Classical Mechanics

노트 제목 2015-03-13

Object

Grad. level "analytical"

phenomena ← Newton's laws of motion ← principle → least action

↑ Solution; math; dynamics

1. EoM (Equation of Motion)

classical ~~mechanics~~ ⟷ quantum

mechanics ⟷ Field theory (E & M)

particles (point) (positions) $\vec{r}(t) = (x, y, z)$

Field (場) [infinitely many d.o.f]

in 3D : 3 coordinates per each particle

if N particles; $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N = \vec{r}_j(t)$, $j = 1 \dots N$

3N finite coordinates : deg of freedom

$\vec{r}(j, t)$

(ex) $\vec{E} = (E_x, E_y, E_z) = \vec{E}(\vec{r}, t)$

[classical : least action principle

[quantum : " is not necessary

	mechanics	field theory
classic	classical mechanics	E & M, General Relativity
quantum	quantum mechanics	QFT

§1.1. generalized coordinates

(x, y, z) orthogonal coordinates

is NOT always best.

Advantage of orthogonal coordinates:

Kinetic energy
(velocity)

$$\dot{\vec{r}} = \vec{v} = \underbrace{(\dot{x}, \dot{y}, \dot{z})}$$

$$= \dot{x}\,\vec{i} + \dot{y}\,\vec{j} + \dot{z}\,\vec{k}$$

indep of time

Advantage of generalized coordinates:

Potential energy

(ex) spherical coordinates (r, θ, ϕ)

$$\vec{r} = r\,\hat{r} \qquad \dot{\hat{r}} \neq 0$$

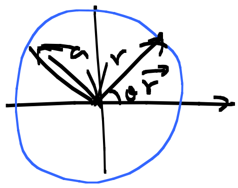
$$\vec{v} = \dot{\vec{r}} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta} + r\sin\theta\,\dot{\phi}\,\hat{\phi}$$

$$\vec{v}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2$$

complicated EoM

$\boxed{\text{Constraints}}$

2D
orthog. (x, y)



(ex) on circle

polar (r, θ)

$$x^2 + y^2 = a^2 \quad ; \quad \underbrace{r = a}$$

θ is only dyn. variable

d.o.f : $2 - 1 = 1$

If naive $3N$ d.o.f.
& n constraints system:

d.o.f $= S = 3N - n$

$$\vec{r}_1, \cdots, \vec{r}_N \longrightarrow q_1(t), \cdots, q_s(t)$$

differentiation

$$\dot{\vec{r}}_1, \cdots, \dot{\vec{r}}_N \longrightarrow \dot{q}_1, \cdots, \dot{q}_s$$

Dynamical system: laws of motion are 2nd order time differential equation

↳ integrate twice → two integration constants per each d.o.f.

initial condition.

$q_i(0)$, $\dot{q}_i(0)$ $i=1,\cdots,s$

§2. least action principle (Hamilton's)

⌜ dynamical system changes to make "action" minimized. ⌟

Lagrangian $L = T - V = L(q_i, \dot{q}_i)$ function of 2S variables

Action $S = \int_{t_1}^{t_2} dt \, L(q_i, \dot{q}_i, t)$ ← explicit.

$q_i(t)$: implicit time dependence

$$L = \frac{m}{2}\dot{q}_1^2 - \frac{1}{2}k(t)q_1^2$$

E. of M: $q_i(t)$ are determined in such a way that S is minimum.

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if $f(x)$ is min. at $x = x_0 \longmapsto$ $f(x_0+\epsilon) > f(x_0)$ for any small ϵ

Taylor expansion

$$f(x_0+\epsilon) = f(x_0) + \epsilon f'(x_0) + \frac{1}{2}\epsilon^2 f''(x_0) \underbrace{+ \cdots}_{\text{negligible}}$$

$$\therefore \quad \underline{\epsilon f'(x_0) + \frac{1}{2}\epsilon^2 f''(x_0) > 0} \quad \text{if } \epsilon \text{ is small enough.}$$

for any ϵ

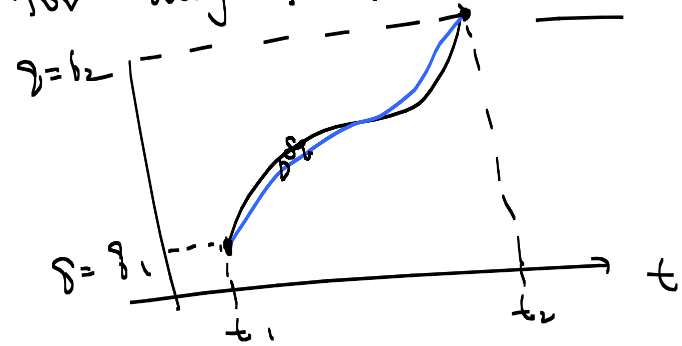
$$\Rightarrow \underline{f'(x_0) = 0 \ , \ f''(x_0) > 0}$$

$$S(q_i, \dot q_i, t) \quad \text{minimum}$$

if $q_i(t), \dot q_i(t)$ are sol. of EoM,

$$S(q_i(t) + \delta q_i(t), \dot q_i + \delta \dot q_i, t) > S(q_i, \dot q_i, t)$$

for any funct of $\underline{\delta q_i}$



$$\underline{\delta q(t_1) = \delta q(t_2) = 0}$$

$$S(q_i + \delta q_i) \geq S[q_i] \quad \text{for any } \delta q.$$

$$\delta S = \underbrace{\int_{t_1}^{t_2} dt \, L(q + \delta q, \dot q + \delta \dot q, t)}_{} - \int_{t_1}^{t_2} dt \, L(q, \cancel{\dot q}, t)$$

$= \text{minimum}$; δq is not allowed (coeff. of $\delta q = 0$)

Taylor expansion

$$L(q + \delta q, \dot q + \delta \dot q, t) = L(\cancel{q, \dot q}, t) + \delta q \frac{\partial L}{\partial q} + \delta \dot q \frac{\partial L}{\partial \dot q} + \cdots$$

$$\delta S = \int_{t_1}^{t_2} dt \left[\boxed{\delta q} \frac{\partial L}{\partial q} + \delta \dot q \frac{\partial L}{\partial \dot q} \right]$$

$$= \int_{t_1}^{t_2} dt \left[\delta q \frac{\partial L}{\partial q} + \boxed{\frac{d}{dt}\left(\delta q \frac{\partial L}{\partial \dot q}\right) - \delta q \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)} \right]$$

$$\left[\frac{d}{dt}(f\, g) = f g' + f' g \qquad f' g = (f g)' - f g' \right]$$

$$\int_{t_1}^{t_2} \frac{d}{dt}\left(\delta q \frac{\partial L}{\partial \dot q}\right) dt = \delta q \frac{\partial L}{\partial \dot q}\Big|_{t_1}^{t_2}$$

$$= \underbrace{\delta q(t_2)}_{=0} \frac{\partial L}{\partial \dot q} - \underbrace{\delta q(t_1)}_{=0} \frac{\partial L}{\partial \dot q}$$

$$\therefore = 0$$

$$\delta S = \int_{t_1}^{t_2} \delta q(t) \underbrace{\left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) \right]}_{=0} + (\delta q)^2 \geqq 0$$

Euler-Lagrange eq: $\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q} = 0}$

$s > 1$: $q_1, \dots, q_s$; $L(q_1, \dots q_s, \dot q_1, \dots \dot q_s; t)$

$$q_i \to q_i + \delta q_i$$

$$\delta S \geqq 0 ; \quad \delta S = \sum_{i=1}^{s} \int_{t_1}^{t_2} \left[\delta q_i \frac{\partial L}{\partial q_i} + \underbrace{\delta \dot q_i \frac{\partial L}{\partial \dot q_i}}_{= \frac{d}{dt}\left(\delta q_i \frac{\partial L}{\partial \dot q_i}\right) - \delta q_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right)} \right] dt$$

$$\int \text{〃} = \delta q_i \frac{\partial L}{\partial \dot q_i}\Big|_{t_1}^{t_2} = 0$$

$$\because \delta q_i(t_1) = \delta q_i(t_2) = 0 \quad i = 1, \dots, s$$

$$\delta S = \sum_{i=1}^{s} \int_{t_1}^{t_2} \delta q_i \underbrace{\left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot q_i} \right]}_{=0} dt + (\delta q_i)^2$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0}$$

$$L = T - V$$

quadratic of $\dot{q}_i$; $\dot{q}_1 \dot{q}_2 + \dot{q}_3^2 + \cancel{\dot{q}_1 \dot{q}_2 \dot{q}_3}$

$$\frac{\partial L}{\partial \dot{q}_i} \propto \dot{q}_i,$$

2nd order time diff eq

$$\ddot{q}_i + \cdots = 0$$

two integral constants per each d.o.f

(2s)

2s initial conditions $q_i(0), \dot{q}_i(0)$ $i = 1, \cdots, s$

Variation of L.

$$\boxed{L' = L + \frac{df}{dt}}$$

$$f(q(t); t)$$

$$S' = \int_{t_1}^{t_2} L' \, dt = \int_{t_1}^{t_2} \left(L + \frac{df}{dt}\right) dt$$

$$= S + \underbrace{f(t_2) - f(t_1)}_{\text{depend on } q(t_2), q(t_1), t_1, t_2}$$

$$\underset{q \to q + \delta q}{\delta S' = \delta S + 0}$$

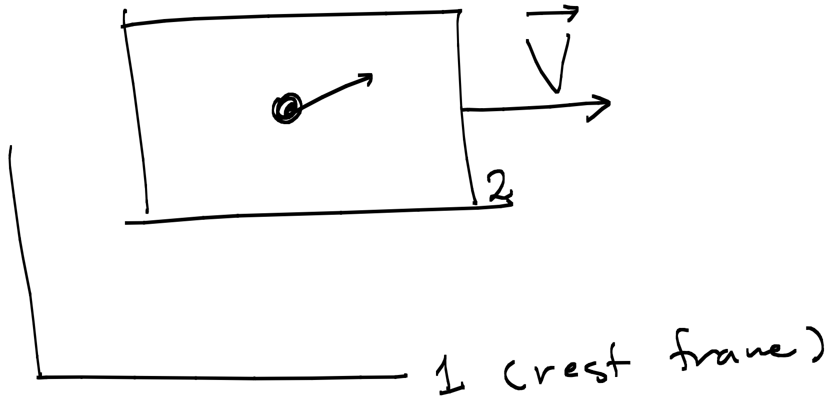
E of M : identical

L & L' : equivalent.

§3. Galileo's relativity principle

If a ref. frame is moving with constant

velocity, the EoM is identical to that of original frame



If $\vec{F}=0$ in rest frame, $\vec{F}=0$ in a constant moving frame. (inertial frame)

isotropic + homogeneous

↳ independent of position & time.

$$L(\cancel{\vec{r}}, \underset{\vec{v}}{\underset{\shortparallel}{\dot{\vec{r}}}}, \cancel{t}) = L(\vec{v}\cdot\vec{v})$$

(ex) $L = v_x \longrightarrow v_z$ ~~isotropic~~

$L = v_x^2 + v_y^2 + v_z^2 \rightarrow v_x'^2 + v_y'^2 + v_z'^2$

$$\frac{\partial L}{\partial \vec{r}} = 0 \longrightarrow \frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot{\vec{r}}}}_{} = 0$$

$$\frac{\partial L(\vec{v}^2)}{\partial \vec{v}} = L'(v^2)\underbrace{\frac{\partial \overbrace{v^2}^{\vec{v}\cdot\vec{v}}}{\partial \vec{v}}}_{2\vec{v}}$$

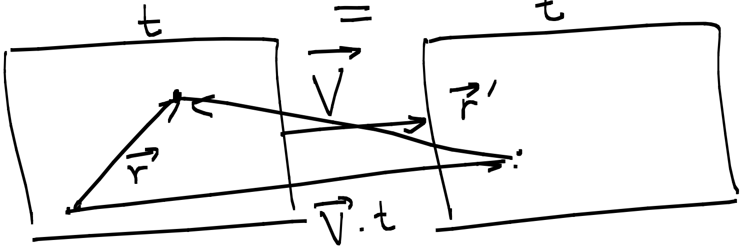
$$\downarrow$$

$$\underbrace{2\frac{d}{dt}\left(L'(v^2)\,\vec{v}\right) = 0}_{} \longleftrightarrow \underbrace{\frac{d\vec{v}}{dt} = 0}_{}$$

inertial frame ; $\vec{v}$ = const

for consistency ; $L'(v^2) = $ 상수 $\equiv \frac{m}{2}$

$$L'(v^2) = \frac{m}{2} \longrightarrow L = \frac{m}{2} v^2 = T$$



$$\vec{r} = \vec{r}' + \vec{V} t$$

Galilean transf.

$L(v^2)$ $\qquad$ $L(\vec{v}'^2)$

$$\vec{v} = \vec{v}' + \underbrace{\vec{V}}_{-\vec{\epsilon}} \qquad \vec{v}' = \vec{v} + \vec{\epsilon}$$

$$L(\vec{v}'^2) \equiv \cancel{L(v^2)}$$

$$= L((\vec{v} + \vec{\epsilon})^2)$$

$$= L(v^2 + 2\vec{v} \cdot \vec{\epsilon} + \cancel{\epsilon^2})$$

$$\overset{\text{T. exp.}}{=} \cancel{L(v^2)} + \underbrace{\frac{\partial L}{\partial v^2} 2(\vec{v} \cdot \vec{\epsilon})}_{0 \text{ or } \frac{df}{dt}}$$

$$\underbrace{\left(\frac{\partial L}{\partial v^2}\right)}\cdot \underbrace{\frac{d\vec{r}}{dt} \cdot \vec{\epsilon}}_{\frac{d}{dt}(\vec{r} \cdot \vec{\epsilon})} = \frac{df}{dt}$$

$$\longrightarrow \frac{\partial L}{\partial v^2} = \text{const} = \frac{m}{2}$$

$$L = \frac{m}{2} \left(\frac{d\vec{r}}{dt}\right)^2 \qquad \frac{d\vec{r}}{dt} = \dot{\vec{r}}$$

$$L'(\dot{\vec{r}}') = L(\dot{\vec{r}}) = L(\dot{\vec{r}}' - \vec{V})$$

$$= \frac{m}{2} (\dot{\vec{r}}' - \vec{V})^2$$

$$= \frac{m}{2} \dot{\vec{r}}'^2 - m \dot{\vec{r}}' \cdot \vec{V} + \frac{m}{2} \vec{V}^2$$

$$= \frac{m}{2}\dot{\vec r}'^2 + \frac{d}{dt}\Big(\underbrace{-m\vec r'\cdot\vec V + \frac{m}{2}\vec V^2 t}_{f(t)}\Big)$$

$$\equiv \frac{m}{2}\dot{\vec r}'^2$$

$$L'(\dot{\vec r}') = \frac{m}{2}\dot{\vec r}'^2, \quad L(\dot{\vec r}) = \frac{m}{2}\dot{\vec r}^2$$

$$L' = L$$

$L \longrightarrow \overset{L'}{\equiv} \underbrace{\alpha}\, L(q,\dot q)$: equivalent

$L(q,\dot q)$ $\qquad \underline{\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} = 0}$

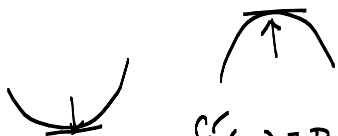
$$\alpha\underbrace{\left[\frac{\partial(L)}{\partial q} - \frac{d}{dt}\frac{\partial(L)}{\partial \dot q}\right]}=0$$

N

$$L = \frac{1}{2}m_1\dot{\vec r}_1^2 + \cdots = \frac{1}{2}\sum_{a=1}^{N} m_a \dot{\vec r}_a^2$$

$$\{m_1, \dots, m_N\} \longrightarrow \{\alpha m_1, \alpha m_2, \dots, \alpha m_N\}$$

unit: $\quad S = \int dt\, L$



$\delta S \underset{\sim}{\geq} 0$ $\qquad$ minimum

$L = \frac{m}{2}\dot{\vec r}^2$ $\quad (m>0)$

$f'(x_0) = 0$

$\underline{\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} = 0}$ $\quad \begin{cases} f(x) > f(x_0) \\ f(x) < f(x_0) \end{cases}$

$f''(x_0) \begin{matrix} >0 \\ <0 \end{matrix}$

$$\dot{\vec r}^2 = \vec v^2 = \begin{cases} \dot x^2 + \dot y^2 + \dot z^2 & \text{Cartesian} \\ \dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\phi^2 & \text{spherical} \\ \dot r^2 + r^2\dot\phi^2 + \dot z^2 & \text{cylindrical} \end{cases}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot x}\right) = \frac{d}{dt}(2\dot x) = 0 \longrightarrow \dot x = \text{const}$$

$$\frac{\partial L}{\partial r} \neq 0 \qquad \underbrace{\frac{\partial L}{\partial r}}_{\neq 0} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = 0$$

physical quantity : dimensionful

$\downarrow$

unit (SI) : kg, m, s

M L T

$[v] = [L][T]^{-1}$

$[a] = [L][T]^{-2}$

$[l] = [L]^2[M][T]^{-1}$; $\vec{l} = \vec{r} \times \underbrace{\vec{p}}_{m\vec{v}}$

$[K.E] = [M][L]^2[T]^{-2}$

$[\text{Action}] = [M][L]^2[T]^{-1}$

Planck constant : (ex) $\vec{L}^2 = \hbar^2 \underline{l(l+1)}$

$E = \hbar\omega(n+\frac{1}{2})$

$\dfrac{S}{\hbar}$: dimensionless

$f(x) =$ sin$x$, log$x$, e^x, x^2, ...

(physics) dimensionless quantity only

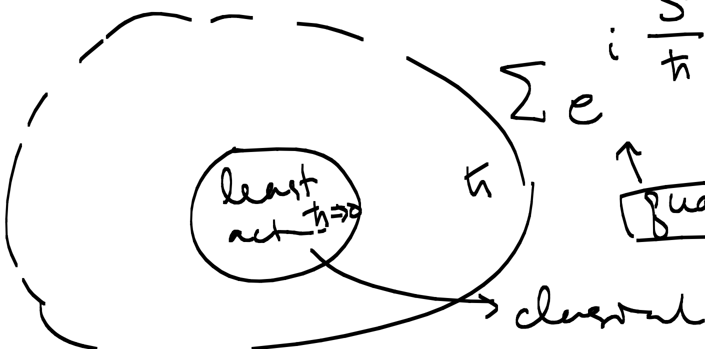
(ex) e^{ikx} $\quad k = \dfrac{2\pi}{\lambda}$

$$\sum_{\text{all cases}} e^{i\frac{S}{\hbar}} \approx e^{i\frac{S_{min}}{\hbar}}$$

$\underline{\hbar \to 0}$; quantum $\to$ classical limit

(steepest descent method)

$[e^{i\theta} = \cos\theta + i\sin\theta]$



$\sum e^{i\frac{S}{\hbar}}$

$\uparrow$

$\boxed{\text{quant}}$ Dirac Feynman Path Integral

§5. Interaction

$$L = \sum_a \frac{1}{2} m_a \dot{\vec{r}}_a^2 - U(\vec{r}_1, \cdots, \vec{r}_N)$$

$$0 = \frac{\partial L}{\partial \vec{r}_a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}_a} = -\frac{\partial U}{\partial \vec{r}_a} - m_a \ddot{\vec{r}}_a = 0$$

$$m_a \ddot{\vec{r}}_a = -\frac{\partial U}{\partial \vec{r}_a} = \vec{F}_a$$

$U \to U + \text{const}$

$$\vec{r}_1, \cdots, \vec{r}_N = x_1, \cdots, x_{3N}$$

If constraints n ; d.o.f. $S = 3N - n$

$q_1, \cdots, q_S$

$$x_a = f_a(q_1, \cdots, q_S)$$

$a = 1, \cdots, 3N$

$$\dot{x}_a = \sum_{k=1}^{S} \frac{\partial f_a}{\partial q_k} \dot{q}_k$$

$$\frac{1}{2}\sum_{a=1}^{3N} M_a \dot{x}_a^2 \leftarrow \frac{1}{2}\sum_a m_a \dot{\vec{r}}_a^2$$

(x_1, x_2, x_3)

$= \vec{r}_1 \to m_1$

$m_1 = m_2 = m_3 = M_1$

$\vec{r}_1, \vec{r}_2$

$\downarrow$

$\underbrace{x_1, x_2 x_3}_{m_1} \underbrace{x_4 x_5 x_6}_{m_2}$

$M_1 M_2 M_3 M_4 M_5 M_6$

$$\frac{1}{2}\sum_a M_a \dot{x}_a \dot{x}_a$$

$$\left(\sum_{k=1}^{S} \frac{\partial f_a}{\partial q_k} \dot{q}_k\right) \quad \sum_{j=1}^{S} \frac{\partial f_a}{\partial q_j} \dot{q}_j$$

$$= \sum_k \sum_j \frac{\partial f_a}{\partial q_k} \frac{\partial f_a}{\partial q_j} \dot{q}_k \dot{q}_j$$

$$(x_1 + x_2 + x_3)^2 = x_1^2 + x_2^2 + x_3^2 + 2(x_1x_2 + x_2x_3 + x_3x_1)$$

$$\left(\sum_{j=1}^{3} x_j\right)\left(\sum_{k=1}^{3} x_k\right) = \sum_{j=1}^{3}\sum_{k=1}^{3} x_j x_k$$

$$= \frac{1}{2}\sum_k \sum_j \underbrace{\left(\sum_a M_a \frac{\partial f_a}{\partial q_k} \frac{\partial f_a}{\partial q_j}\right)}_{\equiv a_{kj}(q_1, \cdots, q_S) = a_{jk}} \dot{q}_k \dot{q}_j$$

$$T = \frac{1}{2}\sum_{k,j=1}^{s} a_{kj}(q_1,\cdots,q_s)\,\dot q_k \dot q_j$$

complicated

$U(q_1,\cdots,q_s)$ is simple

$$\boxed{L = \frac{1}{2}\sum_{k,j=1}^{s} a_{kj}(q_1,\cdots,q_s)\,\dot q_k \dot q_j - U(q_1\cdots q_s)}$$

$$\frac{\partial L}{\partial q_\ell} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_\ell} = 0$$

$$\frac{\partial L}{\partial q_\ell} = -\frac{\partial U}{\partial q_\ell} + \frac{1}{2}\sum_{k,j}\frac{\partial a_{kj}}{\partial q_\ell}\dot q_k \dot q_j$$

