substack{\downarrow \\ \cos\omega t}}{\underset{\sim}{e^{i\omega t}}} \overbrace{\underset{\sim}{(A + B e^{i\epsilon t})}}^{C}$$

$$(A = a e^{i\alpha} \quad B = b e^{i\beta})$$





over T : $\epsilon \cdot \frac{2\pi}{\omega} \ll 1$

$$|C| = |A + B e^{i\epsilon t}|$$

$$= \sqrt{a^2 + b^2 + 2ab \underbrace{\cos(\epsilon t + \beta - \alpha)}_{-1 \leq \quad \leq 1}}$$

$$= \begin{matrix} a + b & \text{max} \\ |a - b| & \text{min} \end{matrix}$$

in general F(t) $$\ddot{x}+\omega^2 x=\frac{F}{m}=\frac{d}{dt}(\dot{x}+\underset{\sim}{i\omega x})-i\omega\underbrace{(\dot{x}+i\omega x)}_{\xi}=\dot{\xi}-i\omega\xi$$ $\underset{\sim}{\xi\equiv\dot{x}+i\omega x}$

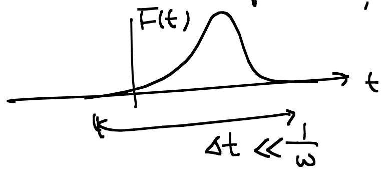
assume $\underline{\xi(t)=A(t)e^{i\omega t}}$ $$\rightarrow \dot{\xi}=\dot{A}e^{i\omega t}+i\omega A\cancel{e^{i\omega t}}$$
$$+\big)\; -i\omega\xi=-i\omega A\cancel{e^{i\omega t}}$$
$$\frac{F}{m}=\dot{A}e^{i\omega t}\rightarrow\frac{dA}{dt}=\frac{F(t)}{m}e^{-i\omega t}$$

$$\xi=e^{i\omega t}\left(A_0+\int_0^t\frac{F(t)}{m}e^{-i\omega t}dt\right)\leftarrow A=\underset{\overset{\shortparallel}{A(0)}}{A_0}+\int_0^t\frac{F(t)}{m}e^{-i\omega t}dt \leftarrow$$
$$\shortparallel$$
$$\underbrace{\dot{x}}_{\text{real}\,\xi}+i\omega\underbrace{x}_{\text{Im}\,\xi}$$
$$\underline{x=\frac{1}{\omega}\,\text{Im}\left[e^{i\omega t}\left(A_0+\int_0^t\frac{F(t)}{m}e^{-i\omega t}dt\right)\right]}$$

with initial condition at $t=-\infty$ ; $\xi(-\infty)=0\rightarrow\dot{x}(-\infty)=x(-\infty)=0$

$$\xi=e^{i\omega t}\left(\overset{0}{\cancel{A_0}}+\int_{-\infty}^t\frac{F(t)}{m}e^{-i\omega t}dt\right)=\dot{x}+i\omega x$$
$$|\xi(\infty)|^2=\left|\int_{-\infty}^{\infty}\frac{F(t)}{m}e^{-i\omega t}dt\right|^2=\dot{x}^2+\omega^2x^2\Big|_{t=\infty}$$
$$=\underbrace{\left[\frac{m}{2}(\quad)\right]}_{E(\infty)}\frac{2}{m}=\frac{2E}{m}$$
$$E=\frac{1}{2m}\Bigg|\underbrace{\int_{-\infty}^{\infty}F(t)e^{-i\omega t}dt}_{\tilde{F}(\omega)}\Bigg|^2$$

in particular, $F(t)\neq0$ for small period ;



$$e^{-i\omega t}\sim1$$
$$\uparrow$$
$$\omega t\ll1$$

$$\underline{E\simeq\frac{1}{2m}\left(\int_{-\infty}^{\infty}F(t)\,dt\right)^2}$$
$$\uparrow$$
$$\int_{-\infty}^{\infty}F(t)\,dt=\Delta p$$

H.W. on pages 64, 65.

§23. many d.o.f. $x_i$ $i=1,\cdots,s$

$$T=\frac{1}{2}\sum_{i,k=1}^{s}m_{ik}\underset{\sim}{\dot{x}_i}\underset{\sim}{\dot{x}_k}\qquad L=T-U$$
$$U=\frac{1}{2}\sum_{i,k=1}^{s}k_{ik}x_ix_k$$

$$\underset{\sim}{\frac{d}{dt}}\frac{\partial L}{\partial\dot{x}_j}=\frac{\partial L}{\partial x_j}\longrightarrow\boxed{\sum_{k=1}^{s}(m_{jk}\ddot{x}_k+k_{jk}x_k)=0}$$
$$\Uparrow$$
$$\frac{1}{2}\sum_{ⓘk=1}^{s}m_{ik}(\underset{\sim}{\delta_{ij}}x_k+\delta_{kj}\dot{x}_i)$$
$$\underline{\sum_k m_{jk}\dot{x}_k}+\underline{\sum_i m_{ij}\dot{x}_i}$$
$$m_{ij}=m_{ji}$$

Ansatz ; $$x_k=A_ke^{i\underset{=}{\omega}t}\qquad k=1,2,\cdots,s$$
$\uparrow$ indep of k

$$\ddot{x}_k=-\omega^2A_ke^{i\omega t}\quad\Longrightarrow\boxed{\sum_{k=1}^{s}\underbrace{(-\omega^2m_{jk}+k_{jk})}\underset{\sim}{A_k}=0}$$
$$\Downarrow$$
$$(\mathbb{X}\vec{A})=0$$

$$|\mathbb{X}|=|\underset{\uparrow}{\mathbb{k}}-\omega^2\underset{\uparrow}{\mathbb{M}}|=0\longrightarrow\omega_1^2\;\cdots\;\overset{}{ⓦ}\;\omega_s^2$$
$\hookrightarrow$ $\underline{\text{deg } s}$ $\underline{\text{polynomial of } \omega^2}$ $\qquad \omega_\alpha^2$

$$\det\mathbb{X}=0\;\;\swarrow\text{for}\;\vec{A}\neq0$$

$$s\left(\underbrace{\begin{vmatrix} k_{11}-\omega^2 m_{11} & k_{12}-\omega^2 m_{12} & & \\ k_{12}-\omega^2 m_{12} & k_{22}-\omega^2 m_{22} & & \\ & & \omega^1 & \\ & & & \omega^2 \end{vmatrix}}_{s}\right) = (\omega^2)^{s}$$

$\omega^2 = \omega_\alpha^2$

$$\underbrace{(\mathbb{k} - \omega_\alpha^2 \mathbb{m})}_{\mathbb{M}} \cdot \vec{A} = 0 \quad \rightarrow \sum_{j=1}^{s} \underline{\mathbb{M}_{kj}} A_j^{(\alpha)} = 0 \text{ for all } k=1,\cdots,s \qquad \rightarrow A_k = \Delta_{k\alpha} \text{ "minor"}$$

$m_{kj} = m_{jk}$

$$\mathbb{M} = \underbrace{\begin{pmatrix} m_{11} & \cdots & m_{1\alpha} & \cdots & m_{1s} \\ \vdots & & m_{2\alpha} & & \vdots \\ \vdots & & & & \vdots \\ m_{s1} & \cdots & m_{s\alpha} & \cdots & m_{ss} \end{pmatrix}}$$

$$\rightarrow \underset{\underset{0}{\parallel}}{|\mathbb{M}|} = \left(m_{1\alpha} \underbrace{\Delta_{1\alpha}}_{\uparrow} + m_{2\alpha}\Delta_{2\alpha} + \cdots + m_{s\alpha}\Delta_{s\alpha}\right) \nwarrow k$$

$(-1)^{1+\alpha} \det (s-1)\times(s-1)$ matrix

$$0 = m_{1k}A_1 + m_{2k}A_2 + \cdots + m_{sk}A_s$$

$$\boxed{\Delta_{j\alpha} = A_j^{(\alpha)}}$$

$$\vec{A}^{(\alpha)} = \vec{\Delta}_\alpha \quad \leftarrow \omega^2 = \omega_\alpha^2$$

$$(\vec{\Delta}_\alpha)_j \equiv \Delta_{j\alpha}$$

$$\Rightarrow \boxed{\vec{x}^{(\alpha)} = C_\alpha \vec{A}^{(\alpha)} e^{i\omega_\alpha t}}$$ normal mode

$\alpha = 1, \cdots, s$

$$\vec{x} = \sum_{\alpha=1}^{s} C_\alpha \vec{A}^{(\alpha)} e^{i\omega_\alpha t} \quad \text{or} \quad x_k = \mathrm{Re}\sum_{\alpha=1}^{s} C_\alpha \underbrace{\vec{A}_k^{(\alpha)}}_{\Delta_{k\alpha}} e^{i\omega_\alpha t}$$

$$= \sum_\alpha \underbrace{\Delta_{k\alpha}} \underset{\underset{\mathrm{Re}\left(\underline{\underline{C_\alpha}} e^{i\omega_\alpha t}\right)}{\parallel\!\parallel}}{\Theta_\alpha}$$