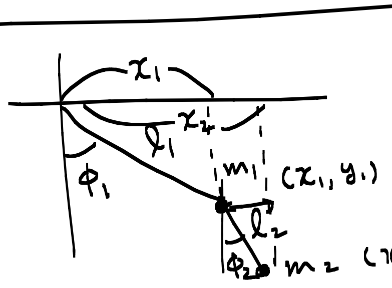
$$\frac{\partial L}{\partial \dot q_\ell} = \frac{1}{2}\sum_j a_{\ell j}\dot q_j + \frac{1}{2}\sum_k \underbrace{a_{k\ell}}_{=a_{\ell k}}\dot q_k$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_\ell}\right) = \frac{d}{dt}\left(\sum_j a_{\ell j}(q_1 \cdots q_s)\,\dot q_j\right)$$

$$= \sum_j\left[\left(\sum_k \frac{\partial a_{\ell j}}{\partial q_k}\dot q_k\right)\dot q_j + a_{\ell j}\ddot q_j\right]$$

$$\boxed{\sum_j\left[\sum_k\left(\frac{\partial a_{\ell j}}{\partial q_k}\dot q_k\right)\dot q_j + a_{\ell j}\ddot q_j\right] = -\frac{\partial U}{\partial q_\ell} + \frac{1}{2}\sum_{k,j}\frac{\partial a_{kj}}{\partial q_\ell}\dot q_k \dot q_j}$$

(ex)



Orthogonal

$\downarrow$

2 Constraints

$(x_1^2 + y_1^2 = l_1^2$

(ϕ_1, ϕ_2) : $q_1, \dots, q_s$; $2 \cdot 2 - 2 = 2$

$s = 2$

$$x_1 = l_1 \sin\phi_1$$
$$y_1 = -l_1 \cos\phi_1$$
$$x_2 = l_1 \sin\phi_1 + l_2 \sin\phi_2$$
$$y_2 = -l_1 \cos\phi_1 - l_2 \cos\phi_2$$

$$U = m_1 g y_1 + m_2 g y_2$$
$$= -m_1 g l_1 \cos\phi_1 - m_2 g (l_1 \cos\phi_1 + l_2 \cos\phi_2)$$
$$= U(\phi_1, \phi_2)$$

$$\dot{x}_1 = \underbrace{l_1 \cos\phi_1 \dot{\phi}_1}, \quad \dot{y}_1 = \underbrace{l_1 \sin\phi_1 \dot{\phi}_1}$$
$$\dot{x}_2 = l_1 \cos\phi_1 \dot{\phi}_1 + l_2 \cos\phi_2 \dot{\phi}_2$$
$$\dot{y}_2 = l_1 \sin\phi_1 \dot{\phi}_1 + l_2 \sin\phi_2 \dot{\phi}_2$$
$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$
$$= \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) \right]$$

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① least action : $\left.\begin{array}{l} S \text{ minimized} \\ \shortparallel \\ \int dt\, L \end{array}\right\}$ E-L eq

$$\frac{\partial L}{\partial q_a} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_a} = 0$$

② inertial frame ; $\vec{F} = 0$

$$L = L(\dot{\vec{r}}^2) \rightarrow \frac{\partial L}{\partial (\dot{\vec{r}}^2)} = \text{const} = \frac{m}{2}$$
$$L = T = \frac{m}{2} \dot{\vec{r}}^2$$

$L = L'$ up to total time derivative

③ generalized coordinate for constraint system.

3N particle $\qquad$ n #

$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{d.o.f. } s = 3N - n}$$

$U(q_1, \cdots, q_s)$

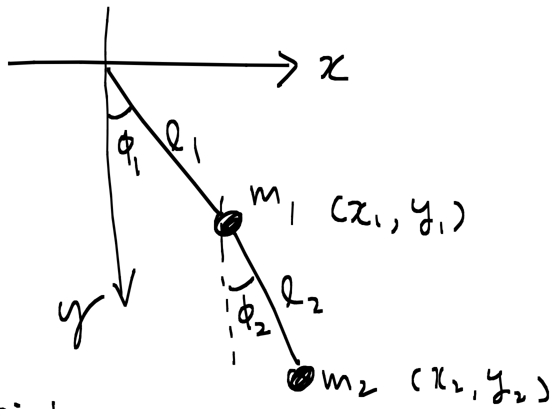
T ; orthogonal coordinate

$x_a = f_a(q_1, \cdots, q_s)$

$a = 1 \cdots 3N$

$$L = \frac{1}{2} \sum_{j,k=1}^{s} a_{jk}(q_1 \cdots q_s)\, \dot{q}_j \dot{q}_k - U(q_1 \cdots q_s)$$

prob. 1



$x_1 = l_1 \sin\phi_1$

$y_1 = + l_1 \cos\phi_1$

$\longrightarrow \dot{x}_1 = l_1 \dot{\phi}_1 \cos\phi_1$

$\dot{y}_1 = - l_1 \dot{\phi}_1 \sin\phi_1$

$x_2 = x_1 + l_2 \sin\phi_2 \qquad \dot{x}_2 = \dot{x}_1 + l_2 \dot{\phi}_2 \cos\phi_2$

$y_2 = y_1 + l_2 \cos\phi_2 \qquad \dot{y}_2 = \underline{\dot{y}_1} - \underline{l_2 \dot{\phi}_2 \sin\phi_2}$

$$T_1 = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) = \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2$$

$$T_2 = \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2} m_2 \Big(\underline{\dot{x}_1^2 + \dot{y}_1^2} + l_2^2 \dot{\phi}_2^2 + 2 \overset{l_1\dot{\phi}_1\cos\phi_1}{\dot{x}_1}\, l_2 \dot{\phi}_2 \cos\phi_2 - 2 \underset{-l_1\dot{\phi}_1\sin\phi_1}{\dot{y}_1}\, l_2 \dot{\phi}_2 \sin\phi_2 \Big)$$

$$T_2 = \frac{1}{2} m_2 \Big(l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \underbrace{(\cos\phi_1\cos\phi_2 + \sin\phi_1 \sin\phi_2)}_{\cos(\phi_1 - \phi_2)} \Big)$$

$$T = \frac{1}{2}(m_2+m_1)l_1^2\dot{\phi}_1^2 + \frac{m_2}{2}\left[l_2^2\dot{\phi}_2^2 + 2l_1l_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)\right]$$

$$U = m_1 g(-y_1) + m_2 g(-y_2)$$

$$= -(m_1+m_2)g l_1\cos\phi_1 - m_2 g l_2 \cos\phi_2$$

$$L = \frac{1}{2}(m_2+m_1)l_1^2\dot{\phi}_1^2 + \frac{m_2}{2}\left[l_2^2\dot{\phi}_2^2 + 2l_1l_2\dot{\phi}_1\dot{\phi}_2\underline{\underline{\cos(\phi_1-\phi_2)}}\right]$$

$$+(m_1+m_2)g l_1\cos\phi_1 + m_2 g l_2\cos\phi_2$$

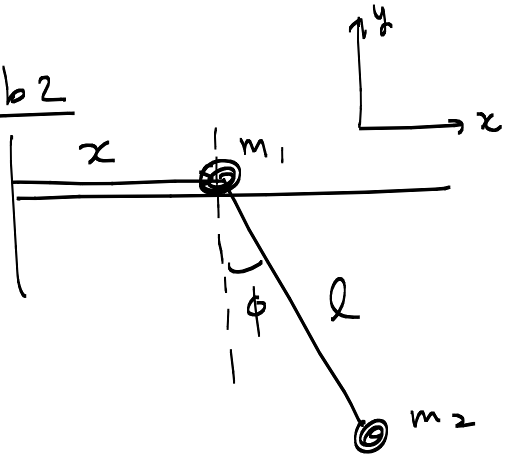
$$\frac{\partial L}{\partial \phi_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_1} = 0$$

$$\frac{\partial L}{\partial \phi_1} = -m_2 l_1 l_2 \dot{\phi}_1\dot{\phi}_2 \sin(\phi_1-\phi_2) - (m_1+m_2)g l_1 \sin\phi_1$$

$$\parallel$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\phi}_1} = (m_1+m_2)l_1^2\dot{\phi}_1 + m_2 l_1 l_2 \dot{\phi}_2\cos(\phi_1-\phi_2)\right]$$

<u>Prob 2</u>



$m_1 \Rightarrow (x, 0)$

$m_2 \Rightarrow (x + l\sin\phi, -l\cos\phi)$

$$\vec{v}_1 = (\dot{x}, 0), \quad \vec{v}_2 = (\dot{x} + l\dot{\phi}\cos\phi, \ l\dot{\phi}\sin\phi)$$

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x}^2 + l^2\dot{\phi}^2 + 2\dot{x}l\dot{\phi}\cos\phi)$$

$$V = m_2 g y_2 = -m_2 g l\cos\phi$$

$$\therefore L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x}^2 + l^2\dot{\phi}^2 + 2\dot{x}l\dot{\phi}\cos\phi)$$

$$+ m_2 g l\cos\phi$$

$$\frac{\partial L}{\partial \phi} = -m_2 l \dot{x}\dot{\phi} \sin\phi - m_2 g l \sin\phi$$
$$= -m_2 l \sin\phi\ (\dot{x}\dot{\phi} + g)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{d}{dt}\left(m_2 l^2 \dot{\phi} + m_2 l \dot{x} \cos\phi\right)$$
$$= m_2 l^2 \ddot{\phi} + m_2 l \left(\ddot{x}\cos\phi - \cancel{\dot{x}\dot{\phi}\sin\phi}\right)$$
$$= -m_2 l \sin\phi\ (\cancel{\dot{x}\dot{\phi}} + g)$$
$$\therefore\ \underline{l\ddot{\phi} + \ddot{x}\cos\phi = g\sin\phi}$$

$$\underline{\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}} = 0 \longrightarrow \frac{\partial L}{\partial \dot{x}} = \text{constant}$$

$$\frac{\partial L}{\partial x} = 0,\quad \frac{\partial L}{\partial \dot{x}} = \underbrace{m_1\dot{x} + m_2\dot{x} + m_2 l \dot{\phi}\cos\phi}_{= \text{const.}}$$

$$\therefore L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\dot{x}^2 + l^2\dot{\phi}^2 + 2\dot{x}l\dot{\phi}\cos\phi\right)$$
$$+ m_2 g l \cos\phi$$

$$(m_1 + m_2)\dot{x} + m_2 l \frac{d}{dt}(\sin\phi) = P_x$$

$$\therefore\ \underline{(m_1+m_2)x + m_2 l \sin\phi = P_x t + c}$$

$$\ddot{x} = -\frac{m_2 l}{m_1+m_2}\ \ddot{\sin\phi}$$

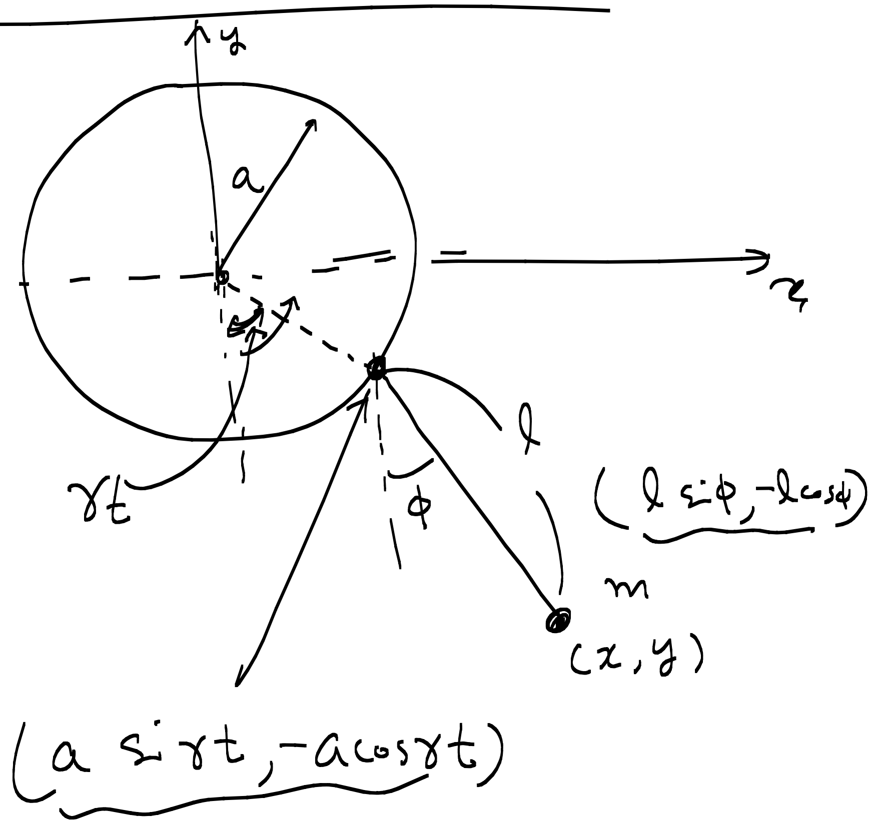
$$\underline{l\ddot{\phi} + \ddot{x}\cos\phi = g\sin\phi}$$

$$l\ddot{\phi} - \frac{m_2 l}{m_1+m_2}\ \ddot{\sin\phi}\ \cos\phi = g\sin\phi$$

$$\frac{d}{dt}(\dot{\phi}\cos\phi)$$
$$\shortparallel$$
$$-\dot{\phi}^2\sin\phi + \ddot{\phi}\cos\phi$$

$$l\left(1-\frac{m_2}{m_1+m_2}\cos^2\phi\right)\ddot{\phi}+\frac{m_2 l}{m_1+m_2}\dot{\phi}_1^2\sin\phi\cos\phi$$
$$=g\sin\phi$$

Prob 3



$$x = a\sin\gamma t + l\sin\phi$$
$$y = -a\cos\gamma t - l\cos\phi$$

$$T=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)=\frac{1}{2}m\Big((a\gamma\cos\gamma t+l\dot{\phi}\cos\phi)^2$$
$$+(a\gamma\sin\gamma t+l\dot{\phi}\sin\phi)^2\Big)$$

$$=\frac{1}{2}m\Big(a^2\gamma^2+l^2\dot{\phi}^2$$
$$+2al\gamma\dot{\phi}\cos(\phi-\gamma t)\Big)$$

$$V = mgy = -mg(a\cos\gamma t + l\cos\phi)$$

$$\boxed{\begin{aligned} L = \frac{1}{2}m\Big(&\cancel{a^2\gamma^2}_{\text{상수}}+l^2\dot{\phi}^2\\ &+2al\gamma\dot{\phi}\cos(\phi-\gamma t)\Big)\\ +\,mg(&\cancel{\quad\quad}+l\cos\phi)\end{aligned}}$$

$$\frac{df}{dt}$$

explicit dependence on t

$\boxed{\phi \text{ implicit} \quad \text{"} \qquad \text{"}}$

$$\frac{\partial L}{\partial t} \neq 0 \qquad \frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{d\phi}{dt}\frac{\partial L}{\partial \phi} + \frac{d\dot\phi}{dt}\frac{\partial L}{\partial \dot\phi}$$

$$L \equiv L + \frac{df}{dt}$$

$$L = \frac{1}{2} m \left(\ell^2 \dot\phi^2 + 2 a \ell \gamma \dot\phi \cos(\phi - \gamma t) \right) - m g \ell \cos\phi$$

$$\frac{\partial L}{\partial \phi} = -m a \ell \gamma \dot\phi \sin(\phi - \gamma t) + m g \ell \sin\phi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = \frac{d}{dt}\left(m \ell^2 \dot\phi + m a \ell \gamma \cos(\phi - \gamma t) \right)$$

$$= m \ell^2 \ddot\phi - m a \ell \gamma (\dot\phi - \gamma) \sin(\phi - \gamma t)$$

$$S = \int L \, dt$$

$$L' = L + \frac{df}{dt}$$

$$0 = \delta S = \int \left(\delta L + \delta \frac{df}{dt} \right) dt$$

$$= \delta \int L \, dt + \int \frac{d}{dt}(\delta f) \, dt$$

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} = 0$$

$$\frac{d}{dt} \delta f = \frac{d}{dt} f(t, q + \delta q, \dot q + \delta \dot q) - \frac{d}{dt} f(t, q, \dot q) = \delta \left(\frac{df}{dt} \right)$$

$$\boxed{\delta f \Big|_{t_1}^{t_2} = 0}$$

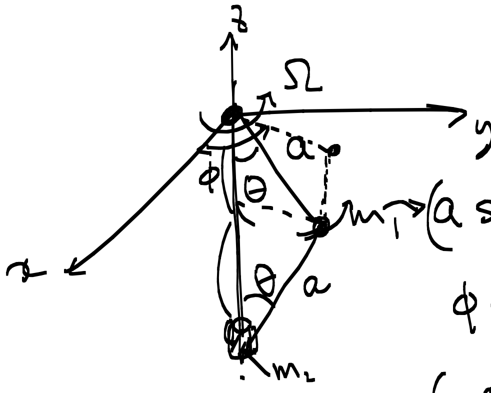
$$\dot\phi \cos(\phi - \gamma t) \neq \frac{d}{dt} g$$

$$\frac{d}{dt}\left(y = \sin(\phi - \gamma t)\right)$$

$$\parallel$$

$$(\dot{\phi} - \gamma)\cos(\phi - \gamma t)$$

Prob 4



$m_1 \to (a\sin\theta\cos\phi, a\sin\theta\sin\phi, -a\cos\theta)$

$\phi = \Omega t$

$(0, 0, -2a\cos\theta)$

$$\vec{v}_1 = a\left(\dot{\theta}\cos\theta\cos\phi - \Omega\sin\theta\sin\phi,\ \dot{\theta}\cos\theta\sin\phi + \Omega\sin\theta\cos\phi,\ \dot{\theta}\sin\theta\right)$$

$$\vec{v}_2 = (0, 0, 2a\dot{\theta}\sin\theta)$$

$$T = \frac{1}{2}m_1 a^2\left(\dot{\theta}^2 + \Omega^2\sin^2\theta\right) + 2m_2 a^2\dot{\theta}^2\sin^2\theta$$

$$V = -m_1 g a\cos\theta - 2m_2 g a\cos\theta = -(m_1+2m_2)ga\cos\theta$$

$$T - V = L$$

$$\frac{\partial L}{\partial \theta} = m_1 a^2\Omega^2\sin\theta\cos\theta + 4m_2 a^2\dot{\theta}^2\sin\theta\cos\theta - (m_1+2m_2)ga\sin\theta$$

$$\parallel$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = m_1 a^2\ddot{\theta} + 4m_2 a^2\left(\ddot{\theta}\sin^2\theta + 2\dot{\theta}^2\sin\theta\cos\theta\right)$$

End of Chap 1

Note Title 2015-03-20

Chap 2. Conservation Laws

d.o.f = s ; $q_1, \cdots, q_s$

E.of.M $\leftarrow$ q_i 2nd order d.e.

$\ddot{q}_i$

Initial Conditions are required.

$$q_i(0) = \underline{\underline{C_i}} \qquad i = 1, \cdots, s$$

$$\dot{q}_i(0) = \underline{\underline{D_i}} \qquad i = 1, \cdots, s$$

$$\left\{ q_i(t, C_i, D_i),\ \dot{q}_i(t, C_i, D_i) \right\}$$

$$\Downarrow$$

$$\left.\begin{array}{l} \text{const} = C_i(\{q_i, \dot{q}_i\}, t) \\ \text{const} = D_i(\{q_i, \dot{q}_i\}, t) \end{array}\right\} \text{conserved}$$

(ex) 1d H.O. $\quad L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2$

$$\ddot{q} + \omega^2 q = 0 \leftarrow$$

$$q = A \cos \omega t + B \sin \omega t$$

$$q(0) = A \leftarrow$$

$$\dot{q}(0) = \omega B \leftarrow$$

$$\frac{\dot{q}}{\omega} = -A \sin \omega t + B \cos \omega t$$

$$\text{const} = \boxed{q \cos \omega t - \frac{\dot{q}}{\omega} \sin \omega t} = A$$

$$\text{const} = \boxed{q \sin \omega t + \frac{\dot{q}}{\omega} \cos \omega t} = B$$

$$q = \underbrace{\sqrt{A^2+B^2}}\cos(\omega t - \alpha)$$

$$q = C\cos(\omega\underbrace{(t-t_0)}_{\downarrow\ t}) \qquad \alpha = \omega t_0$$

$$\Uparrow$$

$$s=1 \longrightarrow A, B\ (2s) \xrightarrow[t\text{의 기준점}]{} C$$

$$\Uparrow$$

$$2s-1$$

Conserved quantities $\longleftrightarrow$ Symmetries

$$\delta L = 0$$

$t \to t + \alpha$ time translation invariance

$$L(t, q_i, \dot{q}_i) \qquad \frac{\partial L}{\partial t} = 0$$

$$\frac{dL}{dt} = \underbrace{\frac{\partial L}{\partial t}}_{= 0} + \sum_i \dot{q}_i \underbrace{\frac{\partial L}{\partial q_i}}_{= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}} + \sum_i \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i}$$

$$= \sum_i \left(\dot{q}_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right)$$

$$= \frac{d}{dt}\left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right)$$

$$\therefore \frac{d}{dt}\underbrace{\left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \right)}_{\text{constant} \equiv E = H} = 0$$

$$\frac{\partial L}{\partial \dot{q}_i} = \text{generalized momentum} \equiv p_i$$

$$\boxed{H = \sum_i p_i \dot{q}_i - L} = E$$

$$\frac{dE}{dt} = 0 \rightarrow E = \text{conserved}$$

$$L = \underbrace{\frac{1}{2} \sum_{j,k=1}^{s} a_{jk}(q_1 \cdots q_s)\, \dot{q}_j \dot{q}_k}_{T} - U(q_1, \cdots q_s)$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{1}{2} \sum_k a_{ik} \dot{q}_k + \frac{1}{2} \sum_j a_{ji} \dot{q}_j$$

$$= \sum_k a_{ik} \dot{q}_k$$

$$H = \sum_i p_i \dot{q}_i - L$$

$$= \sum_{i,k} a_{ik} \dot{q}_i \dot{q}_k - L$$

$$= \underbrace{\frac{1}{2} \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k}_{T} + U(q_1, \cdots, q_s)$$

$$= T + U$$

§7. momentum.

spatial translation invariant

$$q_i \longrightarrow q_i + \alpha_i$$

$$L = T - U(\cancel{q_1, \cdots, q_s})$$

$$\delta L = \sum_i \frac{\partial L}{\partial q_i} \alpha_i$$

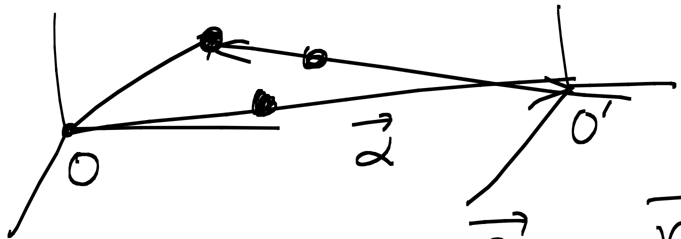
$$= 0$$

$$f(x_1, x_2)$$

$$x_1 \rightarrow x_1 + \alpha_1$$
$$x_2 \rightarrow x_2 + \alpha_2$$

$$\delta f = \underbrace{f(x_1+\alpha_1,\, x_2+\alpha_2)}_{\cancel{f(x_1,x_2)} + \alpha_1 \frac{\partial f}{\partial x_1} + \alpha_2 \frac{\partial f}{\partial x_2}} - \cancel{f(x_1,x_2)}$$

$$\boxed{\sum_i \frac{\partial L}{\partial q_i}\,\alpha_i = 0}$$



$$\vec{r}_1 \rightarrow \vec{r}_1 + \vec{\alpha}$$
$$\vec{r}_2 \rightarrow \vec{r}_2 + \vec{\alpha}$$

If α_i are indep of i

$$\rightarrow \alpha \sum_i \frac{\partial L}{\partial q_i} = 0$$

$$\rightarrow \sum_i \underbrace{\frac{\partial L}{\partial q_i}}_{\text{"}\frac{d}{dt}\text{"} \frac{\partial L}{\partial \dot{q}_i}} = 0$$

$$\therefore \frac{d}{dt} \sum_i \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i} = 0$$

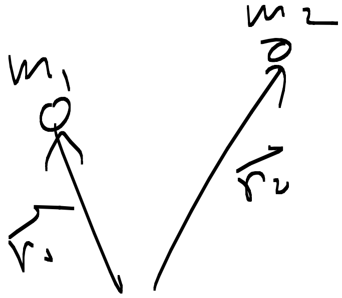
$$\therefore \boxed{\sum_i p_i = \text{const}}$$

total generalized momentum

= conserved

$\updownarrow$

<u>global</u> coordinate translation invariance



$$U(|\vec{r}_1 - \vec{r}_2|)$$

$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(|\vec{r}_1 - \vec{r}_2|)$$

$$\vec{r}_1 \to \vec{r}_1 + \vec{\alpha}$$

$$\vec{r}_2 \to \vec{r}_2 + \vec{\alpha}$$

$$m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2 = \text{const}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

$$\sum_i \underbrace{\frac{\partial L}{\partial q_i}}_{\downarrow} = 0$$

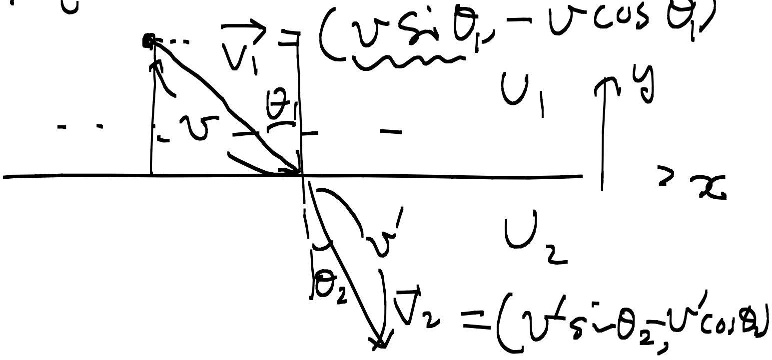
$$L = T - U$$

if $T = T(\dot{q}_i)$ i.e. $\frac{\partial T}{\partial q_i} = 0$

$$\frac{\partial L}{\partial q_i} = -\frac{\partial U}{\partial q_i} = F_i$$

$$\therefore \sum_i F_i = 0$$

<u>Prob.</u>



$x \to x+a$ ✓ $\to U_x =$ conserved

$y \to y+b$ ✗

$$\underline{v\sin\theta_1 = v'\sin\theta_2}$$

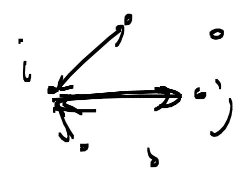
$$\frac{1}{2}mv^2 + U_1 = \frac{1}{2}mv'^2 + U_2$$

$$v' = \sqrt{v^2 + \frac{2}{m}(U_1 - U_2)}$$

$$\sin\theta_2 = \frac{v\sin\theta_1}{\sqrt{v^2 + \frac{2}{m}(U_1 - U_2)}}$$

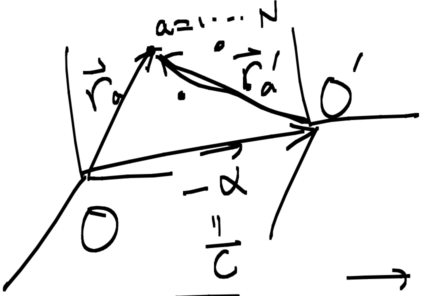
$$\therefore \quad \frac{\sin\theta_1}{\sin\theta_2} = \frac{\cancel{\sin\theta_1}}{\frac{v\cancel{\sin\theta_1}}{\sqrt{}}} = \frac{\sqrt{}}{v} = \sqrt{1 + \frac{2(U_1 - U_2)}{mv^2}} \quad //$$

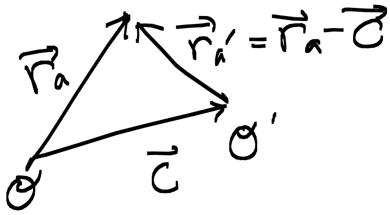
§8. Center of mass



$U(r_{ij})$ $\quad i \neq j = 1, \cdots, s$ (ex) $\quad -\frac{Gm_1m_2}{|\vec{r}_i - \vec{r}_j|}$

$$\vec{r}_i \to \vec{r}_i + \vec{\alpha} \Rightarrow \underline{\sum_i \vec{P}_i = \text{const}}$$





$$\vec{R} \equiv \frac{\sum_a m_a \vec{r}_a}{\sum_a m_a}$$

$$\equiv \vec{c} \neq 0 \qquad \underbrace{\vec{r}_a \to \overset{\vec{r}_a'}{\overset{=}{\vec{r}_a + \vec{\alpha}}}}$$

$$\Rightarrow \frac{\sum_a m_a(\vec{r}_a + \vec{\alpha})}{\sum_a m_a}$$

$$= \vec{c} + \vec{\alpha}\,\frac{\cancel{\sum_a m_a}}{\cancel{\sum_a m_a}}$$

$$= \vec{c} + \vec{\alpha} = 0$$

$$\therefore \quad \vec{\alpha} = -\vec{c}$$

CM frame : $\vec{R} = 0$

$$\vec{v}_a' = \frac{d\vec{r}_a'}{dt} = \frac{d}{dt}(\vec{r}_a - \vec{c})$$

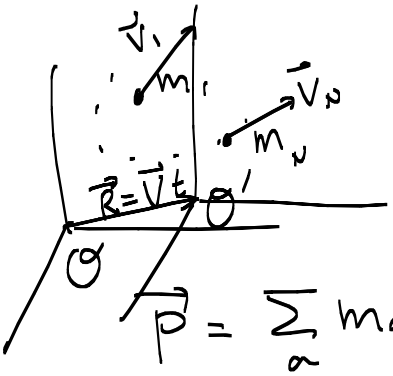
$$= \vec{v}_a - \underbrace{\frac{d\vec{c}}{dt}}_{\vec{V}}$$

$$\vec{P} = \sum_a m_a \vec{v}_a$$

$$= \sum_a m_a(\vec{v}_a' + \vec{V})$$

$$= \vec{P}' + \vec{V} \sum_a m_a$$

$$\vec P' = 0 \quad \to \quad \vec V = \frac{\vec P}{\sum_a m_a} = \frac{\vec P}{M}$$



$$\vec v_a' = \vec v_a - \vec V$$

$$a = 1 \cdots N$$

$$\vec P = \sum_a m_a \vec v_a$$

$$\vec P' = \sum_a m_a \vec v_a' = 0$$

$$\vec V = \frac{\vec P}{M} = \frac{\sum_a m_a \vec v_a}{\sum m_a}$$

$$\downarrow$$

$$\frac{d\vec R}{dt} = \frac{\sum_a m_a \frac{d\vec r_a}{dt}}{\sum_a m_a} = \frac{d}{dt}\left(\frac{\sum_a m_a \vec r_a}{\sum_a m_a}\right)$$

$$\therefore \vec R = \frac{\sum_a m_a \vec r_a}{\sum_a m_a} = \text{c.o.m.}$$

$$\frac{d\vec R}{dt} = \vec V = \frac{\vec P}{M}$$

$$E = \sum_a \frac{1}{2} m_a \vec v_a^{\,2} + U(\{\vec r_a\}) \qquad \vec r_a = \vec r_a' + \vec R$$

$$\vec v_a = \vec v_a' + \vec V$$

$$= \sum_a \frac{1}{2} m_a (\vec v_a' + \vec V)^2 + U$$

$$= \frac{1}{2} \underbrace{\left(\sum_a m_a\right)}_{M} \vec V^{\,2} + \sum_a \frac{1}{2} m_a \left(\vec v_a'^{\,2} + 2\vec v_a' \cdot \vec V\right) + U$$

$$= \frac{1}{2} M V^2 + E_i$$

internal energy

$$E_i = \sum_a \frac{1}{2} m_a \vec{v}_a'^2 + U$$

$$+ \underbrace{\sum_a m_a \vec{v}_a'}_{\vec{p}'} \cdot \vec{V}$$

$$= \cancel{\vec{p}' \cdot \vec{V}} + U + \sum_a \frac{1}{2} m_a \vec{v}_a'^2$$

CM frame: $\vec{p}' = 0$

$$E_i = U + \sum_a \frac{1}{2} m_a \vec{v}_a'^2$$

$$E = E_i + \frac{1}{2} M V^2$$

Prob. $L = T - U$

$$= \underbrace{\frac{1}{2} \sum_a m_a \vec{v}_a'^2}_{} + \sum_a m_a \vec{v}_a' \cdot \vec{V} + \frac{1}{2} M \vec{V}^2 \underbrace{- U}_{}$$

$$= L' + \frac{1}{2} M \vec{V}^2 + \vec{p}' \cdot \vec{V}$$

$$\vec{p}' = \sum_a m_a \underbrace{\vec{v}_a'}_{} \qquad \vec{v}_a' = \frac{d\vec{r}_a'}{dt}$$

$$S = S' + \frac{1}{2} M \vec{V}^2 \cdot t + \underbrace{\int dt \, \underbrace{\sum_a m_a}_{} \vec{V} \cdot \frac{d\vec{r}_a'}{dt}}_{\sum_a m_a \vec{V} \cdot \int dt \frac{d\vec{r}_a'}{dt}}$$

$$\vec{V}\cdot\Big(\overbrace{\sum_a m_a \vec{r}_a{}'}^{\vec{r}_a{}'}\Big)$$

$$\sum_a m_a \vec{r}_a{}' = M\vec{R}'$$

$$\vec{R}' \equiv \frac{\sum_a m_a \vec{r}_a{}'}{\sum_a m_a}$$

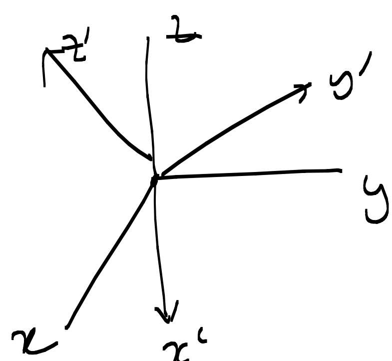
$$\underline{S = S' + \frac{1}{2}MV^2 t + M\vec{V}\cdot\vec{R}'}$$

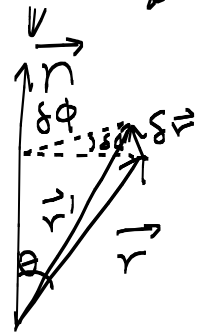
§9. Angular Momentum

isotropy

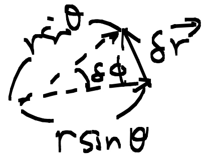
$\updownarrow$

rotational invariance





top view



$$\delta\vec{r} = \vec{r}' - \vec{r}$$

$$|\delta\vec{r}| = \underset{\substack{\uparrow \\ |\vec{r}|}}{r}\sin\theta\, \underset{\substack{\| \\ |\delta\vec{\phi}|}}{\delta\phi}$$

$$\delta\phi\,\vec{n} = \delta\vec{\phi}$$

$$|\delta\vec{r}| = |\vec{r}|\,|\delta\vec{\phi}|\sin\theta$$

$$\delta\vec{r} \perp \vec{n}(\delta\vec{\phi}),\ \vec{r}$$

$$\boxed{\delta\vec{r} = \delta\vec{\phi}\times\vec{r}}$$

$$\frac{d}{dt}\delta\vec{r} = \delta\vec{\phi}\times\frac{d\vec{r}}{dt} \longrightarrow \delta\vec{v} = \delta\vec{\phi}\times\vec{v}$$

$$L = L(\{\vec r_a, \vec v_a\})$$

$$\delta L = \sum_a \left(\frac{\partial L}{\partial \vec r_a} \cdot \delta \vec r_a + \underbrace{\frac{\partial L}{\partial \vec v_a}}_{\vec p_a} \cdot \delta \vec v_a \right)$$

$$\frac{\partial L}{\partial \vec r_a} = \frac{d}{dt} \underbrace{\frac{\partial L}{\partial \dot{\vec r}_a}}_{\vec p_a}$$

$$\vec p_a = \frac{\partial L}{\partial \dot{\vec r}_a}$$

$$= \sum_a \left(\dot{\vec p}_a \cdot \delta \vec r_a + \vec p_a \cdot \delta \vec v_a \right)$$

$$\delta \vec r_a = \delta \vec\phi \times \vec r \qquad \delta \vec v_a = \delta \vec\phi \times \vec v_a$$

$$= \sum_a \dot{\vec p}_a \cdot (\delta \vec\phi \times \vec r_a) + \vec p_a \cdot (\delta \vec\phi \times \vec v_a)$$

$$\vec A \cdot (\vec B \times \vec C) = \vec B \cdot (\vec C \times \vec A) = \vec C \cdot (\vec A \times \vec B)$$

$$= \sum_a \delta \vec\phi \cdot (\vec r_a \times \dot{\vec p}_a) + \cancel{\delta \vec\phi \cdot (\vec v_a \times \vec p_a)}$$

$$\vec p_a = m_a \vec v_a$$

$$\delta L = \delta \vec\phi \cdot \underbrace{\sum_a \vec r_a \times \frac{d}{dt} \vec p_a} = 0$$

For any $\delta \vec\phi$, $\delta L = 0 \Rightarrow \sum_a \vec r_a \times \frac{d \vec p_a}{dt} = 0$

$$\sum_a \vec r_a \times \frac{d \vec p_a}{dt} = \frac{d}{dt}\left(\sum_a \vec r_a \times \vec p_a \right) = \underbrace{\sum_a \vec v_a \times \vec p_a}_{0} + \sum_a \vec r_a \times \dot{\vec p}_a$$

$$\therefore \sum_a \underbrace{\vec r_a \times \vec p_a}_{\vec M_a} = \text{constant}$$

$$\boxed{\vec{M} = \sum_a \vec{m}_a = \text{conserved}}$$

$$\vec{r}_a \longrightarrow \vec{r}_a - \vec{\alpha} \quad \text{global translati}$$

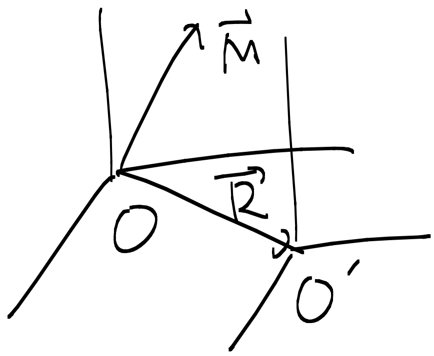
$$\vec{r}_a - \vec{\alpha} = \vec{r}_a{}' \qquad \vec{r}_a = \vec{r}_a{}' + \vec{\alpha}$$

$$\dot{\vec{r}}_a = \vec{v}_a \longrightarrow \vec{v}_a \qquad \vec{M} \qquad \vec{\alpha} = \vec{R}$$

$$\vec{M}' = \sum_a (\vec{r}_a - \vec{\alpha}) \times \vec{p}_a = \sum_a \vec{r}_a \times \vec{p}_a - \vec{\alpha} \times \underbrace{\sum_a \vec{p}_a}_{\vec{P}}$$

$$\vec{M} = \vec{M}' + \vec{\alpha} \times \vec{P}, \qquad \vec{\alpha} = \vec{R}$$

$$\boxed{\vec{M} = \vec{M}' + \vec{R} \times \vec{P}}$$



$$\vec{M}' = \vec{M} - \vec{R} \times \vec{P}$$

If Rotational invariance for $\delta\vec{\phi} = \delta\phi\,\hat{z}$,

$$\delta L = 0 = \delta\vec{\phi} \cdot \frac{d\vec{M}}{dt}$$

$$= \delta\phi \frac{dM_z}{dt} \qquad \hat{z} \cdot \vec{M} \equiv M_z$$

$$M_z = \text{constant} = \sum_a (\vec{r}_a \times \vec{p}_a)_z, \qquad \vec{p}_a = m_a \vec{v}_a, \quad \vec{v}_a = \dot{\vec{r}}_a$$

$$= \sum_a m_a (\vec{r}_a \times \dot{\vec{r}}_a)_z$$

$$= \sum_a m_a (x_a \dot{y}_a - y_a \dot{x}_a)$$

$$x_a = r_a \cos\phi_a$$
$$y_a = r_a \sin\phi_a$$
$$\dot{x}_a = \dot{r}_a \cos\phi_a - r_a \dot{\phi}_a \sin\phi_a$$
$$\dot{y}_a = \dot{r}_a \sin\phi_a + r_a \dot{\phi}_a \cos\phi_a$$
$$x_a \dot{y}_a = \dot{r}_a r_a \cancel{\cos\phi_a \sin\phi_a} + r_a^2 \dot{\phi}_a \cos^2\phi_a$$
$$-\; y_a \dot{x}_a = \dot{r}_a r_a \cancel{\cos\phi_a \sin\phi_a} - r_a^2 \dot{\phi}_a \sin^2\phi_a$$
$$= r_a^2 \dot{\phi}_a$$

$$M_z = \sum_a m_a r_a^2 \dot{\phi}_a \checkmark$$

$$L = \frac{1}{2} \sum_a m_a \left(\dot{r}_a^2 + r_a^2 \dot{\phi}_a^2 + \dot{z}_a^2\right) - U$$

$$\frac{\partial L}{\partial \phi_a} = 0 \rightarrow \frac{\partial L}{\partial \dot{\phi}_a} = \text{conserved}$$

$$M_z = \sum_a m_a r_a^2 \dot{\phi}_a = \text{conser}$$

(H.W) Due next week

Problem 1. Obtain expressions for the Cartesian components and the magnitude of the angular momentum of a particle in cylindrical co-ordinates r, ϕ, z.

Solution.
$$M_x = m(r\dot{z} - z\dot{r}) \sin\phi - mrz\dot{\phi}\cos\phi,$$
$$M_y = -m(r\dot{z} - z\dot{r}) \cos\phi - mrz\dot{\phi}\sin\phi,$$
$$M_z = mr^2\dot{\phi},$$
$$M^2 = m^2r^2\dot{\phi}^2(r^2+z^2) + m^2(r\dot{z} - z\dot{r})^2.$$

Problem 2. The same as Problem 1, but in spherical co-ordinates r, θ, ϕ.

Solution.
$$M_x = -mr^2(\dot{\theta}\sin\phi + \dot{\phi}\sin\theta\cos\theta\cos\phi),$$
$$M_y = mr^2(\dot{\theta}\cos\phi - \dot{\phi}\sin\theta\cos\theta\sin\phi),$$
$$M_z = mr^2\dot{\phi}\sin^2\theta,$$
$$M^2 = m^2r^4(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta).$$

Problem 3. Which components of momentum **P** and angular momentum **M** are conserved in motion in the following fields?

(a) the field of an infinite homogeneous plane, (b) that of an infinite homogeneous cylinder, (c) that of an infinite homogeneous prism, (d) that of two points, (e) that of an infinite homogeneous half-plane, (f) that of a homogeneous cone, (g) that of a homogeneous circular torus, (h) that of an infinite homogeneous cylindrical helix.