External force : $x_i \rightarrow$ Add $\sum_k F_k(t) x_k$ to $L$

$$L = \frac{1}{2} \sum_{k,j} m_{jk} \dot{x}_j \dot{x}_k - \frac{1}{2}\sum_{k} k_{jk} x_j x_k + \sum_j F_j(t) x_j \qquad x_k = \sum_\alpha \Delta_{k\alpha} \Theta_\alpha$$

$$\Theta_\alpha \uparrow e^{i\omega_\alpha t}, \quad \underline{\underline{\ddot{\Theta}_\alpha = -\omega_\alpha^2 \Theta}}$$

$$= \underline{\frac{1}{2} \sum_\alpha \left(\dot{\Theta}_\alpha^2 - \omega_\alpha^2 \Theta_\alpha^2\right)} + \sum_\alpha \boxed{\sum_j F_j \Delta_{j\alpha}} \Theta_\alpha - \rightarrow \sum_\alpha f_\alpha(t) \Theta_\alpha$$

$\swarrow \parallel\!\parallel\; f_\alpha(t)$

$$\rightarrow \ddot{\Theta}_\alpha + \omega_\alpha^2 \Theta_\alpha = f_\alpha(t)$$

H.W. Prob. on 69, 70.

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§25. damped. $\alpha \dot{x}$ $\longrightarrow$ $m\ddot{x} + kx + \underline{\alpha\dot{x}} = 0$

$$\underline{\boxed{\overset{\dagger}{\ddot{x}} + 2\lambda\dot{x} + \omega_0^2 x = 0}} \qquad \frac{k}{m} \equiv \omega_0^2, \ \frac{\alpha}{m} \equiv 2\gamma$$

$$\therefore r = i\lambda \pm \sqrt{\omega_0^2 - \lambda^2}$$

$$x = e^{irt} \rightarrow \underline{-r^2 + 2i\lambda r + \omega_0^2 = 0}$$

$$r^2 - 2i\lambda r - \lambda^2 = \omega_0^2 - \lambda^2$$

$$\overset{\parallel}{(r - i\lambda)^2}$$

$$\underline{x = e^{-\lambda t}\left(c_1 e^{i\sqrt{\omega_0^2 - \lambda^2}\,t} + c_2 e^{-i\sqrt{\omega_0^2-\lambda^2}\,t}\right)}$$

if $\omega_0 > \lambda$ : underdamped $\quad x = e^{-\lambda t} a \cos(\omega t + \alpha)$

where $\omega = \sqrt{\omega_0^2 - \lambda^2}$

$$\bar{E} \propto A^2 = e^{-2\lambda t} a^2 \Rightarrow \underline{\bar{E} = E_0 e^{-2\lambda t}}$$

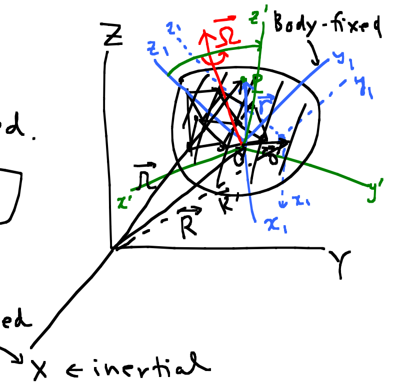
if $\lambda > \omega_0$ : overdamped $\rightarrow r = i\left(\lambda \underline{\pm} \sqrt{\lambda^2 - \omega_0^2}\right) \rightarrow x = e^{-(\lambda + \sqrt{\ })t} + e^{-(\lambda - \sqrt{\ })t}$

if $\lambda = \omega_0$ $r = i\lambda$ $c_1 e^{-\lambda t} + \underline{\underline{c_2 t e^{-\lambda t}}}$

$$\underbrace{\left( m_{ij} r^2 + \alpha_{ij} r + k_{ij} \right)} \cdot A_j = 0$$

$$\det |\quad| = 0 \longrightarrow r$$

# Chap 6. Rigid Body



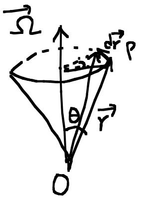
relative distances between atoms are fixed.

⇒ $\boxed{\text{6 d.o.f left: 3 for CM. 3 for rotation}}$

$$\vec{\eta} = \vec{R} + \vec{r}$$

$$\vec{v} = \frac{d\vec{\eta}}{dt} = \underset{\sim}{\frac{d\vec{R}}{dt}} + \frac{d\vec{r}}{dt} = \vec{V} + \underbrace{\vec{\Omega}\times\vec{r}}_{\frac{d\vec{r}}{dt}}$$

$$\underline{\vec{v} = \vec{V} + \vec{\Omega}\times\vec{r}}$$

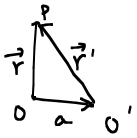


$d\vec{r} \perp \vec{\Omega}$ & $\vec{r}$

$|d\vec{r}| = (\Omega dt) r\sin\theta$

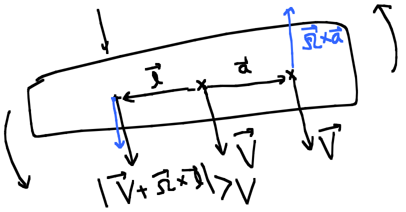
$\frac{d\vec{r}}{dt} = \vec{\Omega}\times\vec{r}$

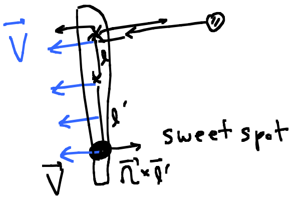
another Body-Fixed frame : $\vec{r} = \vec{a} + \vec{r}'$



$$\vec{v} = \frac{d\vec{R}}{dt} + \vec{\Omega}\times\vec{r} = \vec{V} + \vec{\Omega}\times(\vec{r}'+\vec{a}) = \underbrace{\vec{V}+\vec{\Omega}\times\vec{a}}_{\vec{V}'} + \vec{\Omega}\times\vec{r}' \qquad \boxed{\vec{\Omega} = \vec{\Omega}'}$$

we can chouse $\vec{a}$ such that $\vec{V}' = 0 = \vec{V} + \vec{\Omega}\times\vec{a}$ : instant axis of rotation





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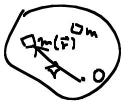
## inertia tensor



F. frame: $T = \sum \frac{1}{2} m\vec{v}^2 = \sum_{\vec{r}} \frac{1}{2} m(\vec{r})(\vec{V} + \vec{\Omega}\times\vec{r})^2$

$$= \frac{1}{2}\sum_{r} m(\vec{r})\left[V^2 + (\vec{\Omega}\times\vec{r})^2 + 2\vec{V}\cdot(\vec{\Omega}\times\vec{r})\right]$$

$$T = \frac{1}{2}\mu V^2 + \vec{V}\cdot\Big(\vec{\Omega}\times \underbrace{\sum_{\vec{r}} m(\vec{r})\,\vec{r}}_{=\,0}\Big) + \frac{1}{2}\sum_{\vec{r}} m(\vec{r})\underbrace{(\vec{\Omega}\times\vec{r})}^{2}$$

∵ O is CM.

$(\vec{\Omega}\times\vec{r})_i = \varepsilon_{ijk}\Omega_j r_k$

$$\sum_{jj'kk'}\underset{\sim}{\sum_{i}} \varepsilon_{ijk}\Omega_j r_k\, \varepsilon_{ij'k'}\Omega_{j'} r_{k'}$$

$$\underbrace{\delta_{jj'}\delta_{kk'}}_{\Omega^2 r^2} - \underbrace{\delta_{jk'}\delta_{j'k}}_{(\vec{\Omega}\cdot\vec{r})^2}$$

$$= \underbrace{\frac{1}{2}\mu V^2}_{\text{translation}} + \underbrace{\frac{1}{2}\sum_{\vec{r}} m\left(\Omega^2 r^2 - (\vec{\Omega}\cdot\vec{r})^2\right)}_{\text{rotation. } T_{rot}}$$

$$T_{rot} = \frac{1}{2}\sum_{j,k}\Omega_j\Omega_k \underset{\sim}{I_{jk}} \quad ; \quad \Omega^2 r^2 = \sum_j \Omega_j \underbrace{\Omega_j}_{\sum_k \delta_{jk}\Omega_k} r^2 = \sum_{j,k}\Omega_j\Omega_k(\delta_{jk}r^2)$$

↑ Inertia Tensor

$\vec{r} = (x_1, x_2, x_3)$
$= (x, y, z)$

$$(\vec{\Omega}\cdot\vec{r})^2 = \Big(\sum_j \Omega_j x_j\Big)\Big(\sum_k \Omega_k x_k\Big) = \sum_{j,k}\Omega_j\Omega_k(x_j x_k)$$