§10. Mechanical similarity

$$c \cdot L(t, q, \dot{q}) = L'(t, q, \dot{q})$$

$$0 = \frac{\partial L'}{\partial q} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{q}} = c\underbrace{\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)}_{= 0}$$

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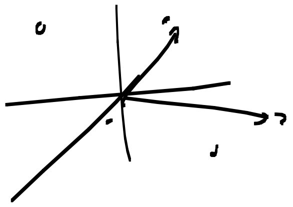
① $U(\alpha \vec{r}_1, \cdots \alpha \vec{r}_n) = \alpha^k U(\vec{r}_1, \cdots, \vec{r}_n)$

k: degree



$f(x) = x^2$ polynomial with deg $= 2$

$x \to \alpha x$

$\underline{\alpha^2} x^2$

$$f(x, y, z) = 2xy + 3y^2 - 4zx$$

$x \downarrow \alpha x$, $y \downarrow \alpha y$, $z \downarrow \alpha z$ $\longrightarrow \alpha^2 f(x, \cdots z)$

$$\cancel{f(x, y, z) = 2xy + z^3}$$

$2xy \downarrow \alpha^2 2xy$, $z^3 \downarrow \alpha^3 z^3$

$$L = T - U$$

how T transforms? when $\vec{r}_a \longrightarrow \alpha \vec{r}_a$

$$U \to \alpha^k U$$

$$T \propto \dot{\vec{r}}_a^{\,2} \quad \text{if } t \to \beta t$$

$$\vec{v}_a = \dot{\vec{r}}_a \longrightarrow \frac{\alpha\, d\vec{r}_a}{\beta\, dt} = \frac{\alpha}{\beta}\dot{\vec{r}}_a$$

$$\dot{\vec{r}}_a = \frac{d\vec{r}_a}{dt}$$

$$T \longrightarrow \left(\frac{\alpha}{\beta}\right)^2 T$$

$$L \longrightarrow L' = \left(\frac{\alpha}{\beta}\right)^2 T - \alpha^k U$$

$$= \alpha^k (\underbrace{T - U}_{L})$$

$$\left(\frac{\alpha}{\beta}\right)^2 = \alpha^k$$

$$\beta = \alpha^{1-\frac{k}{2}}$$

$$t \to t' = \beta t \to \beta = \frac{t'}{t}$$

$$\vec{r}_a \to \vec{r}_a{}' = \alpha \vec{r}_a \to \alpha = \frac{l'}{l}$$

$$l = \sqrt{\vec{r}_a \cdot \vec{r}_a}$$

$$\frac{1}{\beta} = \alpha^{-1+\frac{k}{2}}$$

$$\frac{v'}{v} = \frac{l'/t'}{l/t} = \underbrace{\frac{l'}{l}}_{\alpha} \underbrace{\left(\frac{t}{t'}\right)}_{\beta^{-1}} = \frac{\alpha}{\beta}$$

$$\frac{v'}{v} = \frac{l'}{l}\left(\frac{l'}{l}\right)^{-1+\frac{k}{2}}$$

$$= \left(\frac{l'}{l}\right)^{\frac{k}{2}}$$

$$\frac{E'}{E} = \frac{v'^2}{v^2} = \left(\frac{l'}{l}\right)^{k}$$

$$\frac{M'}{M} = \frac{v' \cdot l'}{v \cdot l} = \left(\frac{l'}{l}\right)^{\frac{k}{2}+1}$$

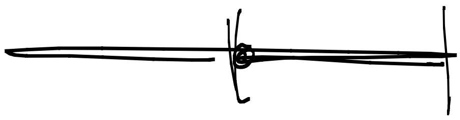
(ex) $U = \frac{1}{2}\left(k_1 \vec{r}_1^{\,2} + k_2 \vec{r}_2^{\,2} + \cdots + k_n \vec{r}_n^{\,2}\right)$

$\vec{r}_a \to \alpha \vec{r}_a \qquad U \to \alpha^2 U$

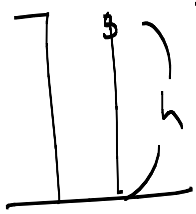
$\therefore \underline{k=2}$

$$\frac{t'}{t} = \beta = \alpha^{1-\frac{k}{2}} = 1$$

$l \to l' = \alpha l$



$U = mgx$



$v^2 = 2gh$

$v = \sqrt{2gh}$

$t = \frac{h}{v} = \sqrt{h}$

$x \to \alpha x = x'$

$U \to \alpha U \quad \therefore k=1$

$$\beta = \alpha^{1-\frac{k}{2}} = \alpha^{\frac{1}{2}} = \frac{t'}{t}$$

$$\frac{t'}{t} = \sqrt{\frac{l'}{l}} \to t \propto \sqrt{h}$$

$U = -\frac{k}{r} \qquad r = \sqrt{x^2+y^2+z^2}$

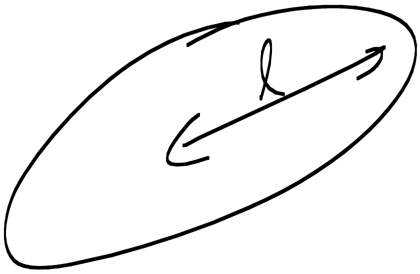
$$x, y, z \to \alpha x, \alpha y, \alpha z$$

$$r \to \alpha r$$

$$U \to \alpha^{-1} U \quad \therefore k = -1$$

$$\frac{t'}{t} = \beta = \alpha^{1-\frac{k}{2}} = \left(\frac{l'}{l}\right)^{\frac{3}{2}}$$

$$t \propto l^{\frac{3}{2}} \leftarrow \text{Kepler's 3rd law}$$



$\boxed{\text{Virial theorem}}$

$$L = T(\vec{v}_a) - U(\vec{r}_a)$$

$$\vec{p}_a = \frac{\partial L}{\partial \underbrace{\dot{\vec{r}}_a}_{=\vec{v}_a}} = \frac{\partial T}{\partial \vec{v}_a}$$

$$\vec{p}_a = m_a \vec{v}_a$$

$$T = \sum_a \frac{1}{2} m_a \vec{v}_a^{\,2}$$

$$\sum_a \frac{1}{2} \vec{v}_a \cdot \vec{p}_a = \sum_a \frac{\vec{v}_a}{2} \cdot \frac{\partial T}{\partial \vec{v}_a} = T$$

$$\underbrace{\sum_a \vec{v}_a \cdot \underbrace{\frac{\partial T}{\partial \vec{v}_a}}_{\vec{p}_a}}_{\frac{d}{dt}\left(\sum_a \vec{p}_a \cdot \vec{r}_a\right) - \sum_a \dot{\vec{p}}_a \cdot \vec{r}_a} = 2T$$

$$\to \dot{\vec{r}}_a \cdot \vec{p}_a = \vec{v}_a \cdot \vec{p}_a$$

$$\left(\begin{aligned} \frac{d}{dt}(g \cdot f) &= \dot{g} f + g \dot{f} \\ \therefore \quad \dot{g} f &= \frac{d}{dt}(g f) - g \dot{f} \end{aligned} \right)$$

$$2T = \frac{d}{dt}\left(\sum_a \vec{p}_a \cdot \vec{r}_a \right) - \sum_a \dot{\vec{p}}_a \cdot \vec{r}_a$$

time average

$$\boxed{\lim_{\tau \to \infty} \frac{\int_0^\tau f(t)\, dt}{\tau} = \overline{f}}$$

$$\overline{T} = \frac{1}{2}\left(\cancel{\overline{\frac{d}{dt}\left(\sum_a \vec{p}_a \cdot \vec{r}_a\right)}} - \overline{\sum_a \dot{\vec{p}}_a \cdot \vec{r}_a} \right)$$

$$\overline{\left(\frac{df}{dt}\right)} = \lim_{\tau \to \infty} \frac{\int_0^\tau \left(\frac{df}{dt}\right) dt}{\tau}$$

$$= \lim_{\tau \to \infty} \frac{\overbrace{f(\tau) - f(0)}^{\text{finite}}}{\underbrace{\tau}_{\text{infinite}}}$$

$$= 0$$

$$\dot{\vec{p}}_a = m_a \dot{\vec{v}}_a = m_a \vec{a}_a = \vec{F}_a$$

$$= -\frac{\partial U}{\partial \vec{r}_a}$$

$$\overline{T} = \overline{\frac{1}{2} \sum_a \vec{r}_a \cdot \frac{\partial U}{\partial \vec{r}_a}} = k\, \overline{U}$$

$$\vec{r}_a \to \alpha \vec{r}_a$$

$$\underline{U \to \alpha^k U} \longrightarrow k U$$

$$U = x^k$$

$$x \frac{\partial U}{\partial x} = x\, k\, x^{k-1} = k \underbrace{x^k} = k U$$

$$U = \left(\sqrt{x^2+y^2+z^2}\right)^k = \left(x^2+y^2+z^2\right)^{\frac{k}{2}}$$

$$x \to \alpha x, \quad y \to \alpha y \quad z \to \alpha z .$$

$$x \frac{\partial U}{\partial x} = \frac{k}{\cancel{2}} \cancel{2} x^2 \left(\quad\right)^{\frac{k}{2}-1}$$

$$y \frac{\partial U}{\partial y} = \frac{k}{\cancel{2}} \cancel{2} y^2 \left(\quad\right)^{\frac{k}{2}-1}$$

$$+)\; z \frac{\partial U}{\partial z} = \frac{k}{\cancel{2}} \cancel{2} z^2 \left(\quad\right)^{\frac{k}{2}-1}$$

$$\sum_a x_a \frac{\partial U}{\partial x_a} = k\left(x^2+y^2+z^2\right)\left(x^2+y^2+z^2\right)^{\frac{k}{2}-1}$$

$$= k \underbrace{\left(x^2+y^2+z^2\right)^{\frac{k}{2}}}_{U}$$

$$2\overline{T} = k\overline{U}$$

$$E = T + U = \overline{E} = \overline{T} + \overline{U}$$

$$E = \overline{T} + \frac{2}{k}\overline{T} = \left(1+\frac{2}{k}\right)\overline{T}$$

$$= \frac{k}{2}\overline{U} + \overline{U} = \left(1+\frac{k}{2}\right)\overline{U}$$

$$E = \left(1+\frac{2}{k}\right)\overline{T} = \left(1+\frac{k}{2}\right)\overline{U}$$

$\underline{\underline{k=2}}$ $\rightarrow$ $E = 2\overline{T} = 2\overline{U}$

<u>Virial theorem</u> $\overline{T} = \overline{U}$

<u>$k=-1$</u> $$U = -\frac{c}{r}$$

$$E = -\overline{T} = \frac{1}{2}\overline{U}$$

$$E, \overline{U} < 0$$

H-atom.

$$U = \frac{1}{4\pi\varepsilon_0}\frac{-e^2}{r}$$

$$\overline{U} = -\frac{e^2}{4\pi\varepsilon_0}\overline{\frac{1}{r}} = 2\times(-13.6)$$

3장. Eq of M

§11. 1d.

$$L = \frac{1}{2} a(q)\, \dot{q}^2 - U(q)$$

$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$ 의 special case

$$E = \frac{1}{2} m \dot{x}^2 + U(x) = \text{const}$$

$$\frac{dx}{dt} = \pm \sqrt{\frac{2}{m}\left(E - U(x)\right)} \quad \begin{pmatrix} + : R \\ - : L \end{pmatrix}$$

"+"

$$t - t_0 = \pm \int_{x_0}^{x} \frac{dx}{\sqrt{\quad " \quad}}$$

$$\boxed{t = \int \frac{dx}{\sqrt{\quad}} + C}$$

$$= G(x) + c$$

$$x = G^{-1}(t - c)$$

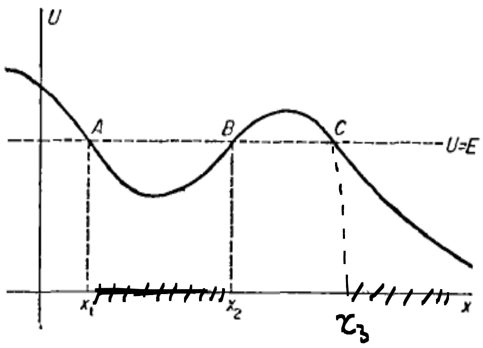


FIG. 6

$$E = T + U(x)$$

$$T \geqq 0$$

$$E \geqq U(x)$$

$$\frac{1}{\sqrt{}} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\sin^2\frac{\phi_0}{2} - \sin^2\frac{\phi}{2}}}$$

$$= \frac{1}{\sqrt{2}\sin\frac{\phi_0}{2}} \frac{1}{\sqrt{1 - \underbrace{\frac{1}{\sin^2\frac{\phi_0}{2}}\sin^2\frac{\phi}{2}}_{\sin^2\xi}}}$$

$$\underbrace{\sin\xi = \frac{\sin\frac{\phi}{2}}{\sin\frac{\phi_0}{2}}} \qquad \underbrace{}_{\frac{1}{\cos\xi}}$$

$$d\xi\cos\xi = \frac{1}{\sin\frac{\phi_0}{2}}\frac{1}{2}\cos\frac{\phi}{2}\,d\phi$$

$$d\phi = \frac{\cos\xi\, d\xi\, 2\sin\frac{\phi_0}{2}}{\sqrt{1 - \underbrace{\sin^2\frac{\phi}{2}}_{\sin^2\frac{\phi_0}{2}\sin^2\xi}}}$$

$$d\phi = \frac{\cos\xi\; 2\sin\frac{\phi_0}{2}\; d\xi}{\sqrt{1 - \sin^2\frac{\phi_0}{2}\sin^2\xi}}$$

$$\int_0^{\phi_0} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}} = \int \frac{\cancel{\cos\xi}\; 2\cancel{\sin\frac{\phi_0}{2}}\; d\xi}{\sqrt{1 - \sin^2\frac{\phi_0}{2}\sin^2\xi}}$$

$$\times \frac{1}{\sqrt{2}} \frac{1}{\cancel{\sin\frac{\phi_0}{2}}} \frac{1}{\cancel{\cos\xi}}$$

$$\overset{\shortparallel}{\sqrt{\frac{2g}{l}} \cdot \frac{T}{4}}$$

$$= \sqrt{2} \int_0^{\frac{\pi}{2}} \frac{d\xi}{\sqrt{1 - k^2\sin^2\xi}}$$

$$\frac{t'}{t} = \beta = \alpha^{1-\frac{k}{2}} = \left(\frac{l'}{l}\right)^{1-\frac{k}{2}}$$

$$T \propto l^{1-\frac{n}{2}} = \left(\left(\frac{E}{A}\right)^{\frac{1}{n}}\right)^{1-\frac{n}{2}}$$

$$\therefore \quad T \propto E^{\frac{1}{n}-\frac{1}{2}} \quad \checkmark$$

$$\underbrace{\int_0^{\frac{T}{4}} dt}_{\frac{T}{4}} = \int_0^{\left(\frac{E}{A}\right)^{\frac{1}{n}}} \frac{dx}{\sqrt{\frac{2}{m}\left(E - A x^n\right)}}$$

$$x\left(\frac{A}{E}\right)^{\frac{1}{n}} \equiv y \to dx = dy\left(\frac{E}{A}\right)^{\frac{1}{n}}$$

$$= \int_0^1 \frac{\left(\frac{E}{A}\right)^{\frac{1}{n}} dy}{\sqrt{\frac{2E}{m}}\sqrt{1 - \underbrace{\frac{A}{E}x^n}_{\left(\left(\frac{A}{E}\right)^{\frac{1}{n}} \cdot x\right)^n = y^n}}}$$

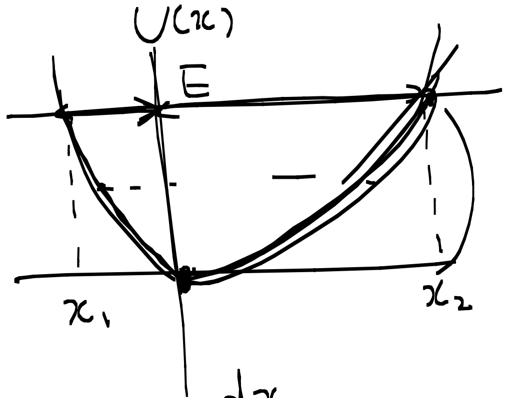
$$= \sqrt{\frac{m}{2}}\frac{1}{A^{\frac{1}{n}}} \underline{\underline{E^{\frac{1}{n}-\frac{1}{2}}}} \int_0^1 \frac{dy}{\sqrt{1-y^n}}$$

$$\int_0^1 \frac{dy}{\sqrt{1-y^n}} = \int_0^1 \frac{\frac{1}{n} u^{\frac{1}{n}-1} du}{\sqrt{1-u}}$$

$$y^n = u \qquad du = n y^{n-1} dy$$

$$y = u^{\frac{1}{n}} \qquad = n\left(u^{\frac{1}{n}}\right)^{n-1} dy$$

$T(E) \longrightarrow U(x)$



$$dt = \frac{dx}{\sqrt{\frac{2}{m}(E-U)}}$$

$$= \frac{dx}{\cancel{dU}} \frac{\cancel{dU}}{\sqrt{\frac{2}{m}(E-U)}}$$

$$t_2 = \int_0^{t_2} dt = \int_0^{x_2} \frac{dx}{\sqrt{\quad ''}}$$

$$= \int_0^E \left(\frac{dx_2}{dU}\right) \frac{dU}{\sqrt{\quad}}$$

$$t_1 = \int_0^{t_1} dt = \int_{x_1}^0 \frac{dx}{\sqrt{\quad ''}}$$

$$= \int_E^0 \left(\frac{dx_2}{dU}\right) \frac{dU}{\sqrt{\quad ''}} \quad \frac{2}{m}(E-U)$$

$$T = 2(t_1 + t_2)$$

$$= \underbrace{\frac{2}{\sqrt{\frac{2}{m}}}}_{\sqrt{2m}} \left(\int_0^E \frac{dx_2}{dU} \frac{dU}{\sqrt{E-U}} + \underbrace{\int_E^0}_{-\int_0^E} \frac{dx_1}{dU} \frac{dU}{\sqrt{E-U}} \right)$$

$$= \sqrt{2m} \int_0^{\alpha} dU_{y} \int_{U}^{\alpha} dE_{x} \, \frac{\boxed{\left(\frac{dx_2}{dU} - \frac{dx_1}{dU}\right)}}{\sqrt{(\alpha - E)(E - U)}}$$

y

$$\int_U^{\alpha} dE \, \frac{1}{\sqrt{(\alpha - E)\underbrace{(E - U)}_{y}}}$$

$$A \equiv \alpha - U$$

$$= \int_0^{\alpha - U} \frac{dy}{\sqrt{y(A - y)}}$$

$E - U = y$

$E = y + U$

$$\frac{y}{A} \equiv z \qquad dy = A\,dz$$

$$= \int_0^1 \frac{dz}{\sqrt{z(1-z)}}$$

$$z = \sin^2\theta$$

$$dz = 2 \sin\theta \cos\theta \, d\theta$$

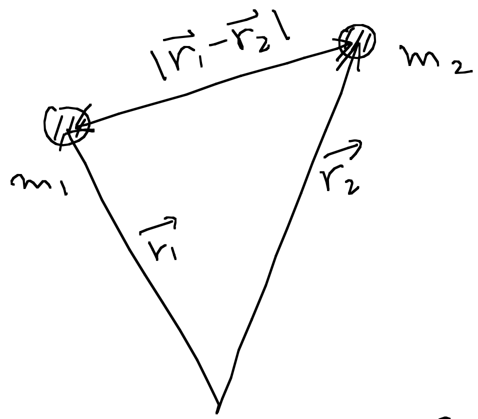
$$= \int_0^{\frac{\pi}{2}} \frac{2 \sin\theta \cancel{\cos\theta} \, d\theta}{\sqrt{\sin^2\theta \cancel{\cos^2\theta}}}$$

$$= 2 \int_0^{\frac{\pi}{2}} d\theta = \pi$$

$$\Rightarrow \boxed{\pi \sqrt{2m} \int_0^{\alpha} dU \left(\frac{dx_2}{dU} - \frac{dx_1}{dU}\right)}$$

§13. reduced mass

relative coordinate.



$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(|\vec{r}_1-\vec{r}_2|)$$

$$\vec{r}_1-\vec{r}_2 \equiv \vec{r}$$

origin = cm $\qquad \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} = 0$

$$m_1\vec{r}_1 + m_2\vec{r}_2 = 0 \longrightarrow \vec{r}_2 = -\frac{m_1}{m_2}\vec{r}_1$$

$$\vec{r}_1 + \frac{m_1}{m_2}\vec{r}_1 = \vec{r} \longrightarrow \vec{r}_1 = \frac{m_2}{m_1+m_2}\vec{r}$$

$$\therefore \vec{r}_2 = -\frac{m_1}{m_1+m_2}\vec{r}$$

$$\therefore \dot{\vec{r}}_1 = \frac{m_2}{m_1+m_2}\dot{\vec{r}} \qquad \dot{\vec{r}}_2 = -\frac{m_1}{m_1+m_2}\dot{\vec{r}}$$

$$\therefore L = \frac{1}{2}\left(m_1\left(\frac{m_2}{m_1+m_2}\right)^2 + m_2\left(\frac{m_1}{m_1+m_2}\right)^2\right)\dot{\vec{r}}^{\,2} - U(|\vec{r}|)$$

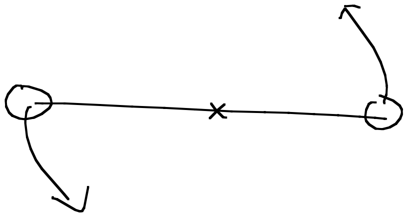
$$\frac{m_1m_2(m_1+m_2)}{(m_1+m_2)^2} = \frac{m_1m_2}{m_1+m_2} \equiv m$$

"reduced mass"

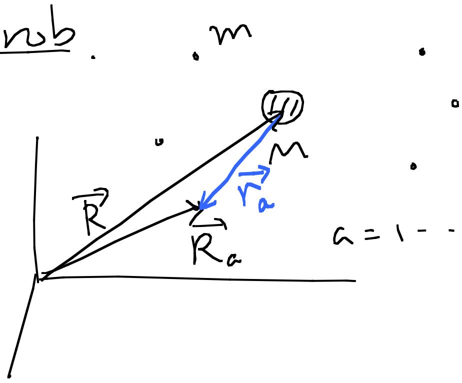
$$L = \frac{1}{2} m \dot{\vec{r}}^{\,2} - U(|\vec{r}|)$$

$m_1 \gg m_2 \qquad m \cong \dfrac{m_1m_2}{m_1} \approx m_2$

$m_1 = m_2 \qquad m = \dfrac{m_1}{2}$



Prob.



$a = 1 \cdots n$

$$\vec{r}_a = \vec{R}_a - \vec{R}$$

$$\vec{R}_a = \vec{r}_a + \vec{R}$$

Origin = CM

$$\frac{M\vec{R} + m\vec{R}_1 + \cdots + m\vec{R}_n}{M + m + \cdots + m} = 0$$

$$(M + nm)\vec{R} + m\sum_a \vec{r}_a = 0$$

$$\therefore \vec{R} = -\frac{m}{M+nm}\sum_a \vec{r}_a$$

$$L = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}m\sum_{a=1}^{n}\dot{\vec{R}}_a^2 - U$$

$$\dot{\vec{R}}_a = \dot{\vec{r}}_a + \dot{\vec{R}}$$

$$\dot{\vec{R}}_a^2 = \dot{\vec{r}}_a^2 + \dot{\vec{R}}^2 + 2\vec{r}_a \cdot \dot{\vec{R}}$$

$$\dot{\vec{R}} = -\frac{m}{M+nm}\sum_b \dot{\vec{r}}_b$$

$$L = \frac{1}{2}(M+nm)\dot{\vec{R}}^2 + \frac{1}{2}m\sum_a \dot{\vec{r}}_a^2$$

$$- \frac{1}{\cancel{2}}\frac{m^2}{M+nm}\sum_{a=1}^{n}\cancel{2}\,\dot{\vec{r}}_a \cdot \sum_b \dot{\vec{r}}_b - U$$

$$\left(\sum_a \dot{\vec r}_a\right)^2$$

$$= \frac{1}{2}\left[\cancel{(M+nm)}\,\frac{m^2}{(M+nm)^{\cancel{2}}} - 2\,\frac{m^2}{(M+nm)}\right]\left(\sum_b \dot{\vec r}_b\right)^2 + \frac{1}{2} m \sum_a \dot{\vec r}_a^{\,2} - U$$

$\dot{\vec r}_a \equiv \vec v_a$

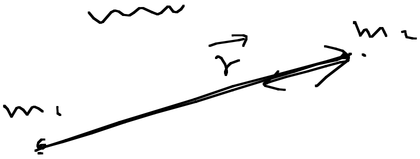
$$= \frac{1}{2} m \sum_a \vec v_a^{\,2} - \frac{1}{2}\,\frac{m^2}{(M+nm)}\left(\sum_a \vec v_a\right)^2 - U$$

§14. central field

$$U(\vec r) = U(|\vec r|) = U(r)$$

$U = U(r, \theta, \phi)$

$$\vec F = -\vec\nabla U = -\hat r \frac{\partial U}{\partial r}$$



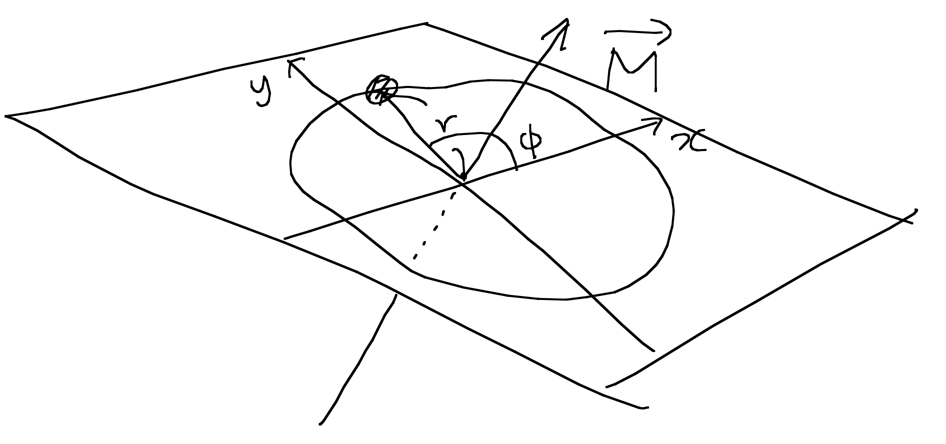
$$\vec M = \vec r \times \vec p$$

$$\frac{d\vec M}{dt} = \cancel{\vec v \times \underbrace{\vec p}_{m\vec v}}^{\,0} + \vec r \times \underbrace{\dot{\vec p}}_{m\vec a = \vec F} = \vec r \times \vec F = \vec\tau$$

$\vec\tau = \vec 0$

$\vec F \propto \vec r$

$\vec M$ = constant



$$x = r\cos\phi \qquad y = r\sin\phi$$

$$\dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi$$

$$\dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi$$

$$v^2 = \dot{r}^2 + r^2\dot{\phi}^2$$

$$\boxed{L = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r)}$$

6 coordinates $\longrightarrow$ 3 coodrinats

$\downarrow$

2 coordinats
(r, ϕ)

$$\frac{\partial L}{\partial \phi} = 0 \longrightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0 \longrightarrow \frac{\partial L}{\partial \dot{\phi}} = \text{const}$$

$$\longrightarrow m r^2 \dot{\phi} = M = \text{const}$$

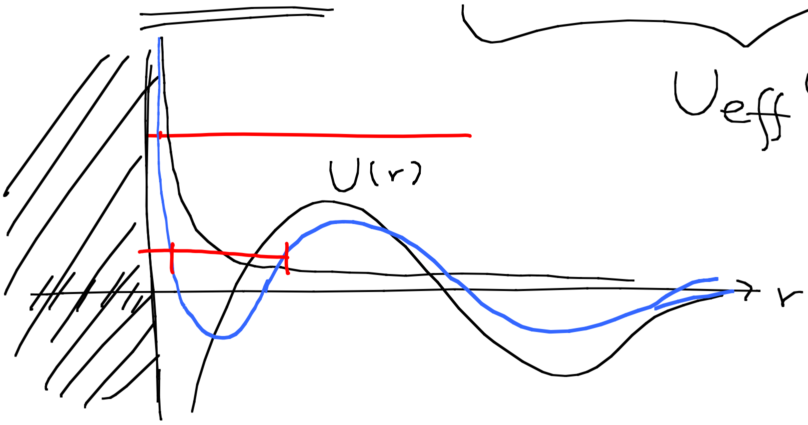
$$\dot{\phi} = \frac{M}{m r^2}$$

$$E = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\phi}^2\right) + U(r)$$

$$\frac{1}{2}\frac{M^2}{m}\frac{1}{r^2}$$

$$= \text{constant}$$

$$E = \underline{\underline{\frac{1}{2} m \dot{r}^2}} + \underbrace{\frac{1}{2} m r^2 \left(\frac{M}{m r^2}\right)^2 + U(r)}_{U_{eff}(r)}$$



$$\dot{r} = \frac{dr}{dt} \longrightarrow r = r(t)$$

$$\downarrow$$

$$\phi = \phi(t)$$

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}\left[E - \frac{1}{2}mr^2\left(\frac{M}{mr^2}\right)^2 - U(r)\right]}$$

$$\int \frac{dr}{\sqrt{\frac{2}{m}\left[E - \frac{1}{2}mr^2\left(\frac{M}{mr^2}\right)^2 - U(r)\right]}} = \int dt$$

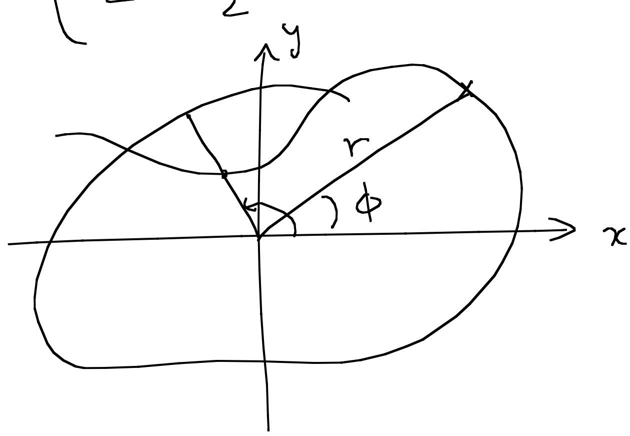
$$= t$$

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{M}{mr^2} = \frac{d\phi}{dr}\frac{dr}{dt} = \frac{M}{mr^2}$$

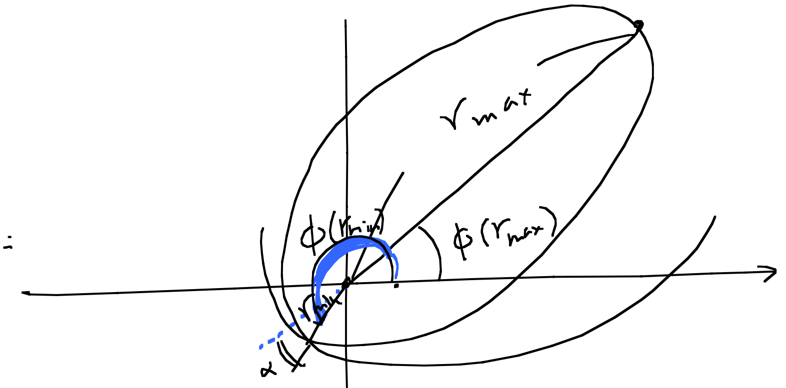
$$\frac{d\phi}{dr} = \frac{M}{mr^2\left(\frac{dr}{dt}\right)}$$

$$= \frac{M}{mr^2\sqrt{\frac{2}{m}\left[E - \frac{1}{2}mr^2\left(\frac{M}{mr^2}\right)^2 - U(r)\right]}}$$

$$\phi = \int \frac{M \quad dr/r^2}{\sqrt{2m\left[E - \frac{1}{2}mr^2\left(\frac{M}{mr^2}\right)^2 - U(r)\right]}}$$



$$\phi = f(r) \longrightarrow r = f^{-1}(\phi)$$



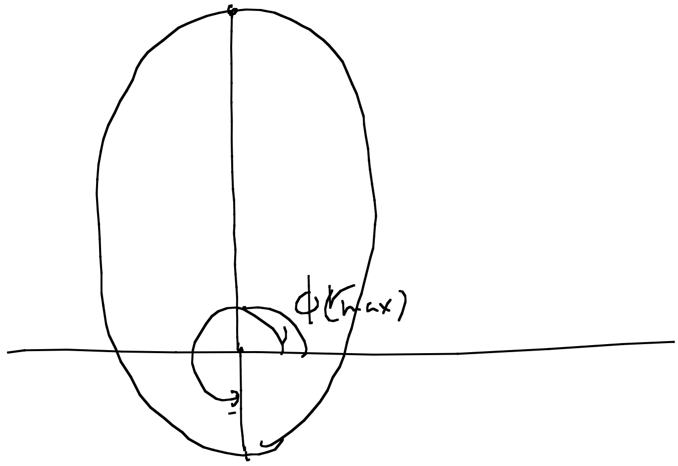
$$-(\pi + \alpha)$$

$$\phi(r_{max}) - \phi(r_{min}) =$$

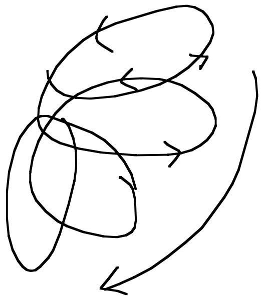
$$= \int_{r_0}^{r_{max}} \frac{M \quad dr/r^2}{\sqrt{2m\left[E - \frac{1}{2} m r^2 \left(\frac{M}{mr^2}\right)^2 - U(r)\right]}} + \cancel{C}$$

$$- \left(\int_{r_0}^{r_{min}} \frac{M \quad dr/r^2}{\sqrt{2m\left[E - \frac{1}{2} m r^2 \left(\frac{M}{mr^2}\right)^2 - U(r)\right]}} + \cancel{C} \right)$$

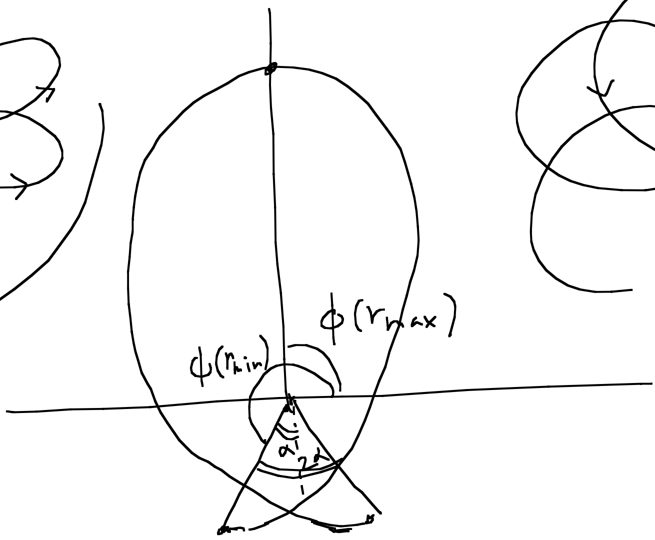
$$= \int_{r_{min}}^{r_{max}} \frac{M \quad dr/r^2}{\sqrt{2m\left[E - \frac{1}{2} m r^2 \left(\frac{M}{mr^2}\right)^2 - U(r)\right]}}$$

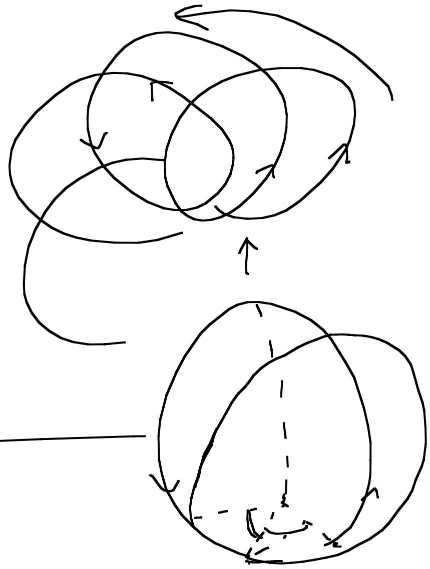


$$\phi(r_{max}) - \phi(r_{min}) = -\pi$$



$\Delta\phi > 0$





$\Delta\phi < 0$

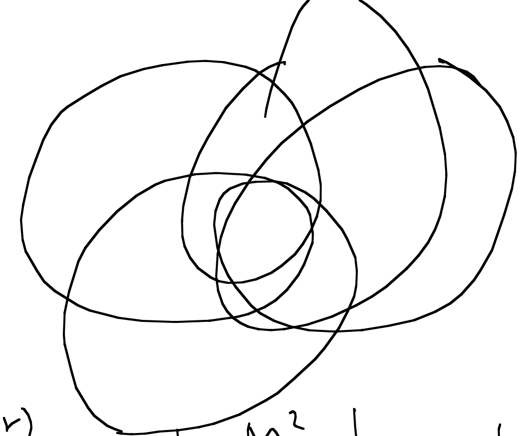
$$2\left(\phi(r_{max}) - \phi(r_{min})\right) = \underline{\underline{-2\pi + 2\alpha}} \equiv 2\alpha$$

deficit angle
$\shortparallel$
$\Delta\phi$

$$\Delta\phi = \int_{r_{min}}^{r_{max}} \frac{2M \quad dr/r^2}{\sqrt{2m\left[E - \frac{1}{2} m r^2 \left(\frac{M}{mr^2}\right)^2 - U(r)\right]}}$$

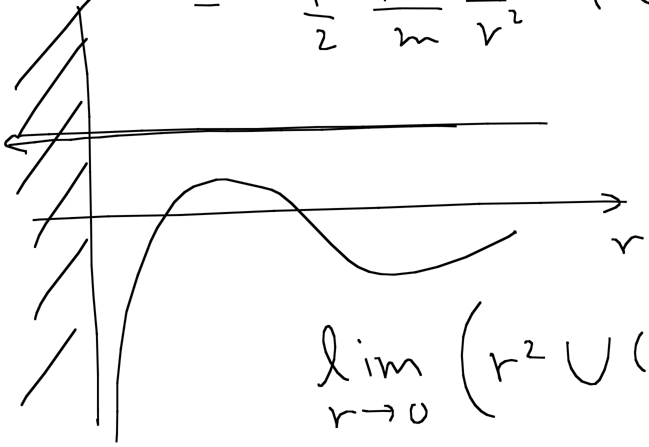
$$\Delta\phi = 2\pi \frac{n}{m} \longrightarrow m\Delta\phi = 2\pi n = 0$$

$$\neq 2\pi \frac{n}{m} \longrightarrow \text{nonclosed orbit}$$



$$U_{eff}(r) = \frac{1}{2}\frac{M^2}{m}\frac{1}{r^2} + U(r)$$

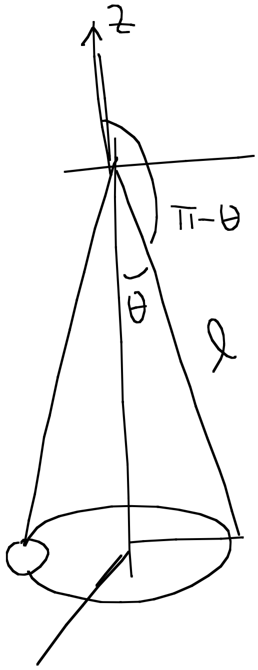
$r \to 0$

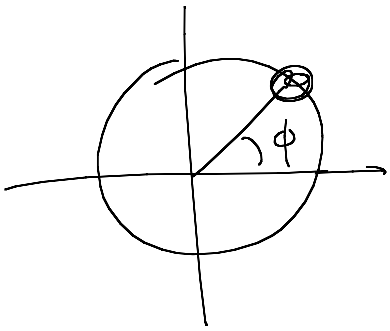


$$\lim_{r\to 0}\left(r^2 U(r)\right) < -\frac{M^2}{2m}$$

$$\frac{1}{2} m \dot{r}^2 = E - U_{eff}(r) > 0$$

Prob.





$$z = l\cos(\pi-\theta) = -l\cos\theta$$

$$x = l\sin(\pi-\theta)\cos\phi = l\sin\theta\cos\phi$$

$$y = \quad " \quad \sin\phi = l\sin\theta\sin\phi$$

$$\dot{z} = l\dot{\theta}\sin\theta$$

$$\dot{x} = l\dot{\theta}\cos\theta\cos\phi - l\dot{\phi}\sin\theta\sin\phi$$

$$\ddot{y} = l\ddot{\theta}\cos\theta\sin\phi + l\ddot{\phi}\sin\theta\cos\phi$$

$$L = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + mgl\cos\theta$$

$$mgl\cos\theta = -mgz$$

$$\frac{\partial L}{\partial \phi} = 0 \longrightarrow \frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi}\sin^2\theta \equiv M_z$$

$$\dot{\phi} = \frac{M_z}{ml^2\sin^2\theta}$$

$$E = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) - mgl\cos\theta$$

$$= \frac{1}{2}ml^2\dot{\theta}^2 + \underbrace{\frac{M_z^2}{2ml^2}\frac{1}{\sin^2\theta} - mgl\cos\theta}_{U_{eff}(\theta)}$$

$$\frac{d\theta}{dt} = \sqrt{\frac{2}{ml^2}\left(E - U_{eff}(\theta)\right)}$$

$$\dot{\phi} = \frac{M_z}{ml^2\sin^2\theta} = \frac{d\phi}{dt} = \frac{d\phi}{d\theta}\frac{d\theta}{dt}$$

$$\frac{d\phi}{d\theta} = \frac{M_z}{\sin^2\theta\sqrt{2ml^2\left(E - V_{eff}(\theta)\right)}}$$

$$\phi = \int_0^\theta \frac{M_z}{\sqrt{2ml^2}}\frac{1}{\sin^2\theta\sqrt{E - V_{eff}(\theta)}}\,d\theta$$

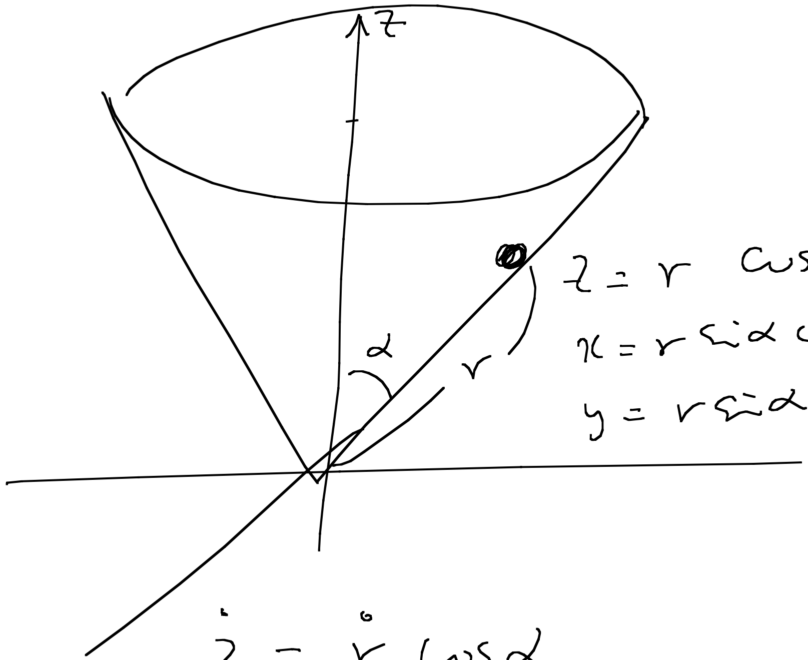
$$\frac{M_z^2}{2ml^2}\frac{1}{\sin^2\theta} - mgl\cos\theta$$

$$\int \frac{d\theta}{\sqrt{-\frac{M_z^2}{2ml^2}\sin^2\theta + E\sin^4\theta - mgl\cos\theta\sin^4\theta}}$$

$$\cos\theta = x \qquad dx = -\sin\theta\,d\theta$$

$$= \int \frac{dx}{\sqrt{E(1-x^2)^3 - \frac{M_z^2}{2ml^2}(1-x^2)^2 - mgl\,x(1-x^2)^3}}$$

<u>Prob 2</u>.



$$z = r\cos\alpha$$
$$x = r\sin\alpha\cos\phi$$
$$y = r\sin\alpha\sin\phi$$

$$\dot{z} = \dot{r}\cos\alpha$$
$$\dot{x} = \dot{r}\sin\alpha\cos\phi - r\dot{\phi}\sin\alpha\sin\phi$$
$$\dot{y} = \dot{r}\sin\alpha\sin\phi + r\dot{\phi}\sin\alpha\cos\phi$$

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\sin^2\alpha\right) - mgr\cos\alpha$$

$$\frac{\partial L}{\partial\dot{\phi}} = M_z = mr^2\dot{\phi}\sin^2\alpha$$
$$\rightarrow \dot{\phi} = \frac{M_z}{mr^2\sin^2\alpha}$$

$$E = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\sin^2\alpha\right) + mgr\cos\alpha$$

$$= \frac{1}{2}m\dot{r}^2 + \underbrace{\frac{M_z^2}{2m\sin^2\alpha}\frac{1}{r^2} + mgr\cos\alpha}_{U_{eff}(r)}$$