$m(x_1,x_2,x_3)$

$$= \frac{1}{2}\sum_{jk}\Omega_j\Omega_k\left(\Big(\sum_{\vec{r}} m \underbrace{r^2}_{\sum_\ell x_\ell^2}\Big)\delta_{jk} - \Big(\sum_{\vec{r}} m\, x_j x_k\Big)\right)$$

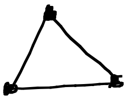
$\longrightarrow \sum_{\vec{r}} m\,(r^2\delta_{jk} - x_j x_k)$ (discrete)

or $\int \rho(\vec{r})(r^2\delta_{jk} - x_j x_k)dV$ (continuous)

$I_{jk} = I_{kj}$ symmetric matrix

I : 3×3 matrix

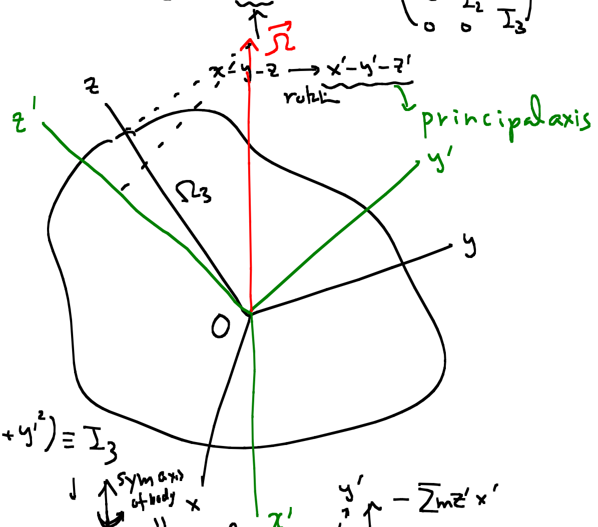
$$= \begin{pmatrix} \sum m(y^2+z^2) & -\sum m xy & -\sum m xz \\ -\sum m xy & \sum m(x^2+z^2) & -\sum m yz \\ -\sum m xz & -\sum m yz & \sum m(x^2+y^2) \end{pmatrix}$$



3×3 sym matrix → always diagonalizable   $O^T I O = I_d = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$

$$T_{rot} = \frac{1}{2} \sum_{jk} \overset{= I_j \delta_{jk}}{(I_d)_{jk}} \Omega_j \Omega_k$$

$$= \frac{1}{2} ( I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2 )$$



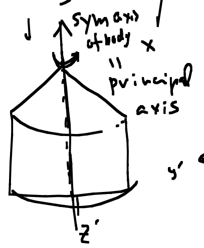
$$I_1 = \sum m ( y'^2 + z'^2 )$$

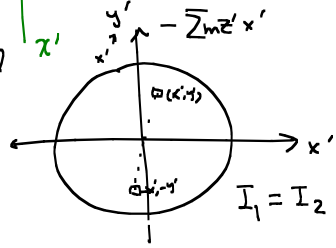
$$I_2 = \sum m ( x'^2 + z'^2 )$$

$$I_1 + I_2 = \sum m ( x'^2 + y'^2 + 2z'^2 ) \geq \sum m ( x'^2 + y'^2 ) \equiv I_3$$

$$I_2 + I_3 \geq I_1$$

If a rigid body has a symmetry:





$I_1 = I_2$

asymmetrical top : $I_1 \neq I_2 \neq I_3$

sym " top ; $I_1 = I_2 \neq I_3$ ←

spherical " : $I_1 = I_2 = I_3$

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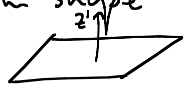
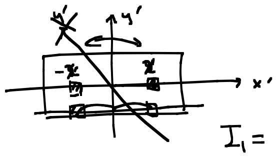
2 dim shape   $I_{zx} = I_{zy} = 0$   $\underset{\uparrow \sum m x' y'}{I_{xy} = 0}$



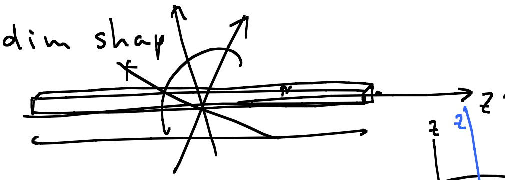


$I_1 = \sum m ( y'^2 + \overset{0}{z'^2} )$

$I_2 = \sum m ( x'^2 + \overset{0}{z'^2} )$

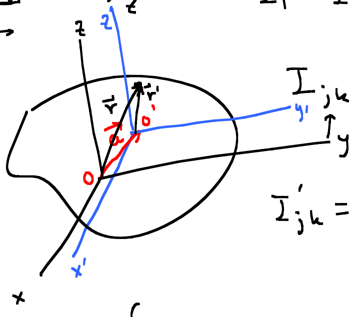
$I_3 = \sum m ( y'^2 + x'^2 ) = I_1 + I_2$

1 dim shap



$\sum m ( \overset{0}{y'^2} + z'^2 )$   $\sum m ( \overset{0}{x'^2} + z'^2 )$

$I_1 = I_2$ , $I_3 = \sum m ( \underset{0}{x'^2} + \underset{0}{y'^2} ) = 0$



$I_{jk}$

$$\vec r = \vec r' + \vec a$$

$$( x_j = x_j' + a_j ) \to x_j' = x_j - a_j$$

$$I'_{jk} = \sum m ( r'^2 \delta_{jk} - x_j' x_k' )$$

$$\sum_j x_j'^2 = \sum_j ( x_j^2 + a_j^2 - 2 x_j a_j )$$

$$= r^2 + a^2 - 2 \cancel{\sum a_j x_j}$$

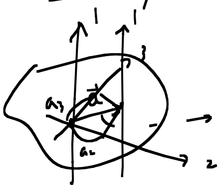
$$x_j' x_k' = ( x_j - a_j ) ( x_k - a_k )$$

$$= x_j x_k + a_j a_k - ( a_j x_k + a_k x_j ) \quad \Bigg| \; - 2 \sum_j a_j \Big[ \underbrace{\sum_{\vec r} m(\vec r) x_j}_{= x^{CM}_j = 0} \Big]$$

$$a_j \Big( \underbrace{\sum_{\vec r} m x_k}_{0} \Big) , \quad a_k \Big( \underbrace{\sum_{\vec r} m x_j}_{= 0} \Big)$$

$$\therefore \; I'_{jk} = \sum_{\vec r} m \Big[ ( r^2 + a^2 ) \delta_{jk} - ( x_j x_k + a_j a_k ) \Big] = I_{jk} + \mu ( \underset{= a_1^2 + a_2^2 + a_3^2}{a^2} \delta_{jk} - a_j a_k )$$

$j = k$   $I_1' = I_1 + \mu ( a_2^2 + a_3^2 )$



→ $a_2^2 + a_3^2$

Prob 1,2 → H.W.

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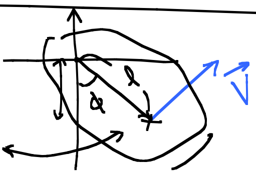
Prob 3.



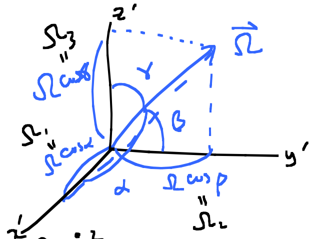
$\vec V = \vec\Omega \times \vec l$

↓

$V = l \dot\phi$

$|\vec\Omega| = \dot\phi$   $\vec\Omega = \dot\phi ( \cos\alpha , \cos\beta , \cos\gamma )$

$$T_{trans} = \frac{1}{2}\mu l^2 \dot{\phi}^2$$



$$T_{rot} = \frac{1}{2}(\Omega_1^2 I_1 + \Omega_2^2 I_2 + \Omega_3^2 I_3)$$

$$= \frac{1}{2}\left( I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right)\dot{\phi}^2$$

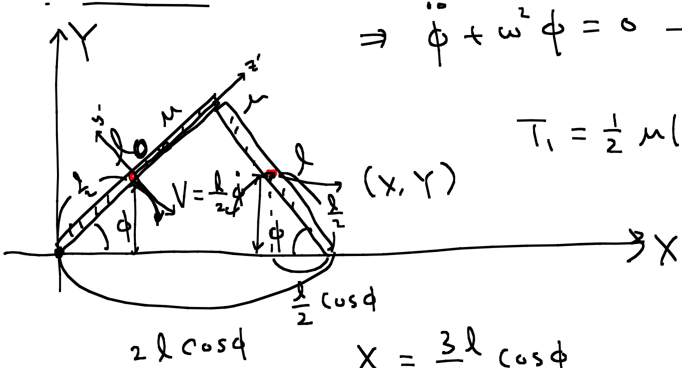
$$Z = -l\cos\phi \rightarrow U = -\mu g l\cos\phi \approx (-\mu g l) + \frac{1}{2}\mu g l\phi^2$$

$$L = T_{trans} + T_{rot} - \frac{1}{2}\mu g l \phi^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} - \frac{\partial L}{\partial\phi} = (\underline{\mu l^2 \quad +} \quad I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma)\ddot{\phi} + \mu g l \phi = 0$$

$$\Rightarrow \ddot{\phi} + \omega^2\phi = 0 \rightarrow \omega^2 = \frac{\mu g l}{(- - - -)}$$

Prob 4.



$$T_1 = \frac{1}{2}\mu\left(\frac{l}{2}\right)^2\dot{\phi}^2 + \frac{1}{2}\underbrace{\sum m z'^2}_{I_1}\dot{\phi}^2 \qquad \Omega = (\dot{\phi}, 0, 0)$$

$$I_1 = \int_{-\frac{l}{2}}^{\frac{l}{2}} \rho\, dz'\, z'^2 = \frac{2\rho}{3}\left(\frac{l}{2}\right)^3 \qquad \rho l = \mu$$

$$I_1 = \frac{\mu}{12}l^2$$

$$X = \frac{3l}{2}\cos\phi$$

$$Y = \frac{l}{2}\sin\phi$$

$$\dot{X} = -\frac{3l}{2}\sin\phi\,\dot{\phi} \qquad \dot{Y} = \frac{l}{2}\cos\phi\,\dot{\phi}$$

$$T_{2,trans} = \frac{1}{2}\mu(\dot{X}^2 + \dot{Y}^2) = \frac{1}{2}\mu\frac{l^2}{4}\dot{\phi}^2(9\sin^2\phi + \cos^2\phi) = \frac{\mu l^2}{8}\dot{\phi}^2(1 + 8\sin^2\phi)$$

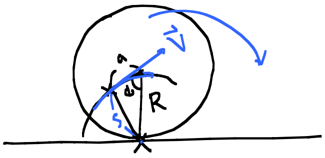
$$T_{2,rot} = T_{1,rot} \qquad \rightarrow T = \frac{\mu l^2}{4}\dot{\phi}^2(1 + 4\sin^2\phi) + \frac{\mu}{12}l^2\dot{\phi}^2$$

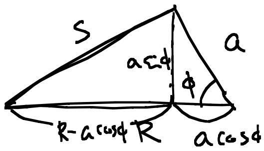
$$= \frac{\mu}{12}l^2\dot{\phi}^2(1 + 3 + 12\sin^2\phi)$$

$$U = \cancel{2}\mu g\frac{l}{\cancel{2}}\sin\phi \qquad = \frac{\mu}{3}l^2\dot{\phi}^2(1 + 3\sin^2\phi)$$

$$\therefore L = \frac{\mu}{3}l^2\dot{\phi}^2(1 + 3\sin^2\phi) - \mu g l\sin\phi$$

Prob 5.



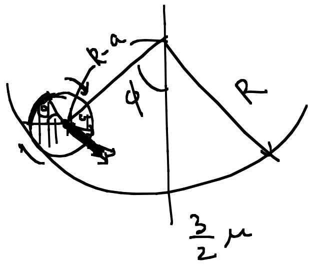


$$s^2 = R^2 + a^2 - 2aR\cos\phi$$

$$s = \sqrt{R^2 + a^2 - 2aR\cos\phi}$$

$$T_{trans} = \frac{1}{2}\mu(s\dot{\phi})^2 \qquad T_{rot} = \frac{1}{2}I_3\dot{\phi}^2$$

Prob 6.



$$T_{trans} = \frac{1}{2}\mu((R-a)\dot{\phi})^2$$

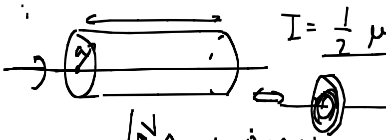
$$T_{rot} = \frac{1}{2}I\dot{\theta}^2 \qquad (R-a)\phi = a\theta$$

$$\dot{\theta} = \dot{\phi}\frac{R-a}{a}$$

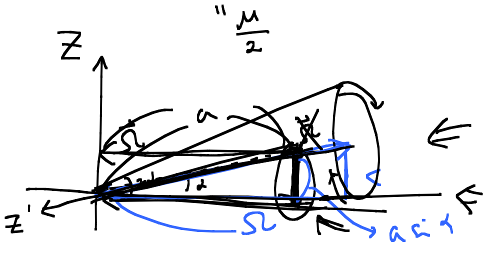
$$\therefore T = \frac{1}{2}\overbrace{\left(\mu + \underset{= \frac{\mu}{2}}{\boxed{\frac{I}{a^2}}}\right)}^{\frac{3}{2}\mu}(R-a)^2\dot{\phi}^2 = \frac{3}{4}\mu(R-a)^2\dot{\phi}^2 \quad \vdots \quad \frac{1}{2}I\frac{(R-a)^2}{a^2}\dot{\phi}^2$$

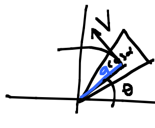
$$I = \frac{1}{2}\mu a^2 \qquad \mu = \pi a^2\rho$$

$$\int_0^a (\rho\, dr\, 2\pi r) r^2 = 2\pi\rho\frac{a^4}{4} = \frac{\mu a^2}{2}$$



Prob 7.



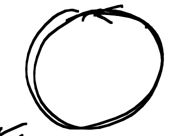


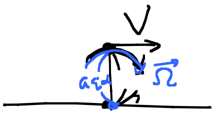
$$V = \dot{\theta}a\cos\alpha$$

top Z view

$$T_{trans} = \frac{1}{2}\mu V^2 = \frac{1}{2}\mu a^2\cos^2\alpha\,\dot{\theta}^2$$

$$T_{rot} = \frac{1}{2} I_3 \dot{\theta}^2$$





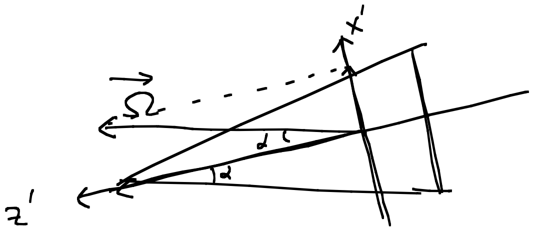
$$V = \Omega a \sin\alpha$$

$$V = \Omega \sin\alpha\, a = a\dot{\theta}\cos\alpha$$

$$\Omega = \frac{V}{a\sin\alpha} = \dot{\theta}\cot\alpha$$

$$V = \dot{\theta}\, a\cos\alpha = \Omega a \sin\alpha$$

$$\Rightarrow \Omega = \dot{\theta}\cot\alpha$$



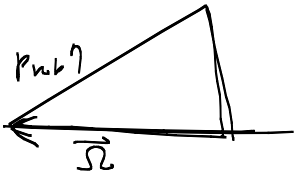
$$T_{rot} = \frac{1}{2}\left(I_1 \Omega^2 \sin^2\alpha + I_3 \Omega^2 \cos^2\alpha\right)$$

$$\vec{\Omega} = (\Omega\sin\alpha, 0, \Omega\cos\alpha) \qquad \overset{\shortparallel}{\frac{\dot{\theta}^2}{2}}\left(I_1\cos^2\alpha + I_3\frac{\cos^4\alpha}{\sin^2\alpha}\right)$$

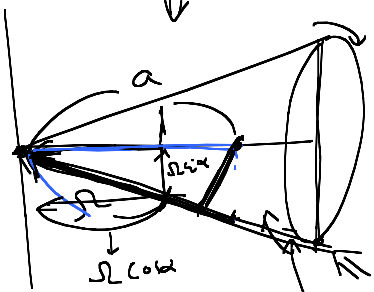
$$T = \frac{1}{2}\mu a^2\cos^2\alpha\,\dot{\theta}^2 + \frac{\dot{\theta}^2}{2}\left(I_1\cos^2\alpha + I_3\frac{\cos^4\alpha}{\sin^2\alpha}\right)$$

$$\Downarrow$$

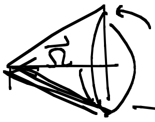
Prob 7

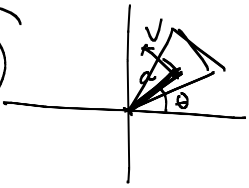


Prob 8.