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}$$

$$\int \frac{dr}{\sqrt{\frac{2}{m}(E - U_{eff}(r))}} = \int dt = t$$

$$\dot{\phi} = \frac{M_z}{mr^2 \sin^2\alpha} = \frac{d\phi}{dt} = \frac{d\phi}{dr}\underbrace{\frac{dr}{dt}}_{\sqrt{\quad}}$$

$$\frac{d\phi}{dr} = \frac{M_z}{mr^2 \sin^2\alpha \sqrt{\quad}}$$

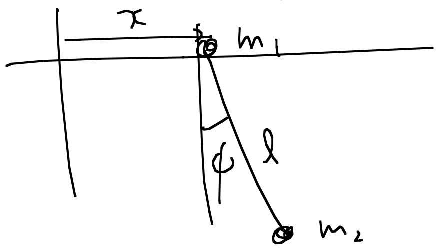
$$\therefore \phi = \int \frac{M_z}{\sqrt{2m}\, \sin^2\alpha\, r^2 \sqrt{(E - U_{eff}(r))}} \, dr$$

$$\int \frac{1}{r^2 \sqrt{E - \frac{a}{r^2} - br}} \, dr$$

$$= \int^r \frac{1}{\sqrt{Er^4 - ar^2 - br^5}} \, dr$$

Prob3.

$$L = \tfrac{1}{2}(m_1 + m_2)\dot{x}^2 + \tfrac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) + m_2 g l \cos\phi.$$



$$\frac{\partial L}{\partial x} = 0 \longrightarrow \frac{\partial L}{\partial \dot{x}} = \text{const} = P_x$$

$$\parallel$$

$$(m_1 + m_2)\dot{x} + m_2 l \dot{\phi} \cos\phi$$

$$\dot{x} = \frac{P_x - m_2 l \dot{\phi} \cos\phi}{m_1 + m_2}$$

$$E = \tfrac{1}{2}(m_1+m_2)\dot{x}^2 + \tfrac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) - m_2 g l\cos\phi.$$

$$E = \frac{1}{2}(m_1+m_2)\left(\frac{P_x - m_2 l\dot{\phi}\cos\phi}{m_1+m_2}\right)^2$$

$$+ \frac{1}{2}m_2 l^2\dot{\phi}^2$$

$$+ m_2 l\dot{\phi}\cos\phi\left(\frac{P_x - m_2 l\dot{\phi}\cos\phi}{m_1+m_2}\right)$$

$$- m_2 g l\cos\phi$$

$$= \frac{1}{2}m_2 l^2\dot{\phi}^2\left(1 - 2\frac{m_2}{m_1+m_2}\cos^2\phi + \frac{m_2}{m_1+m_2}\cos^2\phi\right)$$

$$- m_2 g l\cos\phi + C$$

$$E - C = \tilde{E} = \frac{1}{2}m_2 l^2\dot{\phi}^2\left(1 - \frac{m_2}{m_1+m_2}\cos^2\phi\right)$$

$$- m_2 g l\cos\phi$$

$$\frac{1}{2}m_2 l^2\dot{\phi}^2 = \frac{E + m_2 g l\cos\phi}{1 - \frac{m_2}{m_1+m_2}\cos^2\phi}$$

$$\frac{d\phi}{dt} = \dot{\phi} = \sqrt{\frac{2}{m_2 l^2}\,\frac{E + m_2 g l\cos\phi}{1 - \frac{m_2}{m_1+m_2}\cos^2\phi}}$$

$$\oint dt = t = \int \frac{d\phi}{\sqrt{}}$$

$$= l\sqrt{\frac{m_2}{2(m_1+m_2)}} \int \sqrt{\frac{m_1+m_2\sin^2\phi}{E+m_2 g l\cos\phi}}\, d\phi$$

§15. Kepler's problem

$$U(r) = -\frac{\alpha}{r}$$

$$\phi = \int \frac{M\, dr/r^2}{\sqrt{2m\left(E - \frac{M^2}{2m}\frac{1}{r^2} + \frac{\alpha}{r}\right]}}$$

$$\frac{1}{r} \equiv u \qquad du = -\frac{1}{r^2}dr$$

$$= \frac{M}{\sqrt{2m}} \int \frac{-du}{\sqrt{E - \frac{M^2}{2m}u^2 + \alpha u}}$$

$$E - \frac{M^2}{2m}u^2 + \alpha u$$

$$= E - \frac{M^2}{2m}\left(\underbrace{u^2 - \frac{2m\alpha}{M^2}u + \left(\frac{m\alpha}{M^2}\right)^2}_{\left(u - \frac{m\alpha}{M^2}\right)^2} - \left(\frac{m\alpha}{M^2}\right)^2\right]$$

$$= \frac{\bar{E} + \frac{m\alpha^2}{2M^2} - \frac{M^2}{2m}\left(u - \frac{m\alpha}{M^2}\right)^2}{\sqrt{\bar{E} + \frac{m\alpha^2}{2M^2} - \frac{M^2}{2m}\left(u - \frac{m\alpha}{M^2}\right)^2}}$$

$$= \sqrt{\bar{E} + \frac{m\alpha^2}{2M^2}} \sqrt{1 - \frac{M^2}{2m} \frac{\left(u - \frac{m\alpha}{M^2}\right)^2}{\bar{E} + \frac{m\alpha^2}{2M^2}}}$$

$$= \sqrt{\bar{E} + \frac{m\alpha^2}{2M^2}} \sqrt{1 - v^2}$$

$$v \equiv \frac{\mu^2}{2m\left(E + \frac{m\alpha^2}{2\mu^2}\right)} \left(u - \frac{m\alpha}{\mu^2}\right)$$

$$dv \equiv \frac{\mu^2}{2m\left(E + \frac{m\alpha^2}{2\mu^2}\right)} du$$

§15. Kepler's problem

$$U(r) = -\frac{\alpha}{r}$$

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}\left[E - \frac{1}{2}mr^2\left(\frac{M}{mr^2}\right)^2 + \frac{\alpha}{r}\right]}$$

$$\phi = \int \frac{M \quad dr/r^2}{\sqrt{2m\left[E - \frac{1}{2}mr^2\left(\frac{M}{mr^2}\right)^2 + \frac{\alpha}{r}\right]}}$$

$$\frac{1}{r} \equiv u \qquad du = -\frac{dr}{r^2}$$

$$\phi = \int \frac{(-M)\, du}{\sqrt{2m}\sqrt{E - \frac{M^2}{2m}u^2 + \alpha u}}$$

$$E - \frac{M^2}{2m}u^2 + \alpha u = E - \frac{M^2}{2m}\Big(\underbrace{u^2 - \frac{2m\alpha}{M^2}u}_{\left(u - \frac{m\alpha}{M^2}\right)^2 - \frac{m^2\alpha^2}{M^4}}\Big)$$

$$\sqrt{} = \sqrt{E + \frac{m\alpha^2}{2M^2} - \frac{M^2}{2m}\left(u - \frac{m\alpha}{M^2}\right)^2}$$

$$= \sqrt{E + \frac{m\alpha^2}{2M^2}}\sqrt{1 - \underbrace{\frac{M^2/2m}{E + \frac{m\alpha^2}{2M^2}}\left(u - \frac{m\alpha}{M^2}\right)^2}_{\equiv v^2}}$$

$$v = \sqrt{\frac{M^2/2m}{E + \frac{m\alpha^2}{2M^2}}}\left(u - \frac{m\alpha}{M^2}\right)$$

$$dv \equiv \sqrt{\frac{M^2}{2m\left(E+\frac{m\alpha^2}{2M^2}\right)}}\, du$$

$$\phi = -\frac{\cancel{M}}{\cancel{\sqrt{2m}}} \frac{1}{\cancel{\sqrt{E+\frac{m\alpha^2}{2M^2}}}} \int \frac{\sqrt{\cancel{2m\left(E+\frac{m\alpha^2}{2M^2}\right)}/\cancel{M^2}}}{\sqrt{1-v^2}}\, dv$$

$$= -\int \frac{dv}{\sqrt{1-v^2}} \qquad \underline{v \equiv \sin\theta}$$

$$dv = \cos\theta\, d\theta$$

$$\sqrt{1-v^2} = \cos\theta$$

$$\phi = -\int \frac{\cancel{\cos}\theta\, d\theta}{\cancel{\cos}\theta} = -\theta + \phi_0$$

$$\phi - \phi_0 = -\theta = -\sin^{-1} v$$

$$v = \sqrt{\frac{M^2/2m}{E+\frac{m\alpha^2}{2M^2}}}\left(\frac{1}{r}-\frac{m\alpha}{M^2}\right)$$

$$v = -\sin(\phi-\phi_0)$$

$$\frac{1}{r} = \frac{m\alpha}{M^2} - \sqrt{\frac{E+\frac{m\alpha^2}{2M^2}}{M^2/2m}}\sin(\phi-\phi_0)$$

$$\frac{1}{r} = \frac{m\alpha}{M^2} + \sqrt{\frac{E+\frac{m\alpha^2}{2M^2}}{M^2/2m}}\cos\left(\phi-\phi_0+\frac{\pi}{2}\right)$$

$$\phi_0 = \frac{\pi}{2}, \qquad \frac{M^2}{m\alpha} = p$$

$$= \frac{1}{p}\left(1 + p\sqrt{\frac{E+\frac{m\alpha^2}{2M^2}}{M^2/2m}}\cos\phi\right)$$

$$\overbrace{\qquad\qquad}^{}$$
$$e$$

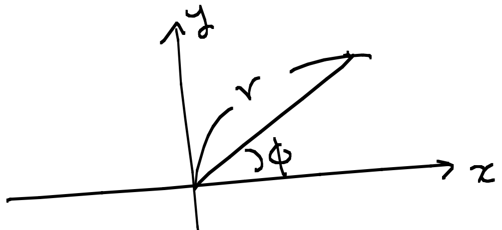
$$e = \frac{M^2}{m\alpha}\sqrt{\frac{E+\frac{m\alpha^2}{2M^2}}{M^2/2m}} = \sqrt{\frac{\frac{EM^4}{m^2\alpha^2}+\frac{M^2}{2m}}{M^2/2m}}$$

$$\equiv \sqrt{1+\frac{2EM^2}{m\alpha^2}}$$

$$\frac{1}{r} = \frac{1}{p}\left(1+e\cos\phi\right)$$

$$\boxed{\frac{p}{r} = 1 + e\cos\phi} \quad (p = r + ex) \quad \rightarrow x = \frac{p-r}{e}$$

$2p$: latus rectum, e: eccentricity



$$x = r\cos\phi$$
$$y = r\sin\phi$$

$$p = r + e\,r\cos\phi$$
$$= \sqrt{x^2+y^2} + ex$$
$$\left(\sqrt{x^2+y^2}\right)^2 = \left(p - ex\right)^2$$
$$x^2+y^2 = p^2 - 2pex + e^2x^2$$
$$\underbrace{(1-e^2)x^2 + 2pex}_{(1-e^2)\left[x^2 + 2\frac{pe}{1-e^2}x + \frac{p^2e^2}{(1-e^2)^2} - \frac{p^2e^2}{(1-e^2)^2}\right]} + y^2 = p^2$$

$$\underbrace{\left(x+\frac{pe}{1-e^2}\right)^2}$$

$$(1-e^2)\left(x+\frac{pe}{1-e^2}\right)^2+y^2=p^2\underbrace{\left(1+\frac{e^2}{1-e^2}\right)}_{\frac{1}{1-e^2}}$$

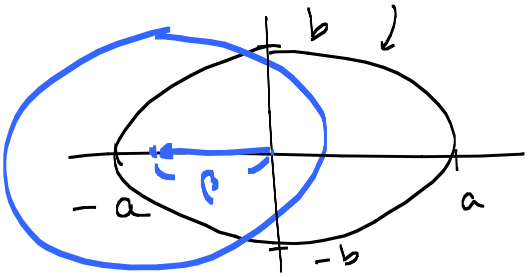
$$\boxed{\frac{\left(x+\frac{pe}{1-e^2}\right)^2}{\frac{p^2}{(1-e^2)^2}}+\frac{y^2}{\frac{p^2}{1-e^2}}=1}$$

(1st law)

① $e<1$; $\quad e=\sqrt{1+\frac{2EM^2}{m\alpha^2}}<1$

$\Rightarrow$ if $E<0$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad x\to x+\beta$$

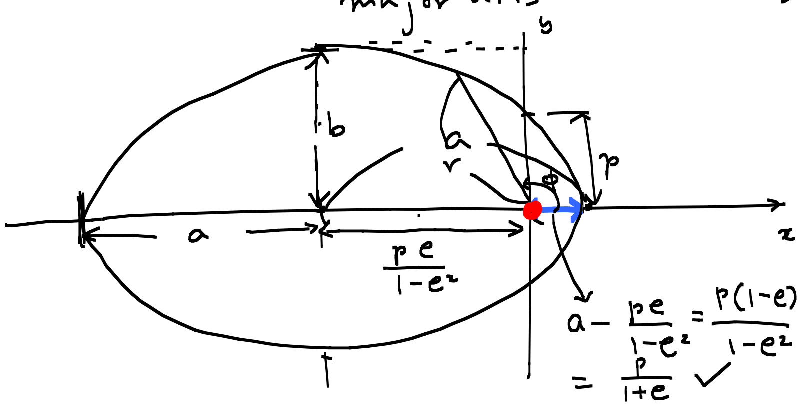


$$\frac{(x+\beta)^2}{a^2}+\frac{y^2}{b^2}=1$$

$(a>b)$

$$\beta:\ \frac{pe}{1-e^2} \qquad a=\frac{p}{1-e^2},\quad b=\frac{p}{\sqrt{1-e^2}}$$

major axis $\quad$ minor axis



$$a-\frac{pe}{1-e^2}=\frac{p(1-e)}{1-e^2}$$
$$=\frac{p}{1+e}\ ✓$$

$$\frac{p}{r} = 1 + e\cos\phi$$

$\underline{\underline{\phi = 0}}$ $\quad r_{min} = \dfrac{p}{1+e}$

perihelion :

$$\begin{cases} p = \dfrac{M^2}{m\alpha} \\ e = \sqrt{1 + \dfrac{2EM^2}{m\alpha^2}} \end{cases}$$

$$1 - e^2 = -\frac{2EM^2}{m\alpha^2} = \frac{2|E|M^2}{m\alpha^2}$$

$$a = \frac{p}{1-e^2} = \frac{m\alpha^2}{2|E|M^2}\,\frac{M^2}{m\alpha} = \frac{\alpha}{2|E|}$$

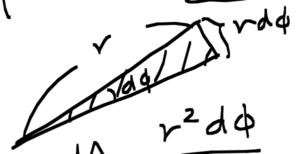
$$b = \frac{p}{\sqrt{1-e^2}} = \sqrt{\frac{m\alpha^2}{2|E|M^2}}\,\frac{M^2}{m\alpha} = \sqrt{\frac{M^2}{2m|E|}}$$

$$\frac{p}{r_{max}} = 1 - e \rightarrow r_{max} = \frac{p}{1-e} = a(1+e)$$

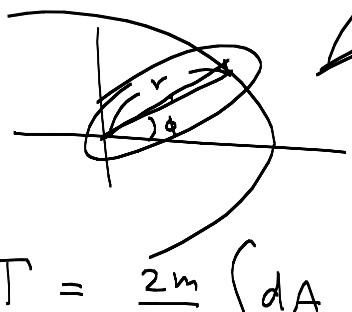
$$r_{min} = \frac{p}{1+e} = a(1-e)$$

$$M = \dot{\phi}\, m r^2 \Rightarrow \frac{d\phi}{dt} = \frac{M}{mr^2}$$

$$dt = \frac{m r^2}{M}\, d\phi = \frac{m}{M}\, r^2 d\phi = \frac{2m}{M} dA$$



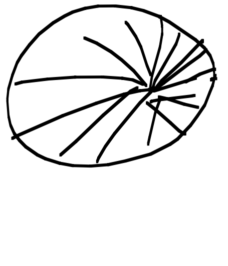
$$dA = \frac{r^2 d\phi}{2}$$



$$\frac{dA}{dt} = \frac{M}{2m}$$

(2nd law)

$$\int_0^T dt = T = \frac{2m}{M}\int dA = \frac{2m}{M}\pi a b$$



$$a = \frac{p}{1-e^2} = \frac{m\alpha^2}{2|E|M^2}\,\frac{M^2}{m\alpha} = \frac{\alpha}{2|E|}$$

$$b = \frac{p}{\sqrt{1-e^2}} = \sqrt{\frac{m\alpha^2}{2|E|M^2}}\,\frac{M^2}{m\alpha} = \sqrt{\frac{M^2}{2m|E|}}$$

$$T = \frac{2m}{\cancel{M}}\,\pi\,\frac{\alpha}{2|E|}\,\frac{\cancel{M}}{\sqrt{2m|E|}}$$

$$= |E|^{-\frac{3}{2}}\,\frac{\sqrt{m}\,\alpha\,\pi}{\sqrt{2}}$$

$$T^2 = \frac{m\alpha^2\pi^2}{2}\,\frac{1}{|E|^3} \qquad |E| = \frac{\alpha}{2a}$$

$$= \frac{m\alpha^2\pi^2}{2}\left(\frac{2a}{\alpha}\right)^3 = \underbrace{\frac{8m\pi^2}{2\alpha}}\,a^3$$

(Kepler's 3rd law)

② $e > 1$ $(E > 0)$

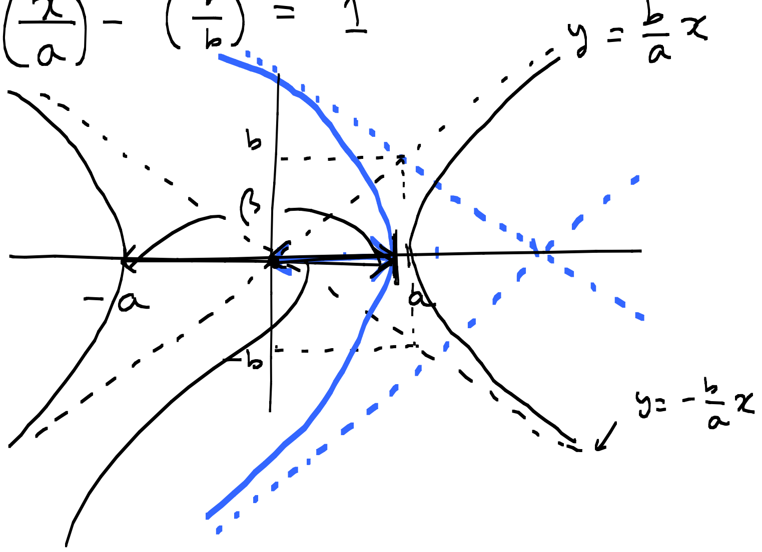
$$\frac{\left(x + \frac{pe}{1-e^2}\right)^2}{\frac{p^2}{(1-e^2)^2}} + \frac{y^2}{\frac{p^2}{1-e^2}} = 1$$

$$\frac{(x-\beta)^2}{a^2} - \frac{y^2}{b^2} = 1$$

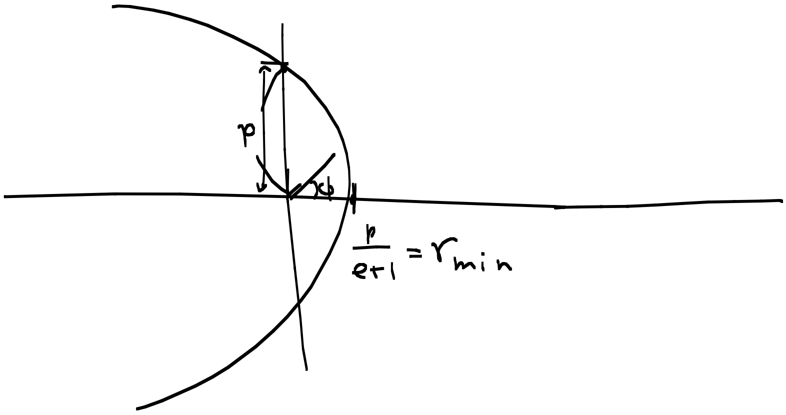
$$a = \frac{p}{e^2-1}, \quad \beta = \frac{pe}{e^2-1}, \quad b = \frac{p}{\sqrt{e^2-1}}$$

$\longrightarrow \beta = ae > a$

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$



$$\beta - a = ae - a = a(e-1) = \frac{p}{e+1}$$



$$\frac{p}{r} = 1 + e\cos\phi \qquad (e > 1)$$

$$r_{min} = \frac{p}{e+1} \quad ✓$$

$$\phi = \frac{\pi}{2} \longrightarrow r = p$$

---

③ $e = 1$

$\downarrow$

$E = 0$

$$\frac{p}{r} = 1 + \cos\phi$$

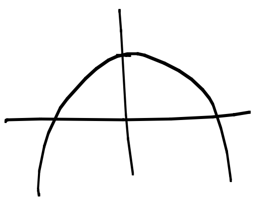
$$p = r + x = \sqrt{x^2+y^2} + x$$

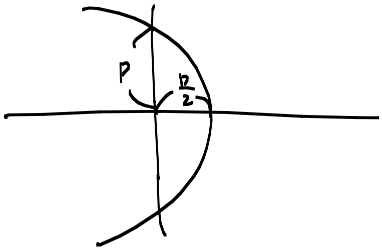
$$p - x = \sqrt{x^2+y^2}$$

$$p^2 + \cancel{x^2} - 2px = \cancel{x^2} + y^2$$

$$x = \frac{p^2 - y^2}{2p} = \frac{p}{2} - \frac{1}{2p}y^2$$

$\underline{y = a - bx^2} \longrightarrow x = a - by^2$





$$r_{min} = \frac{p}{2}$$

dynamics

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}\left[E - \frac{1}{2} m r^2 \left(\frac{M}{mr^2}\right)^2 + \frac{\alpha}{r}\right]}$$