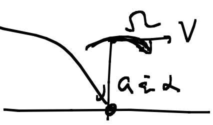


$\neq$





$$V = a\dot{\theta}$$



$$V = a\sin\alpha\,\Omega = a\dot{\theta}$$

$$\Omega = \frac{\dot{\theta}}{\sin\alpha}$$

$$T_{tras} = \frac{1}{2}\mu V^2 = \frac{1}{2}\mu a^2\dot{\theta}^2$$

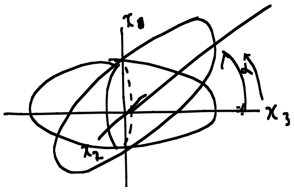
$$T_{rot} = \frac{1}{2}\left(I_1\Omega^2\sin^2\alpha + I_3\Omega^2\cos^2\alpha\right) = \frac{1}{2}\left(I_1\dot{\theta}^2 + I_3\dot{\theta}^2\cot^2\alpha\right)$$

$$T = \frac{1}{2}\mu a^2\dot{\theta}^2 + \frac{1}{2}\left(I_1 + I_3\cot^2\alpha\right)\dot{\theta}^2$$

(cf) prob 7: $T_7 = \frac{1}{2}\mu a^2 \underset{\sim}{\cos^2\alpha}\,\dot{\theta}^2 + \frac{\dot{\theta}^2}{2}\left(I_1 \underset{\sim}{\cos^2\alpha} + I_3\frac{\cos^4\alpha}{\sin^2\alpha}\right) = \cos^2\alpha \times T_{prob8}$ (H.W. consider frictional force to explain "$\cos^2\alpha$")

| |
|---|
| 6/13 (Mon) |
| 2 – 5 PM. |

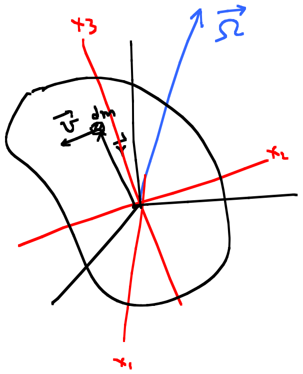
Prob 9.



$$\vec{\Omega} = (\overset{\Omega_1}{\dot{\theta}\cos\phi}, \overset{\Omega_2}{\dot{\theta}\sin\phi}, \overset{\Omega_3}{\dot{\phi}})$$

$$T = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right) \qquad \vec{r} = (x_1, x_2, x_3)$$

## §33. AM.



$$\vec{M} = \int d\vec{M} = \vec{r}\times(dm\,\vec{v}) = \int dm\,\underbrace{\vec{r}\times(\overbrace{\vec{\Omega}\times\vec{r}}^{\vec{v}})}_{\vec{\Omega}r^2 - \vec{r}(\vec{\Omega}\cdot\vec{r})}$$

$$M_i = \int dm\left[\underbrace{\Omega_i}_{\sum_j \Omega_j\delta_{ij}}\big(\overset{r^2}{\sum x_\ell^2}\big) - x_i\sum_j \Omega_j x_j\right]$$

$$= \sum_j \Omega_j \underbrace{\int dm\left(r^2\delta_{ij} - x_i x_j\right)}_{I_{ij}}$$

$$\therefore \quad M_i = \sum_j I_{ij}\Omega_j \quad \Rightarrow \quad \vec{M} = \mathbb{I}\cdot\vec{\Omega}$$

$x_1, x_2, x_3 \to$ Principal axis $\longrightarrow \mathbb{I} = \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{pmatrix}$ diagonal

$$= I_i \delta_{ij}$$

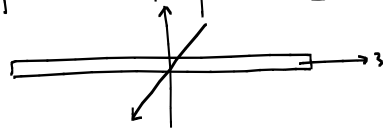
$$\therefore \quad M_i = \sum_j I_i \delta_{ij}\Omega_j = I_i\Omega_i \to \underline{M_1 = I_1\Omega_1,\ M_2 = I_2\Omega_2, M_3 = I_3\Omega_3}$$

without external force (torque) $\longmapsto \vec{M} = \text{constant} \to$

$\vec{M} = I\vec{\Omega} \to \vec{\Omega}$

* $I_1 = I_2 = I_3 = I$ "spherical top" $\quad \mathbb{I} = I\mathbb{1}$

* rotator :



$I_3 = 0$

$I_1 = I_2 = \frac{\mu}{12}l^2$

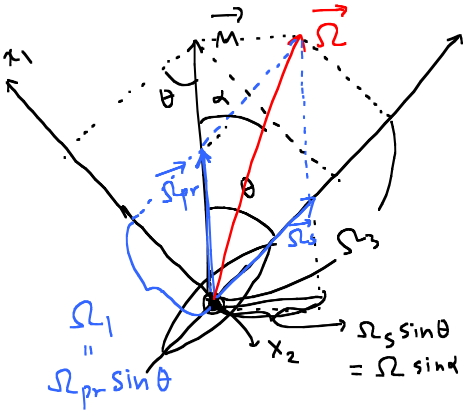
$\underline{M_1} = I_1\underline{\Omega_1}$

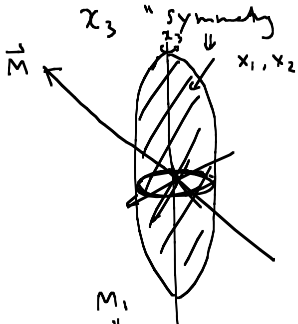
$M_2 = I_2\Omega_2$

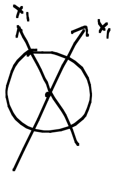
$M_3 = 0$

* $\underline{\text{"symmetrical top" : } I_1 = I_2 \neq I_3}$

$x_3$ "symmetry axis'







$$\vec{M} = (\overset{M_1}{\overset{\shortparallel}{M\sin\theta}}, \underset{\underset{M_2}{\shortparallel}}{0}, \overset{M_3}{\overset{\shortparallel}{\underset{\sim\sim}{M\cos\theta}}})$$

$0 = M_2 = I_2\Omega_2 \longrightarrow \Omega_2 = 0 \Rightarrow \vec{\Omega}$ is also on this plane.

$$\Omega_3 = \frac{M_3}{I_3} = \frac{M\cos\theta}{I_3} \swarrow \quad ; \quad \vec{\Omega} = \underset{\underset{\text{spin}}{\uparrow}}{\underset{\sim}{\vec{\Omega}_s}} + \underset{\underset{\vec{M}}{\parallel}}{\vec{\Omega}_{pr}} \leftarrow \text{precession}$$

$$\Omega_1 = \frac{M_1}{I_1} = \Omega_{pr}\sin\theta = \frac{M\sin\theta}{I_1} \Rightarrow \underset{\sim\sim\sim}{\Omega_{pr} = \frac{M}{I_1}} \swarrow$$

$$\Omega_s = \Omega\frac{\sin\alpha}{\sin\theta} \qquad \cos\alpha = \frac{\vec{M}\cdot\vec{\Omega}}{M\Omega} = \frac{\cancel{M}(\sin\theta\Omega_1 + \cos\theta\Omega_3)}{\cancel{M}\Omega} \to \sin\alpha = \sqrt{1-\cos^2\alpha}$$

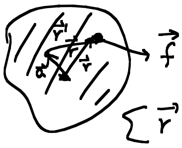
$$= \frac{1}{\sin\theta}\sqrt{\Omega^2 - (\sin\theta\Omega_1 + \cos\theta\Omega_3)^2} \quad = \frac{1}{\sin\theta}\sqrt{\Omega_1^2 + \Omega_3^2 - (\sin^2\theta\Omega_1^2 + 2\cos\theta\sin\theta\Omega_1\Omega_3 + \cos^2\theta\Omega_3^2)}$$

$$= \frac{1}{\sin\theta}\sqrt{\cos^2\theta\,\Omega_1^2 + \sin^2\theta\,\Omega_3^2 - 2\cos\theta\sin\theta\,\Omega_1\Omega_3}$$

$$= \frac{1}{\sin\theta}(\cos\theta\underset{\frac{M_1}{I_1}}{\underset{\shortparallel}{\Omega_1}} - \sin\theta\underset{\frac{M_3}{I_3}}{\underset{\shortparallel}{\Omega_3}}) = \cot\theta\frac{M_1}{I_1} - \frac{M_3}{I_3}$$

## §34. Eq. of Motion

$$\frac{d\vec{M}}{dt} = \vec{K} \qquad \frac{d\vec{p}}{dt} = \vec{F}$$



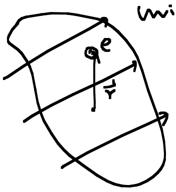
$$\sum\vec{r}\times\vec{f} = \vec{K}$$

$$\vec{r} = \vec{r}' + \vec{a}$$

$$\vec{K} = \sum\vec{r}\times\vec{f} = \sum\vec{r}'\times\vec{f} + \vec{a}\times\underset{\vec{F}}{\underbrace{\sum\vec{f}}} \Rightarrow \vec{K} = \vec{K}' + \vec{a}\times\vec{F}$$

* if $\sum\vec{f} = \vec{F} = 0 \longrightarrow \vec{K} = \vec{K}'$ ;