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left[E - \frac{M^2}{2mr^2} + \frac{\alpha}{r}\right]}}$$

$$dt = \frac{r\,dr}{\sqrt{\frac{2}{m}\left[r^2 E - \frac{M^2}{2m} + \alpha r\right]}}$$

$$dt = \frac{r\,dr}{\sqrt{\frac{2E}{m}\left[\underbrace{r^2 - \frac{M^2}{2mE} + \frac{\alpha}{E} r}_{\left(r+\frac{\alpha}{2E}\right)^2 \underbrace{- \frac{M^2}{2mE} - \frac{\alpha^2}{4E^2}}_{-(ae)^2}}\right]}}$$

$$e = \sqrt{1 + \frac{2EM^2}{m\alpha^2}}, \quad a = \frac{\alpha}{2|E|}$$

$$ae = \sqrt{\frac{\alpha^2}{4E^2} + \underbrace{\frac{2EM^2}{m\cancel{\alpha^2}} \frac{\cancel{\alpha^2}}{4E^2}}_{\frac{M^2}{2mE}}}$$

$$dt = \frac{r\,dr}{\sqrt{\frac{2E}{m}\left(\left(r+\frac{\alpha}{2E}\right)^2 - (ae)^2\right)}}$$

① $E<0 \quad E = -|E|$

$$dt = \frac{r\,dr}{\sqrt{\frac{\alpha/a}{m}}\sqrt{(ae)^2-(r-a)^2}}$$

$$\int dt = \sqrt{\frac{ma}{\alpha}}\int \frac{r\,dr}{\sqrt{(ae)^2-(r-a)^2}}$$

$$r-a \equiv -ae\cos\xi$$

$$\downarrow$$

$$dr = ae\sin\xi\, d\xi$$

$$t = \sqrt{\frac{ma}{\alpha}}\int \frac{(a-ae\cos\xi)\,ae\cancel{\sin\xi}\,d\xi}{ae\ \cancel{\sin\xi}}$$

$$t = \sqrt{\frac{ma}{\alpha}}\int (a-ae\cos\xi)\,d\xi$$

$$= \sqrt{\frac{ma}{\alpha}}\,(a\xi - ae\sin\xi)$$

$$\boxed{t = \sqrt{\frac{ma^3}{\alpha}}\,(\xi - e\sin\xi)}$$

$$r = a(1-e\cos\xi)$$

$$\frac{r}{a} = 1-e\cos\xi \quad\Big|\quad \cos\xi = \frac{1}{e}\left(1-\frac{r}{a}\right)$$

$$\xi = \cos^{-1}\left(\frac{1}{e}\left(1-\frac{r}{a}\right)\right)$$

$$\frac{p-r}{e} = x = \frac{a(\cancel{1}-e^2) - a(\cancel{1}-e\cos\xi)}{e}$$

$$x = a\cos\xi - ae = a(\cos\xi - e)$$

$$x^2+y^2=r^2$$

$$y = \sqrt{r^2-x^2}$$

$$r = a(1-e\cos\xi) \to r^2 = a^2(1-2\cancel{e}\cos\xi + e^2\cos^2\xi)$$

$$x^2 = a^2(\cos^2\xi + e^2 - 2e\cancel{\cos\xi})$$

$$y^2 = r^2 - x^2 = a^2\left(1-\cos^2\xi + e^2\cos^2\xi - e^2\right)$$

$$= a^2(\sin^2\xi - e^2\sin^2\xi) = a(1-e^2)\sin^2\xi$$

② $E>0 \quad (e>1)$

$E = |E|$

$$dt = \frac{r\,dr}{\sqrt{\frac{\alpha/a}{m}}\sqrt{-(ae)^2+(r+a)^2}}$$

$$\int dt = \sqrt{\frac{ma}{\alpha}}\int\frac{r\,dr}{\sqrt{-(ae)^2+(r+a)^2}}$$

$r+a \equiv ae\cosh\xi$

$\downarrow$

$dr = ae\sinh\xi\, d\xi$

$$t = \sqrt{\frac{ma}{\alpha}}\int\frac{(ae\cosh\xi - a)\,ae\,\cancel{\sinh\xi}\,d\xi}{\sqrt{(ae)^2\left[\cosh^2\xi - 1\right]}\ \cancel{ae\sinh\xi}}$$

$$t = \sqrt{\frac{ma}{\alpha}}\int(-a + ae\cosh\xi)\,d\xi$$

$$= \sqrt{\frac{ma}{\alpha}}\,(-a\xi + ae\sinh\xi)$$

$$\boxed{t = \sqrt{\frac{ma^3}{\alpha}}\,(e\sinh\xi - \xi)}$$

$$r = a(e\cosh\xi - 1)$$

$U = \frac{\alpha}{r}$, $\frac{d\vec{M}}{dt} = 0$

$\vec{K} = \vec{v}\times\vec{M} + \alpha\frac{\vec{r}}{r} \longrightarrow \frac{d\vec{K}}{dt} = 0$

$$\dot{\vec{K}} = \dot{\vec{v}} \times \vec{M} + \alpha\left(\frac{\dot{\vec{r}}}{r} + \vec{r}\,\underbrace{\frac{d}{dt}\left(\frac{1}{r}\right)}_{= -\frac{\dot{r}}{r^2}}\right)$$

$$r = |\vec{r}| = \sqrt{\vec{r}\cdot\vec{r}}$$

$$\dot{r} = \frac{d}{dt}(\vec{r}\cdot\vec{r})^{\frac{1}{2}} = \frac{1}{2}(\vec{r}\cdot\vec{r})^{-\frac{1}{2}}\, 2(\dot{\vec{r}}\cdot\vec{r})$$

$$= \frac{\dot{\vec{r}}\cdot\vec{r}}{r}$$

$$\dot{\vec{K}} = \dot{\vec{v}} \times \underbrace{\vec{M}}_{= \vec{r}\times(m\vec{v})} + \alpha\left(\frac{\dot{\vec{r}}}{r} - \frac{\dot{\vec{r}}\cdot\vec{r}}{r^3}\vec{r}\right)$$

$$\dot{\vec{v}} \times \vec{M} = m\,(\dot{\vec{v}} \times (\vec{r}\times\vec{v}))$$

$$= m\left(\vec{r}\,(\dot{\vec{v}}\cdot\vec{v}) - \vec{v}\,(\dot{\vec{v}}\cdot\vec{r})\right)$$

$$\dot{\vec{K}} = m\left(\underbrace{\vec{r}\,(\dot{\vec{v}}\cdot\vec{v})} - \vec{v}\,(\dot{\vec{v}}\cdot\vec{r})\right) + \alpha\left(\frac{\overset{=\vec{v}}{\dot{\vec{r}}}}{r} - \underbrace{\frac{\overset{=\vec{v}}{\dot{\vec{r}}}\cdot\vec{r}}{r^3}\vec{r}}\right)$$

$$m\ddot{\vec{r}} = m\dot{\vec{v}} = -\vec{\nabla}U \quad \left(U = +\frac{\alpha}{r}\right)$$

$$= -\alpha\,\underbrace{\vec{\nabla}\left(\frac{1}{r}\right)}_{-\frac{1}{r^2}\hat{r}} = \frac{\alpha\vec{r}}{r^3}$$

$$\hat{r} = \frac{\vec{r}}{r}$$

$$m\dot{\vec{v}} = \frac{\alpha\vec{r}}{r^3}$$

$$\dot{\vec{K}} = \vec{r}\left[\vec{v}\cdot\overbrace{\left(m\dot{\vec{v}} - \alpha\frac{\vec{r}}{r^3}\right)}^{=0}\right]$$

$$+\,\vec{v}\left(\underbrace{\frac{\alpha}{r}} - m\,\dot{\vec{v}}\cdot\vec{r}\right)$$

$$\frac{\alpha r^2}{r^3} = \frac{\alpha\,\vec{r}\cdot\vec{r}}{r^3}$$

$$\vec{v}\left[\vec{r}\cdot\left(\frac{\alpha\vec{r}}{r^3} - m\dot{\vec{v}}\right)\right]$$

$$\therefore \boxed{\vec{K} = \text{constant}}$$

Prob1: $E = 0$, $U = -\frac{\alpha}{r}$

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left[\quad - \frac{M^2}{2mr^2} + \frac{\alpha}{r} \right]}}$$

$$r \equiv \frac{M^2}{2m\alpha}(1+\eta^2) \rightarrow dr = \frac{M^2}{2m\alpha} 2\eta\, d\eta$$

$$dt = \sqrt{\frac{m}{2}} \cdot r \frac{dr}{\sqrt{\alpha r - \frac{M^2}{2m}}}$$

$$= \sqrt{\frac{m}{2}} \cdot \frac{M^2}{2m\alpha}(1+\eta^2) \frac{\frac{M^2}{2m\alpha} 2\eta\, d\eta}{\sqrt{\frac{M^2}{2m}(1+\eta^2-1)}}$$

$$= \sqrt{\frac{m}{2}}\sqrt{\frac{2m}{M^2}}\left(\frac{M^2}{2m\alpha}\right)^2 (1+\eta^2) \frac{2\cancel{\eta}\, d\eta}{\cancel{\eta}}$$

$$t = \frac{M^3}{4m\alpha^2}\int (1+\eta^2)\, d\eta = \frac{M^3}{2m\alpha^2}\left(\eta + \frac{1}{3}\eta^3\right)$$

$$r = \frac{M^2}{2m\alpha}(1+\eta^2) = \frac{p}{2}(1+\eta^2)$$

$$p = \frac{M^2}{m\alpha} \rightarrow M = \sqrt{m\alpha p}$$

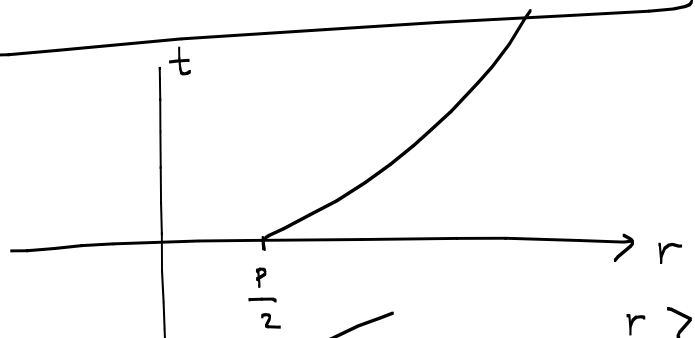
$$t = \frac{(m\alpha p)^{\frac{3}{2}}}{\underbrace{2m\alpha^2}}\left(\eta + \frac{1}{3}\eta^3\right) = \sqrt{\frac{mp^3}{\alpha}} \cdot \frac{\eta + \frac{\eta^3}{3}}{2}$$

$$\frac{m^{\frac{3}{2}}\alpha^{\frac{3}{2}}p^{\frac{3}{2}}}{2m\alpha^2} = \frac{1}{2}\frac{\sqrt{m}}{\sqrt{\alpha}}p^{\frac{3}{2}}$$

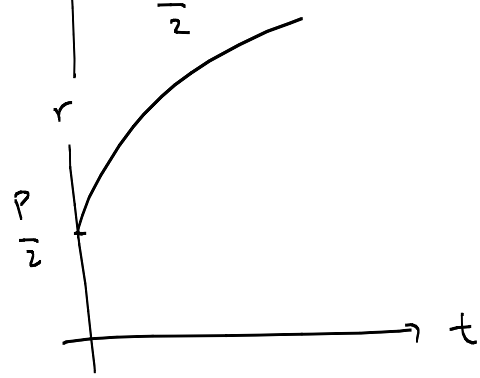
$$\eta = \sqrt{\frac{2r}{p} - 1}$$

$$t = \sqrt{\frac{mp^3}{\alpha}}\ \frac{1}{2}\ \sqrt{\frac{2r}{p} - 1}\left(\underbrace{1 + \frac{1}{3}\left(\frac{2r}{p} - 1\right)}_{\frac{2}{3}\left(\frac{r}{p}+1\right)}\right)$$

$$\boxed{t = \sqrt{\frac{mp^3}{\alpha}}\ \frac{1}{3}\left(\frac{r}{p}+1\right)\sqrt{\frac{2r}{p} - 1}}$$



$$r \geq \frac{p}{2}$$



$$t, r \gg 1 : \quad t \sim \left(\frac{r}{p}\right)^{\frac{3}{2}}$$

$$r \sim t^{\frac{2}{3}}$$

Prob 2. $U = -\frac{\alpha}{r^2} \quad (\alpha > 0)$

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left[E - \frac{M^2}{2mr^2} + \frac{\alpha}{r^2}\right]}}$$

$$= \sqrt{\frac{m}{2}}\ \frac{dr}{\sqrt{E + \frac{(\alpha - M^2/2m)}{r^2}}}$$

$$\phi = \int \frac{M \quad dr/r^2}{\sqrt{2m\left[E + \frac{(\alpha - M^2/2m)}{r^2}\right]}}$$

(a) $E > 0$, $\frac{M^2}{2m} - \alpha \equiv \beta^2$

$$dt = \sqrt{\frac{m}{2}} \frac{dr}{\sqrt{E - \frac{\beta^2}{r^2}}}$$

$$\phi = \int \frac{M \quad dr/r^2}{\sqrt{2mE}\sqrt{1 - \frac{\beta^2/E}{r^2}}}$$

$$\frac{\beta^2/E}{r^2} \equiv \sin^2\theta$$

$$\sin\theta = \sqrt{\frac{\beta^2}{E}} \frac{1}{r} \longrightarrow \cos\theta \, d\theta = \sqrt{\frac{\beta^2}{E}}\left(-\frac{dr}{r^2}\right)$$

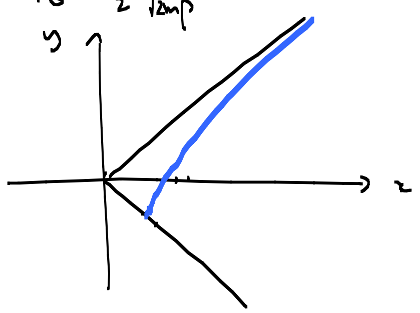
$$\phi = \frac{M}{\sqrt{2m\cancel{E}}} \int \frac{-\sqrt{\frac{\cancel{E}}{\beta^2}} \cancel{\cos\theta} \, d\theta}{\cancel{\cos\theta}} + \phi_0$$

$$= \phi_0 - \frac{M}{\beta\sqrt{2m}} \theta \longrightarrow \theta = \frac{\sqrt{2m}\,\beta}{M}(\phi_0 - \phi)$$

$$\frac{\beta}{\sqrt{E}} \frac{1}{r} = \sin\theta = \sin\left(\frac{\sqrt{2m}\,\beta}{M}(\phi_0 - \phi)\right)$$

$$r = \frac{\beta/\sqrt{E}}{\sin\left(\frac{\sqrt{2m}\,\beta}{M}(\phi_0 - \phi)\right)}$$ ✓

$$\phi_0 - \frac{\pi}{2}\frac{M}{\sqrt{2m}\beta} < \phi < \phi_0$$



$$\phi = \int \frac{M \quad dr/r^2}{\sqrt{2m\left[E + \frac{(\alpha - M^2/2m)}{r^2}\right]}}$$

(b) $E > 0$, $\frac{M^2}{2m} - \alpha \equiv -\beta^2$

$$dt = \sqrt{\frac{m}{2}} \frac{dr}{\sqrt{E + \frac{\beta^2}{r^2}}}$$

$$\phi = \int \frac{M \quad dr/r^2}{\sqrt{2mE}\sqrt{1 + \frac{\beta^2/E}{r^2}}}$$

$$\frac{\beta^2/E}{r^2} \equiv \sinh^2\theta$$

$$\sinh\theta = \sqrt{\frac{\beta^2}{E}} \frac{1}{r} \longrightarrow \cosh\theta \, d\theta = \sqrt{\frac{\beta^2}{E}}\left(-\frac{dr}{r^2}\right)$$

$$\phi = \frac{M}{\sqrt{2m\cancel{E}}} \int \frac{-\sqrt{\frac{\cancel{E}}{\beta^2}} \cancel{\cosh\theta} \, d\theta}{\cancel{\cosh\theta}} + \phi_0$$

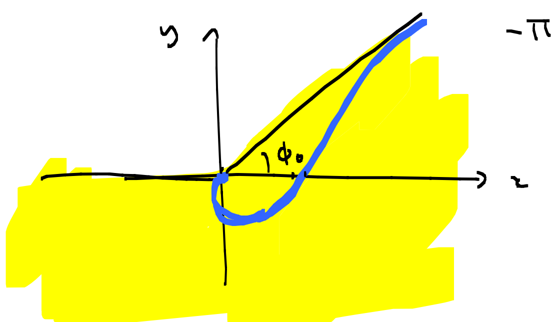
$$= \phi_0 - \frac{M}{\beta\sqrt{2m}} \theta \longrightarrow \theta = \frac{\sqrt{2m}\beta}{M}(\phi_0 - \phi)$$

$$\frac{\beta}{\sqrt{E}} \frac{1}{r} = \sinh\theta = \sinh\left(\frac{\sqrt{2m}\beta}{M}(\phi_0 - \phi)\right)$$

$$r = \frac{\beta/\sqrt{E}}{\sinh\left(\frac{\sqrt{2m}\beta}{M}(\phi_0 - \phi)\right)} \quad \checkmark$$

$$\phi < \phi_0$$

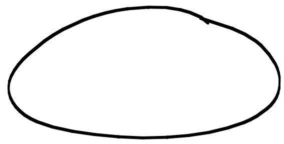
$$-\pi < \phi < \pi$$



Prob 3

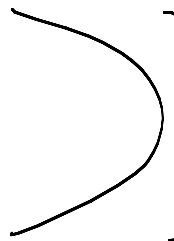
$$U = -\frac{\alpha}{r} + \delta U \quad (\alpha > 0)$$

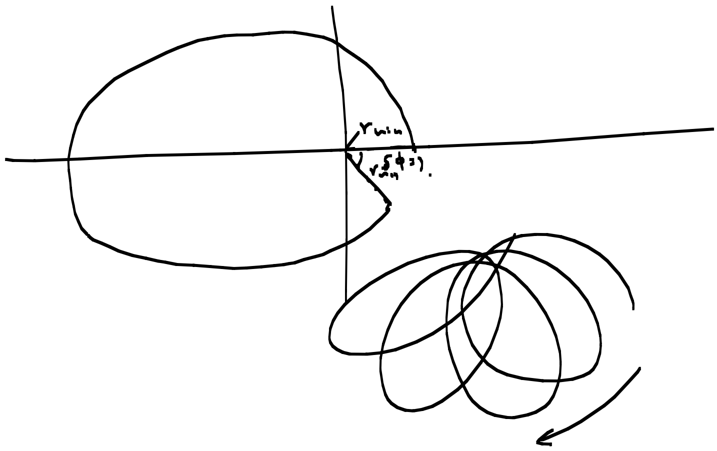
$E < 0$



$\delta U = 0$

$E > 0$





$$\phi = \int \frac{M \, dr/r^2}{\sqrt{2m\Big[\underbrace{E - \frac{M^2}{2m}\frac{1}{r^2} + \frac{\alpha}{r}}_{E - U_{eff}} - \delta U\Big]}}$$

$$U = -\frac{\alpha}{r} + \delta U$$

$$\phi = \int \frac{M \, dr/r^2}{\sqrt{2m(E - U_{eff}(r) - \delta U(r))}}$$

$$= \frac{M}{\sqrt{2m}} \int \frac{dr/r^2}{\sqrt{E - U_{eff}(r)}\,\sqrt{1 - \frac{\delta U}{(E - U_{eff})}}}$$

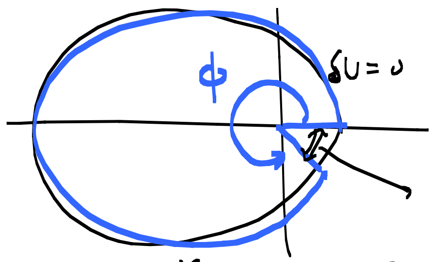
If δU is very small

$$\frac{1}{\sqrt{1-\frac{\delta U}{E-U_{eff}}}} \approx 1+\frac{1}{2}\frac{\delta U}{E-U_{eff}}$$

$$\phi = \frac{M}{\sqrt{2m}}\int \frac{dr/r^2}{\sqrt{E-U_{eff}(r)}}\left(1+\frac{1}{2}\frac{\delta U}{E-U_{eff}}\right)$$

$\Rightarrow$

$$\phi = \underbrace{2\frac{M}{\sqrt{2m}}\int_{r_{min}}^{r_{max}}\frac{dr/r^2}{\sqrt{E-U_{eff}(r)}}}_{2\pi} + 2\frac{M}{2\sqrt{2m}}\int_{r_{min}}^{r_{max}}\frac{\delta U\, dr/r^2}{\left(\sqrt{E-U_{eff}(r)}\right)^3}$$



$\phi - 2\pi = \Delta\phi$

$$\Delta\phi = \frac{M}{\sqrt{2m}}\int_{r_{min}}^{r_{max}}\frac{\delta U \cdot \frac{1}{r^2}\, dr}{\left(E-U_{eff}\right)^{\frac{3}{2}}}$$

$$U_{eff} = \frac{M^2}{2mr^2} \underbrace{- \frac{\alpha}{r}}$$

$$= \frac{M^2}{2mr^2} + U(r)$$

$$\frac{\partial}{\partial M}\left(E-U_{eff}\right)^{-\frac{1}{2}} = -\frac{1}{2}\left(E-U_{eff}\right)^{-\frac{3}{2}}\cdot\left(-\frac{\partial U_{eff}}{\partial M}\right)$$

$$-\frac{\partial U_{eff}}{\partial M} = -\frac{M}{mr^2}$$

$$\frac{\partial}{\partial M}\left(E-U_{eff}\right)^{-\frac{1}{2}} = \frac{M}{2mr^2}\left(E-U_{eff}\right)^{-\frac{3}{2}}$$

$$\Delta\phi = \frac{M}{\sqrt{2m}} \int_{r_{min}}^{r_{max}} \frac{\delta U \frac{1}{r^2}\, dr}{(E-U_{eff})^{\frac{3}{2}}}$$

$$\frac{2m}{M}\frac{\partial}{\partial M}\frac{1}{\sqrt{E-U_{eff}}}$$

$$= \sqrt{2m}\,\frac{\partial}{\partial M} \int_{r_{min}}^{r_{max}} \frac{\delta U}{\sqrt{E-U_{eff}}}\, dr$$

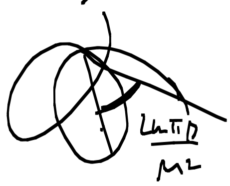
$$\frac{r^2 d\phi \sqrt{2m}}{M} = \frac{dr}{\sqrt{E-U_{eff}}}$$

$$\Delta\phi = 2m\frac{\partial}{\partial M}\left(\frac{1}{M}\int_0^{\pi} r^2\, \delta U\, d\phi\right)$$

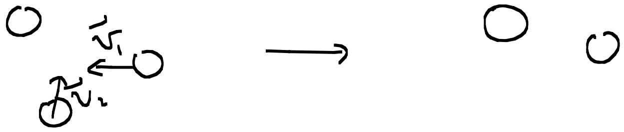
(a) $\delta U = \frac{\beta}{r^2}$

$$\Delta\phi = 2m\frac{\partial}{\partial M}\left(\frac{1}{M}\pi\beta\right) = -\frac{2m\pi\beta}{M^2} < 0$$

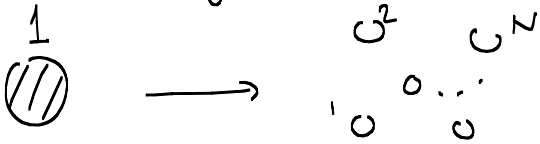
$$\equiv \phi - 2\pi$$



Chap 4. Collision.