* $\vec{F} \perp \vec{K}$ ; one can chouse $\vec{a} \ni: \vec{K}' = 0$



uniform force field $\vec E$

$\vec f = e\vec E$ $\qquad \sum \vec f = \vec F = \underbrace{\vec E \sum e}$

$$\frac{\sum \overset{e}{m}\vec r}{\sum \underset{e}{m}} \equiv \vec r_0$$

$$\vec K = \sum \vec r \times e \vec E = \underset{\overset{\shortparallel}{\vec r_0 \sum e}}{\left(\sum e\vec r\right)} \times \vec E = \vec r_0 \times \underbrace{(\sum e)\vec E}_{\vec F} = \vec r_0 \times \vec F$$

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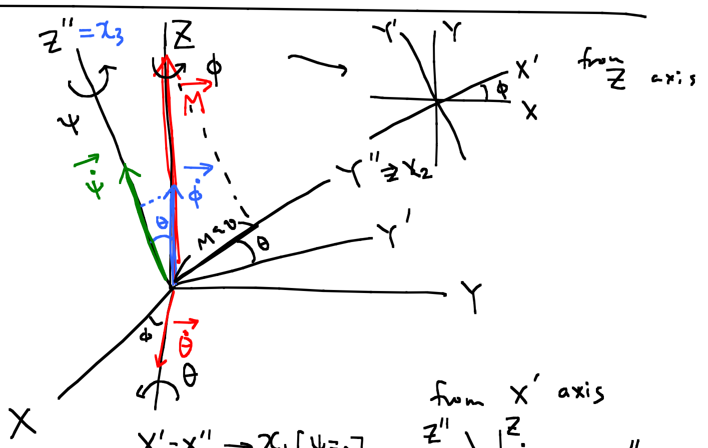
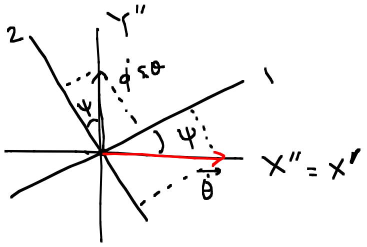
## §35. Euler angle

Rotation in 3D requires

3 angles

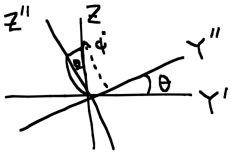


from z'' axis $\quad z'' = x_3$



$X' = X'' \Rightarrow x_1\,[\psi\equiv 0]$

from X' axis



$\dot\phi\, s\theta$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ \vec{\dot\phi} = ( & \dot\phi\, s\theta\, s\psi, & \dot\phi\, s\theta \cos\psi, & \dot\phi\cos\theta ) \\ \vec{\dot\theta} = ( & \dot\theta\cos\psi, & -\dot\theta\, s\psi, & 0 ) \\ \vec{\dot\psi} = ( & 0, & 0, & \dot\psi ) \end{array}$$

$$\vec\Omega = \vec{\dot\phi} + \vec{\dot\theta} + \vec{\dot\psi} = (\underset{\overset{\shortparallel}{\Omega_1}}{\dot\phi\, s\theta\, s\psi + \dot\theta\cos\psi}, \underset{\overset{\shortparallel}{\Omega_2}}{\dot\phi\sin\theta\cos\psi - \dot\theta\, s\psi}, \underset{\overset{\shortparallel}{\Omega_3}}{\dot\phi\cos\theta + \dot\psi})$$

$I_1 = I_2 \neq I_3$

$$T_{rot} = \frac12 I_1\Omega_1^2 + \frac12 I_1\Omega_2^2 + \frac12 I_3\Omega_3^2 = \frac12 I_1(\dot\phi^2 s^2\theta + \dot\theta^2) + \frac12 I_3(\dot\phi\cos\theta + \dot\psi)^2$$

↑ no dependence on ψ
↓
no difference by $\psi\equiv 0$
$\underline{X'' = x_1,\ Y'' = x_2}$

$$\Rightarrow \boxed{\Omega_1 = \dot\theta,\quad \Omega_2 = \dot\phi\, s\theta,\quad \Omega_3 = \dot\phi\cos\theta + \dot\psi}$$

$M_1 = I_1\Omega_1 = I_1\dot\theta$ , $M_2 = I_1\Omega_2 = I_1\dot\phi\, s\theta$ , $M_3 = \underline{I_3\Omega_3} = I_3(\dot\phi\cos\theta + \dot\psi)$

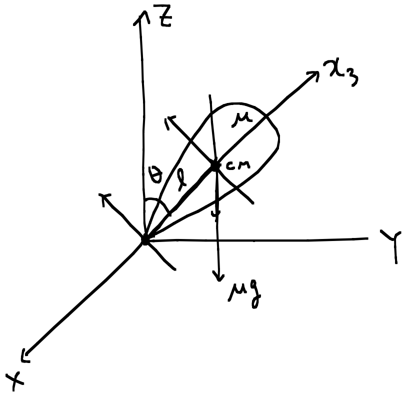
$= 0 \quad \rightarrow \theta = \text{constant}$ $\qquad M_2 = M\cancel{s\theta} = I_1\dot\phi\cancel{s\theta}$ $\qquad = \underline{M\cos\theta}$

$I_1\dot\phi = M \rightarrow \dot\phi = \frac{M}{I_1} = \Omega_{pr}$

$$\Omega_3 = \frac{M\cos\theta}{I_3}$$

---

**Prob. 1.**



$I_1 = I_2 \longrightarrow I_1' = I_2' = I_1 + \mu l^2$

$$L = \frac12 I_1'(\dot\phi^2 s^2\theta + \dot\theta^2) + \frac12 I_3(\dot\phi\cos\theta + \dot\psi)^2 - \mu g l\cos\theta$$

$$\frac{\partial L}{\partial \phi} = 0 \longrightarrow \frac{\partial L}{\partial \dot\phi} = \text{const} = \underbrace{I_1' \sin^2\theta\, \dot\phi + I_3(\dot\phi\cos\theta + \dot\psi)\cos\theta}_{(I_1'\sin^2\theta + I_3\cos^2\theta)\dot\phi + I_3\dot\psi\cos\theta} \equiv M_z$$

$$\frac{\partial L}{\partial \psi} = 0 \longrightarrow \frac{\partial L}{\partial \dot\psi} = \text{''} = M_3 = I_3(\underset{\sim\sim\sim}{\dot\phi\cos\theta} + \dot\psi) \leftarrow$$

$$M_z - M_3\cos\theta = I_1'\sin^2\theta\,\dot\phi \to \dot\phi = \frac{M_z - M_3\cos\theta}{I_1'\sin^2\theta}$$

$$\dot\psi = \frac{M_3}{I_3} - \cos\theta\,\frac{M_z - M_3\cos\theta}{I_1'\sin^2\theta}$$

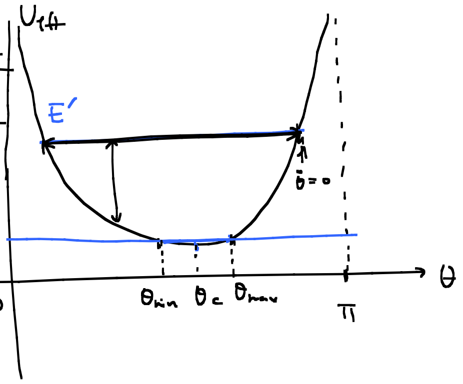
$$E = \frac{1}{2}I_1'(\dot\phi^2 \overset{\downarrow}{\sin^2\theta} + \underset{\sim}{\dot\theta^2}) + \frac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2 + \mu g l\cos\theta \leftarrow$$

$$= \frac{1}{2}I_1'\dot\theta^2 + \underbrace{\frac{1}{2}I_1'\sin^2\theta\left(\frac{M_z - M_3\cos\theta}{I_1'\sin^2\theta}\right)^2}_{} + \frac{1}{2}I_3\left(\frac{M_3}{I_3}\right)^2 + \mu g l - \underbrace{\mu g l(1-\cos\theta)}_{} \leftarrow$$

$$E' \equiv E - \frac{1}{2}I_3\left(\frac{M_3}{I_3}\right)^2 - \mu g l = \frac{1}{2}I_1'\dot\theta^2 + U_{eff}(\theta)$$

$$\dot\theta = \frac{d\theta}{dt} = \pm\sqrt{\frac{2(E' - U_{eff}(\theta))}{I_1'}}$$

$$\int dt = t = \int \frac{d\theta}{\sqrt{\frac{2(E' - U_{eff}(\theta))}{I_1'}}}$$

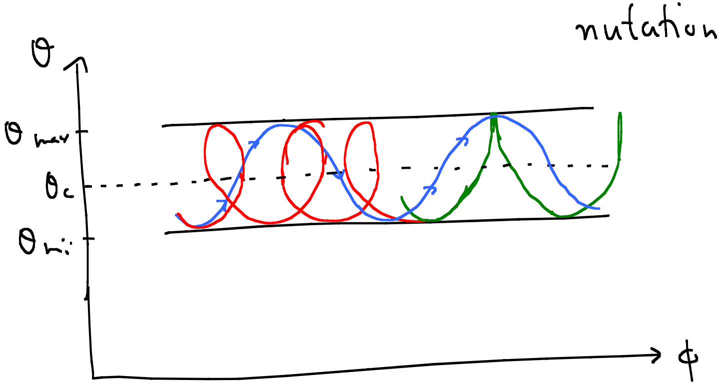


$$\boxed{U_{eff} = \frac{1}{2}\frac{(M_z - M_3\cos\theta)^2}{I_1'\sin^2\theta} - \mu g l(1-\cos\theta)}$$

$\cos\theta \equiv \alpha$

$$U_{eff}(\alpha) = \frac{1}{2I_1'}\frac{(M_z - M_3\alpha)^2}{1-\alpha^2} - \mu g l(1-\alpha) \longrightarrow \frac{dU_{eff}}{d\alpha} = 0 \longrightarrow \alpha$$

nutation



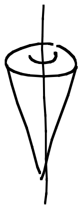
$$\dot\phi = \frac{M_z - M_3\cos\theta}{I_1'\sin^2\theta}$$

if $M_z - M_3\cos\theta > 0$

if $M_z - M_3\cos\theta = 0$ for $\theta = \theta_c$ $\theta_{min} < \theta_c < \theta_{max}$

if $\theta_c = \theta_{max} \longrightarrow \dot\theta = \dot\varphi = 0$

Prob 2.

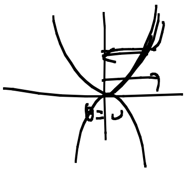


$\theta = 0 \to 3 \equiv z \to M_z = M_3$

$$U_{eff} = \frac{1}{2}\frac{(M_z - M_3\cos\theta)^2}{I_1'\sin^2\theta} - \mu g l(1-\cos\theta)$$

near $\theta=0$ ;

$$U_{eff} \cong \frac{M_z^2}{2I_1'} \frac{\left(\frac{\theta^2}{2}\right)^2}{\theta^2} - \mu g l \frac{\theta^2}{2} = \underbrace{\left(\frac{M_z^2}{8I_1'} - \frac{\mu g l}{2}\right)}_{\vee\, 0} \theta^2$$

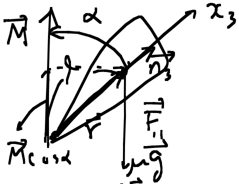


$M_z^2 = M_3^2 > 4I_1'\mu g l$

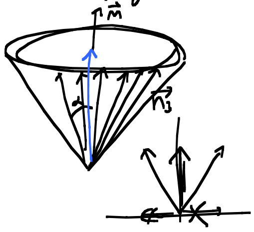
Prob3 : In 1st approx ; neglect pot. ⇒ free rotation

$T_{rot} \gg \mu g l(1-\cos\theta)$

$$\Omega_{pr} = \frac{M}{I_1'}$$



$$\frac{d\vec{M}}{dt} = \vec{K} = \underbrace{l\,\vec{n}_3}_{\vec{r}} \times \underbrace{\mu\vec{g}}_{\vec{F}}$$



average

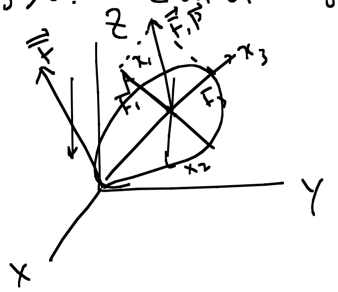
$$\overline{\frac{d\vec{M}}{dt}} = l\,\underbrace{\overline{\vec{n}_3}}_{\cos\alpha\frac{\vec{M}}{M}} \times \mu\vec{g} = \underbrace{\frac{l\mu\cos\alpha}{M}\overbrace{\vec{M}\times\vec{g}}^{-\vec{g}\times\vec{M}}}_{}$$

$$\underset{\sim\sim}{\vec{\Omega}} \times \underline{\underline{\vec{M}}}$$

$$\boxed{\frac{d\vec{X}}{dt} = \vec{\Omega} \times \vec{X}}$$

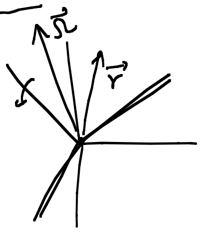
$$\vec{\Omega}_{pr} = -\frac{l\mu\vec{g}\cos\alpha}{M}$$

## §36. Euler eq.



body fixed frame (1-2-3)

$$\underset{\uparrow \text{ fixed frame}}{\frac{d\vec{A}}{dt}} = \underbrace{\frac{d'\vec{A}}{dt}}_{} + \vec{\Omega} \times \vec{A}$$



$\vec{r} = \vec{r}'$

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}'}{dt} + \underbrace{\frac{d\vec{\theta}}{dt}}_{\vec{\Omega}} \times \vec{r}'$$

$\vec{A} = \vec{P}, \vec{M}$ $\quad = \mu\vec{V}$

$$\frac{d\vec{P}}{dt} = \vec{F} = \frac{d'\vec{P}}{dt} + \vec{\Omega} \times \underset{\mu\vec{V}}{\vec{P}} \longrightarrow$$

$$\underline{F_1 = \mu\frac{dV_1}{dt} + \mu(\Omega_2 V_3 - \Omega_3 V_2)}$$

$$\underline{F_2 = \mu\frac{dV_2}{dt} + \mu(\Omega_3 V_1 - \Omega_1 V_3)}$$

$$\underline{F_3 = \mu\frac{dV_3}{dt} + \mu(\Omega_1 V_2 - \Omega_2 V_1)}$$

$$\frac{d\vec{M}}{dt} = \vec{K} = \frac{d'\vec{M}}{dt} + \vec{\Omega} \times \vec{M}$$

$$K_1 = I_1\frac{d\Omega_1}{dt} + \Omega_2 \underset{I_3\Omega_3}{M_3} - \Omega_3 \underset{I_2\Omega_2}{M_2}$$

$$\vec{M} = (I_1\Omega_1, I_2\Omega_2, I_3\Omega_3)$$

Euler Eq.

$$\boxed{\begin{aligned} K_1 &= I_1\frac{d\Omega_1}{dt} + (I_3 - I_2)\Omega_2\Omega_3 \\ K_2 &= I_2\frac{d\Omega_2}{dt} + (I_1 - I_3)\Omega_1\Omega_3 \\ K_3 &= I_3\frac{d\Omega_3}{dt} + (I_2 - I_1)\Omega_1\Omega_2 \end{aligned}}$$

If notorque $\vec{K}=0$ , symmetrical top ; $I_1 = I_2 \neq I_3$

$$\frac{d\Omega_3}{dt} = 0 \longrightarrow \Omega_3 = \text{const.}$$

$$\frac{d\Omega_1}{dt} = \frac{I_1 - I_3}{I_1}\Omega_3 \cdot \Omega_2 = -\omega\Omega_2$$

$$\frac{d\Omega_2}{dt} = \underbrace{\frac{I_3 - I_1}{I_1}\Omega_3}_{=\omega} \cdot \Omega_1 = \omega\Omega_1$$

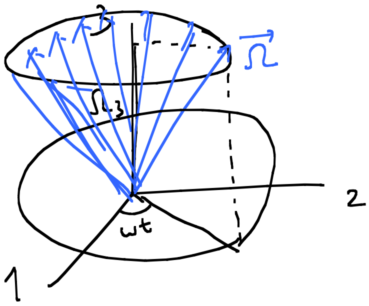
$(I_3 > I_1 = I_2)$

$$\frac{d}{dt}(\underbrace{\Omega_1+i\Omega_2}_{Z}) = -\omega\Omega_2 + i\omega\Omega_1 = i\omega(\underbrace{\Omega_1+i\Omega_2}_{Z})$$

$$\rightarrow Z = Z_0 e^{i\omega t}$$

$$\Omega_1+i\Omega_2 = (\underbrace{\Omega_1(0)+i\Omega_2(0)}_{A})\underbrace{e^{i\omega t}}_{(\cos\omega t+i\sin\omega t)}$$ with $\Omega_1(0)=A$, $\Omega_2(0)=0$

$$\Omega_1 = A\cos\omega t,\quad \Omega_2 = A\sin\omega t$$



§37. asym. top $I_1 \neq I_2 \neq I_3$ $(I_3 > I_2 > I_1)$

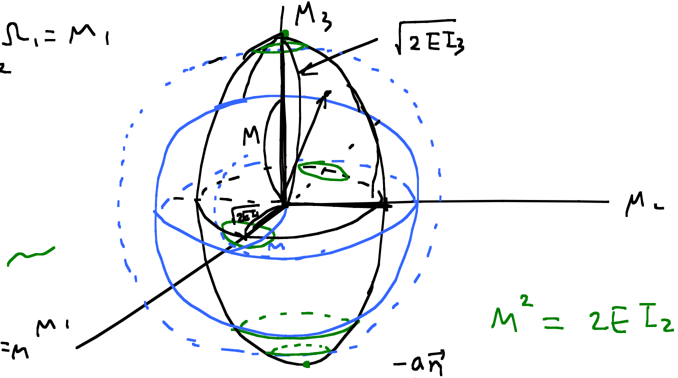
$$E = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2) = \frac{1}{2}\left(\frac{M_1^2}{I_1}+\frac{M_2^2}{I_2}+\frac{M_3^2}{I_3}\right)$$

$I_1\Omega_1 = M_1$

$$M^2 = M_1^2 + M_2^2 + M_3^2$$

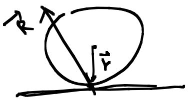
$$2EI_1 < \underset{\sim}{M^2} < 2EI_3$$

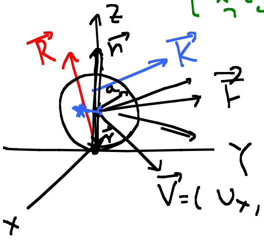
if $M^2 = 2EI_1 \rightarrow M_1 = M_2 = 0,\ M_3 = M$



$M^2 = 2EI_2$

Prob 1.





$\vec{V} = (V_x, V_y, 0)$

[ 작년 노트 참고 ]

$\vec{K} = \sum \vec{r}\times\vec{f}$ $\overset{\vec{r}_{CE}\times\vec{F}}{=}$

$\vec{F} = \sum \vec{f}$

$$I\frac{d\vec{\Omega}}{dt} = \vec{K} + \overset{-a\vec{n}}{\overset{\shortparallel}{\vec{r}}}\times\vec{R}$$

$$\mu\frac{d\vec{V}}{dt} = \vec{F} + \vec{R}$$

$$\vec{M} = I\vec{\Omega}$$
$\updownarrow$
$I\mathbb{1}$

$$\vec{V} = a(\vec{\Omega}\times\vec{n})$$

$$\frac{d\vec{V}}{dt} = a\underset{\sim}{\dot{\vec{\Omega}}}\times\vec{n} = \frac{\vec{F}+\vec{R}}{\mu}$$

$\vec{n}\times(\vec{n}\times\vec{R})$

$$\vec{F}+\vec{R} = \frac{\mu a}{I}(\vec{K} - \overbrace{a\vec{n}\times\vec{R}) \times n}$$

$$= \frac{\mu a}{I}(\vec{K}\times\underset{\sim}{\vec{n}} - a\vec{R} + a\vec{n}\underset{z\text{-compo}}{\underset{\sim}{(\vec{n}\cdot\vec{R})}})$$

$\frac{2}{5}\mu a^2 =$

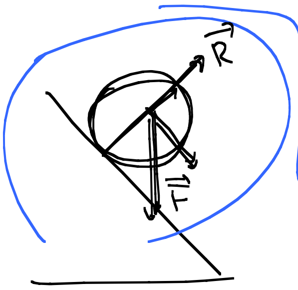
$$\begin{vmatrix} i & j & k \\ K_x & K_y & K_z \\ 0 & 0 & 1 \end{vmatrix} = (K_y, \underline{-K_x}, 0)$$

x-compo.: $F_x + R_x = \frac{5}{2a}\left(K_y - aR_x\right)$

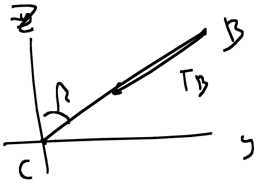
y- 〃 : $F_y + R_y = \frac{5}{2a}\left(-K_x - aR_y\right)$

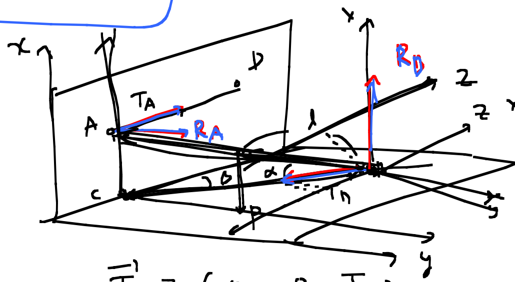
$$R_x = \left(\frac{5}{2a}K_y - F_x\right)\frac{2}{7}$$

$$R_y = \left(-\frac{5}{2a}K_x - F_y\right)\frac{2}{7}$$



Prob3





x-comp. $R_B = P$

y-comp $R_A - T_B\sin\beta = 0$

z-comp $T_A - T_B\cos\beta = 0$

$$P l\cos\alpha - 2lT_A\frac{\cos\alpha}{\sin\beta}$$

$$\vec{T}_A = (0, 0, T_A)$$

$$\vec{r}_A = 2l(\sin\alpha, -\cos\alpha\sin\beta\quad -\cos\alpha\cos\beta)$$

$$\vec{K}_{R_A} = \vec{r}_A \times \vec{R}_A$$
$$= 2l R_A (c_\alpha c_\beta, 0, s_\alpha)$$

$$\vec{K}_A = 2l T_A(-\cos\alpha \sin\beta, -\underline{\sin\alpha}, 0)$$

$$\vec{r}_p = \frac{\vec{r}_A}{2} \qquad \vec{K}_p = \vec{r}_p \times \underset{\sim}{\vec{p}} \overset{pl}{=} (0, \underline{\cos\alpha\cos\beta}, -\cos_\beta)$$
$$(c\beta, 0, 0)$$

$$\vec{R}_A = R_A(0, 1, 0)$$

$$-\cancel{2l} T_A \cancel{c_\alpha} s_\beta + \cancel{2l} R_A \cancel{c_\alpha} c_\beta = 0 \qquad R_A = +\tan\beta\, T_A = T_B \sin\beta$$

$$-2\cancel{l} T_A s_\alpha + 0 + p\cancel{l} c_\alpha c_\beta = 0 \qquad T_A = \underbrace{\tfrac{1}{2} P \cot\alpha}_{T_B''} \cancel{c_\beta} = T_B \cancel{c_\beta}$$

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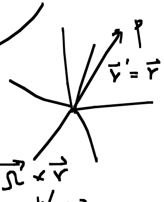
non-inertial frame

$$\vec{r} = \vec{r}'$$

$$\vec{V}_0 = \vec{v}' + \vec{V} \rightarrow \frac{d\vec{r}}{dt} = \frac{\overset{rel.}{d'}\vec{r}'}{dt} + \vec{\Omega} \times \vec{r}'$$

$$\vec{V} = \vec{\Omega} \times \vec{r}'$$

$$\frac{d}{dt} = \frac{d'}{dt} + \vec{\Omega}\times$$



$$m\vec{a} = m\vec{a}' + m\dot{\vec{\Omega}} \times \vec{r}' + 2m\vec{\Omega} \times \dot{\vec{r}}' + m\vec{\Omega} \times (\vec{\Omega} \times \vec{r}')$$

$$\vec{F}' = \vec{F} - (\qquad)$$
$$\vec{F} = m\vec{a}$$

$$\frac{d\vec{v}_0}{dt} = \frac{d\vec{v}'}{dt} + \dot{\vec{\Omega}} \times \vec{r}' + \vec{\Omega} \times \underset{\sim}{\vec{V}'}$$

$$\vec{V}' = \vec{v} + \vec{\Omega} \times \vec{r}$$

$$\frac{d'\vec{v}'}{dt} + \vec{\Omega} \times \vec{v}' = \frac{d'^2}{dt^2}\vec{r}' + \dot{\vec{\Omega}} \times \vec{r}'$$
$$+ 2\vec{\Omega} \times \dot{\vec{r}}'$$
$$+ \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

$-2m\vec{\Omega} \times \dot{\vec{r}}' =$ Coriolis force

$-m\vec{\Omega} \times (\vec{\Omega} \times \vec{r}')$ centrifugal

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H.W. on 129

Chap VII : $H = T + V$ , $H = \sum p_i \dot{q}_i - L = H(p_i, q_i)$

Poisson Bracket.

$$\{q, p\} = 1 \leftarrow \boxed{\{A, B\} \equiv \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial B}{\partial q}\frac{\partial A}{\partial p}}$$

$$\{q_i, p_j\} = \delta_{ij}$$

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \rightarrow \frac{\partial H}{\partial p_i} = \dot{q}_i$$
$$\frac{\partial H}{\partial q_i} = -\dot{p}_i$$

N var. $q_i$ 2nd order DE $\equiv$ 2N variables $p_i, q_i$ 1st order DE

$$\Rightarrow \quad \dot{q}_i = \frac{\partial H}{\partial p_i} = \{q_i, H\}$$
$$\dot{p}_i = \{p_i, H\}$$
$$\frac{dQ}{dt} = \{Q, H\} \qquad Q = Q(q_i, p_i)$$

Q.M. $\{q, p\} i\hbar \equiv [q, p] \rightarrow i\hbar \dot{q} = i\hbar \frac{\partial q}{\partial t} = [q, H]$

1 problem: Kepler problem. (elliptic orbit)

1 " : scattering

{2 " : Rigid Body : modified problems in the text.

1 " : H.W. in this week
on 129