§16. Disintegration



internal energy E_i, total $E = E_i + T + U$

before : $K = 0$ $\boxed{U = 0 \text{ always}}$

$$E_i = E$$

after : $(N=2)$

$$E = E_{1i} + T_1 + E_{2i} + T_2$$

$$\therefore E_i = E_{1i} + \frac{p_1^2}{2m_1} + E_{2i} + \frac{p_2^2}{2m_2}$$

---

also momentum should be conserved.

before : 0

after : $\vec{p}_1 + \vec{p}_2$

$$\therefore \vec{p}_1 + \vec{p}_2 = 0 \longrightarrow \vec{p}_2 = -\vec{p}_1$$

$$\therefore E_i = E_{1i} + \frac{p_1^2}{2m_1} + E_{2i} + \frac{p_1^2}{2m_2}$$

or Define

disintegration Energy $\epsilon \equiv E_i - E_{1i} - E_{2i}$

$$\epsilon = \frac{p_1^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{p_1^2}{2m}$$

$$m \equiv \frac{m_1 m_2}{m_1+m_2} = \text{reduced mass}.$$

speed of m_1: $\dfrac{|\vec{p}_1|}{m_1} = v_{10}$

" m_2: $\dfrac{|\vec{p}_2|}{m_2} = \dfrac{|\vec{p}_1|}{m_2} = v_{20}$

<u>LAB</u> & <u>CM frame</u>

$\sum_i \vec{P}_i \neq 0$

total momentum $= 0$

$\sum_i \vec{P}_{i0} = 0$

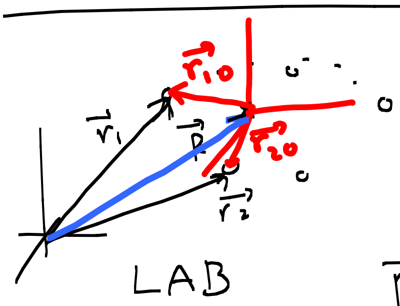
"0" means CM frame

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$$

$$\vec{V} = \frac{\sum_i m_i \vec{v}_i}{\sum_i m_i} = \frac{\sum_i \vec{P}_i}{M}$$

$$\underline{\vec{r}_i = \vec{R} + \vec{r}_{i0}} \longrightarrow \vec{v}_i = \vec{V} + \vec{v}_{i0}$$

$$\vec{p}_i = \underbrace{m_i \vec{V}}_{\frac{m_i}{M}\vec{P}} + \vec{P}_{i0}$$



LAB

$$\vec{P} = M\vec{V}$$

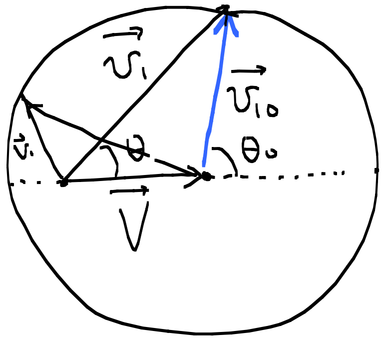
$$\vec{P}_i = \frac{m_i}{M}\vec{P} + \vec{P}_{i0}$$

$$\vec{v}_1 = \vec{V} + \vec{v}_{10}$$

$$\vec{v}_1 - \vec{V} = \vec{v}_{10}$$

$$v_1^2 - 2\underbrace{\vec{v}_1 \cdot \vec{V}}_{v_1 V \cos\theta} + V^2 = v_{10}^2$$

$V < v_{10}$

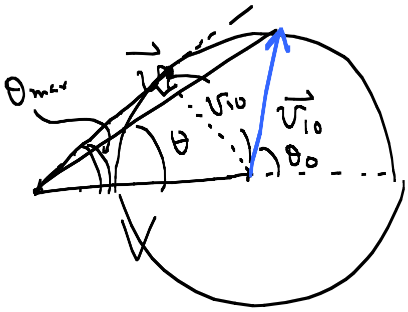


$0 \le \theta_0 \le \pi$

$\downarrow$

$0 \le \theta \le \pi$

$V > v_{10}$



$0 \le \theta_0 \le \pi$

$0 \le \theta \le \theta_{max}$

$$\sin\theta_{max} = \frac{v_{10}}{V}$$

$v_1 \sin\theta = v_{10}\sin\theta_0$

$v_1 \cos\theta = V + v_{10}\cos\theta_0$

$$\therefore \tan\theta = \frac{v_{10}\sin\theta_0}{v_{10}\cos\theta_0 + V}$$

θ in terms θ_0

θ_0 in terms of θ

$$\tan\theta\,(v_{10}\cos\theta_0 + V) = v_{10}\sqrt{1-\cos^2\theta_0}$$

$$t^2\,(v_{10}^2 c_0^2 + V^2 + 2v_{10}Vc_0) = v_{10}^2(1-c_0^2)$$

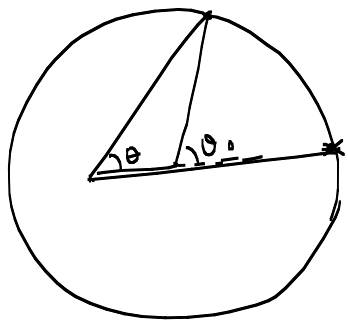
$$0 = \underbrace{(t^2+1)\,v_{10}^2}_{\frac{v_{10}^2}{\cos^2\theta}}\,c_0^2 + 2t^2 v_{10}Vc_0 + t^2V^2 - v_{10}^2$$

$$\cos\theta_0 = -\frac{V}{v_{10}}\sin^2\theta \pm \cos\theta\sqrt{1-\frac{V^2}{v_{10}^2}\sin^2\theta}.$$

if $V < v_{10}$; $\theta = 0 \longrightarrow \theta_0 = 0$

$\cos\theta_0 = \pm\cos\theta$

×



$$\cos\theta_0 = -\frac{V}{v_{10}}\sin^2\theta + \cos\theta\sqrt{1-\frac{V^2}{v_{10}^2}\sin^2\theta}.$$

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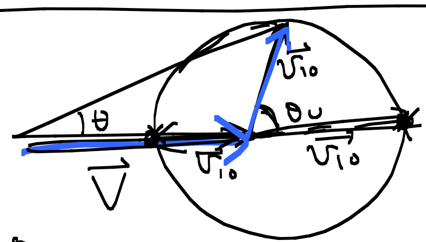
if $V > v_{10}$



$\theta = 0 \longrightarrow \theta_0 = 0$ or π

$$\cos\theta_0 = -\frac{V}{v_{10}}\sin^2\theta \pm \cos\theta\sqrt{1-\frac{V^2}{v_{10}^2}\sin^2\theta}.$$

both signs should be taken

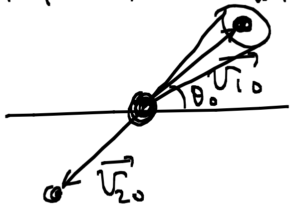
$V > v_{10}$ $\quad \left(\frac{V}{v_{10}}\right)^2 > 1$

θ such that

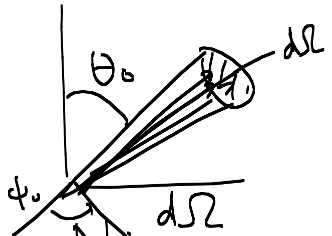
$1 - \frac{V^2}{v_{10}^2}\sin^2\theta < 0$ should be excluded.

$0 \leq \theta \leq \theta_{max} \longrightarrow \sin\theta_{max} = \dfrac{v_{10}}{V}$

CM frame m_1: disintegration happens at angle θ_0 randomly



N integrations $\rightarrow$ dn = # of m_1 in direction of Ω & $\Omega + d\Omega$



$$\int d\Omega = 4\pi$$

$$dn = N \cdot \frac{d\Omega}{4\pi} = N \frac{d\phi_0 \cdot \sin\theta_0 \, d\theta_0}{4\pi}$$

$dn_{\theta_0} \equiv$ # of m_1 between θ_0 and $\theta_0 + d\theta_0$

$$= \int_{0 \leftarrow \phi \text{ integral.}}^{2\pi} dn = \frac{N}{4\pi} \sin\theta_0 \, d\theta_0 \underbrace{\int_0^{2\pi} d\phi_0}_{2\pi}$$

$$= \frac{N}{2} \left| \sin\theta_0 \, d\theta_0 \right|$$

LAB

$$T = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 \left(v_{10}^2 + V^2 + 2 \vec{v}_{10} \cdot \vec{V} \right)$$

$$\vec{v}_1 = \vec{v}_{10} + \vec{V}$$

$v_{10} V \cos\theta_0$

$$|\sin\theta_0 \, d\theta_0| = |d\cos\theta_0|$$

assume v_{10}, V constant.

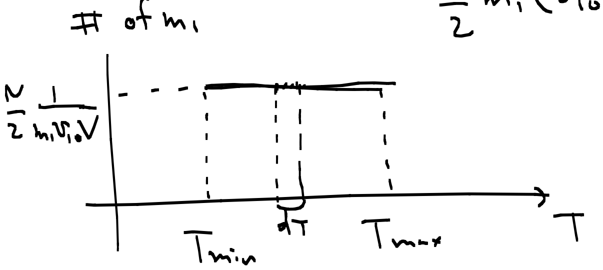
$$dT = m_1 v_{10} V \, d\cos\theta_0$$

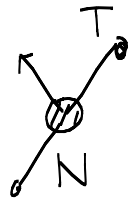
$$dn_T = \frac{N}{2} |d\cos\theta_0| = \frac{N}{2} \frac{dT}{m_1 v_{10} V}$$

of m_1 between T & $T + dT$

T = between T_{min} & T_{max}

$$T_{min} = \frac{1}{2} m_1 (v_{10} - V)^2 \qquad T_{max} = \frac{1}{2} m_1 (v_{10} + V)^2$$





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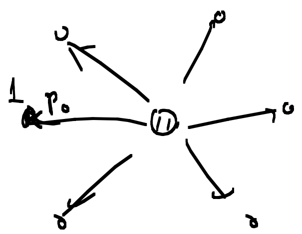
more two-particles



CM frame

internal energy of the rest

$$E_i = E_{1i} + T_1 + E_i' + T_i'$$

$$T_1 = \frac{p_0^2}{2m_1}$$

$$\underbrace{E_i - (E_{1i} + E_i')}_{\equiv\ \epsilon} = T_1 + T_i'$$

$T_{1,max}$ occurs when T_i' is min.

T_i' is minimum when

$\vec{v}_{20}$, $\vec{v}_{30}$, $\vec{v}_{40}$, $\vdots$ $\Big\}$ all particles $m_2, \dots, m_N$ have velocity in the same direction.

$$\vec{v}_{i0} = v_{i0} \vec{n}_0 \qquad i = 2, \dots, N$$

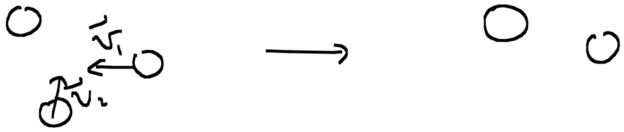
$$\vec{p}_{i0} = m_i v_{i0} \vec{n}_0 \quad (\vec{n}_0^{\,2} = 1)$$

$$0 = m_1 \vec{v}_{10} + \underbrace{m_2 \vec{v}_{20} + \cdots + m_N \vec{v}_{N0}}_{(m_2 v_{20} + \cdots + m_N v_{N0})\vec{n}_0}$$

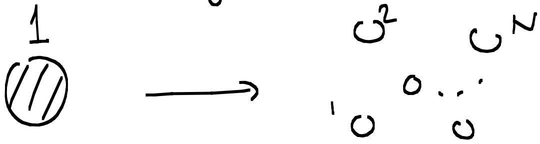
$$T_i' = \frac{1}{2} m_2 v_{20}^2 + \cdots$$

- [illegible]

Chap 4. Collision.



§16. Disintegration



internal energy E_i, total $E = E_i + T + U$

before: $K = 0$ $\boxed{U = 0 \text{ always}}$

$$E_i = E$$

after: $(N = 2)$

$$E = E_{1i} + T_1 + E_{2i} + T_2$$

$$\therefore E_i = E_{1i} + \frac{p_1^2}{2m_1} + E_{2i} + \frac{p_2^2}{2m_2}$$

---

also momentum should be conserved.

before: 0

after: $\vec{p}_1 + \vec{p}_2$

$$\therefore \vec{p}_1 + \vec{p}_2 = 0 \longrightarrow \vec{p}_2 = -\vec{p}_1$$

$$\therefore E_i = E_{1i} + \frac{p_1^2}{2m_1} + E_{2i} + \frac{p_1^2}{2m_2}$$

or Define

disintegration Energy $\epsilon \equiv E_i - E_{1i} - E_{2i}$

$$\epsilon = \frac{p_1^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{p_1^2}{2m}$$

$$m \equiv \frac{m_1 m_2}{m_1+m_2} = \text{reduced mass}.$$

speed of m_1: $\frac{|\vec{p}_1|}{m_1} = v_{10}$

" m_2: $\frac{|\vec{p}_2|}{m_2} = \frac{|\vec{p}_1|}{m_2} = v_{20}$

LAB & CM frame

$\sum_i \vec{P}_i \neq 0$

total momentum $= 0$

$\sum_i \vec{P}_{i0} = 0$

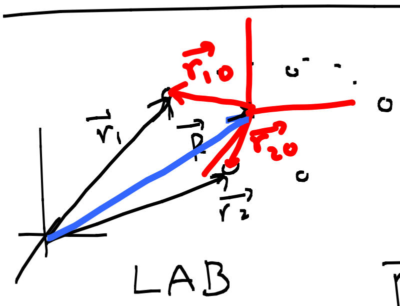
"0" means CM frame

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$$

$$\vec{V} = \frac{\sum_i m_i \vec{v}_i}{\sum_i m_i} = \frac{\sum_i \vec{P}_i}{M}$$

$$\vec{r}_i = \vec{R} + \vec{r}_{i0} \longrightarrow \vec{v}_i = \vec{V} + \vec{v}_{i0}$$

$$\vec{p}_i = \underbrace{m_i \vec{V}}_{\frac{m_i}{M}\vec{P}} + \vec{P}_{i0}$$



$$\vec{P} = M\vec{V}$$

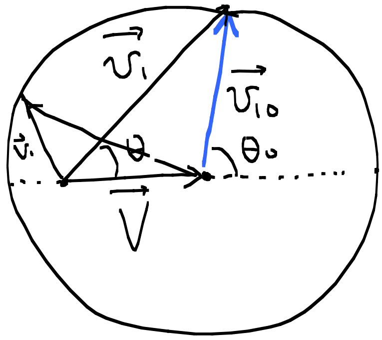
$$\vec{P}_i = \frac{m_i}{M}\vec{P} + \vec{P}_{i0}$$

$$\vec{v}_1 = \vec{V} + \vec{v}_{10}$$

$$\vec{v}_1 - \vec{V} = \vec{v}_{10}$$

$$v_1^2 - 2\underbrace{\vec{v}_1 \cdot \vec{V}}_{v_1 V \cos\theta} + V^2 = v_{10}^2$$

$V < v_{10}$

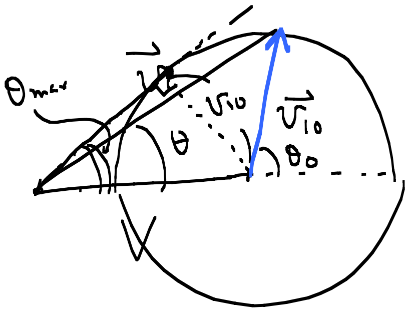


$0 \leq \theta_0 \leq \pi$

$\downarrow$

$0 \leq \theta \leq \pi$

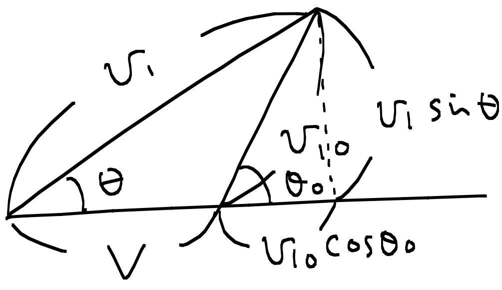
$V > v_{10}$



$0 \leq \theta_0 \leq \pi$

$0 \leq \theta \leq \theta_{max}$

$$\sin\theta_{max} = \frac{v_{10}}{V}$$



$v_1 \sin\theta = v_{10}\sin\theta_0$

$v_1\cos\theta = V + v_{10}\cos\theta_0$

$$\therefore \tan\theta = \frac{v_{10}\sin\theta_0}{v_{10}\cos\theta_0 + V}$$

θ in tems θ_0

θ_0 in tems of θ

$$\tan\theta\,(v_{10}\cos\theta_0 + V) = v_{10}\sqrt{1-\cos^2\theta_0}$$

$$t^2\left(v_{10}^2 c_0^2 + V^2 + 2v_{10}V c_0\right) = v_{10}^2(1-c_0^2)$$

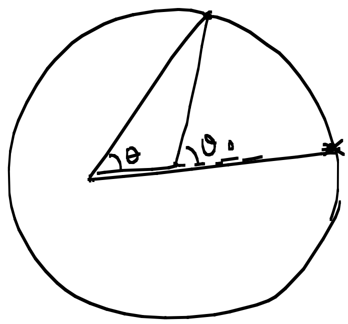
$$0 = \underbrace{(t^2+1)\,v_{10}^2}_{\frac{v_{10}^2}{\cos^2\theta}}\, c_0^2 + 2t^2 v_{10} V c_0 + t^2V^2 - v_{10}^2$$

$$\cos\theta_0 = -\frac{V}{v_{10}}\sin^2\theta \pm \cos\theta\sqrt{1-\frac{V^2}{v_{10}^2}\sin^2\theta}.$$

if $V < v_{10}$; $\theta = 0 \longrightarrow \theta_0 = 0$

$\cos\theta_0 = \pm\cos\theta$

$\times$



$$\cos\theta_0 = -\frac{V}{v_{10}}\sin^2\theta + \cos\theta\sqrt{1-\frac{V^2}{v_{10}^2}\sin^2\theta}.$$

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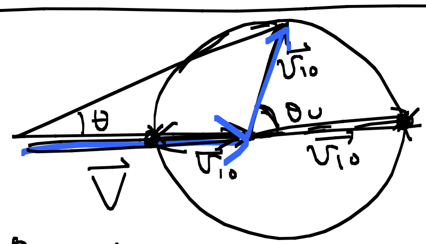
if $V > v_{10}$



$\theta = 0 \longrightarrow \theta_0 = 0$ or π

$$\cos\theta_0 = -\frac{V}{v_{10}}\sin^2\theta \pm \cos\theta\sqrt{1-\frac{V^2}{v_{10}^2}\sin^2\theta}.$$

both signs should be taken

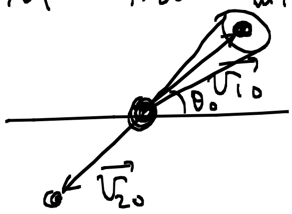
$V > v_{10}$ $\quad \left(\frac{V}{v_{10}}\right)^2 > 1$

θ such that

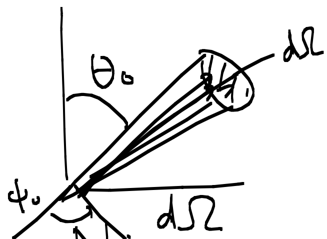
$1-\frac{V^2}{v_{10}^2}\sin^2\theta < 0$ should be excluded.

$$0 \le \theta \le \theta_{max} \rightarrow \sin\theta_{max} = \frac{v_{10}}{V}$$

CM frame m_1: disintegration happens at angle θ_0 randomly



N integrations $\rightarrow$ dn = # of m_1 in direction of Ω & $\Omega + d\Omega$



$$\int d\Omega = 4\pi$$

$$dn = N \cdot \frac{d\Omega}{4\pi} = N\,\frac{d\phi_0 \cdot \sin\theta_0\, d\theta_0}{4\pi}$$

$dn_{\theta_0} \equiv$ # of m_1 between θ_0 and $\theta_0 + d\theta_0$

$$= \int_0^{2\pi} dn = \frac{N}{4\pi}\sin\theta_0\, d\theta_0 \underbrace{\int_0^{2\pi} d\phi_0}_{2\pi}$$

$\leftarrow \phi$ integral.

$$= \frac{N}{2}\left|\sin\theta_0\, d\theta_0\right|$$

LAB

$$T = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 \left(v_{10}^2 + V^2 + 2\underbrace{\vec{v}_{10}\cdot\vec{V}}_{v_{10} V \cos\theta_0}\right)$$

$$\vec{v}_1 = \vec{v}_{10} + \vec{V}$$

$$|\sin\theta_0\, d\theta_0| = |d\cos\theta_0|$$

assume v_{10}, V constant.

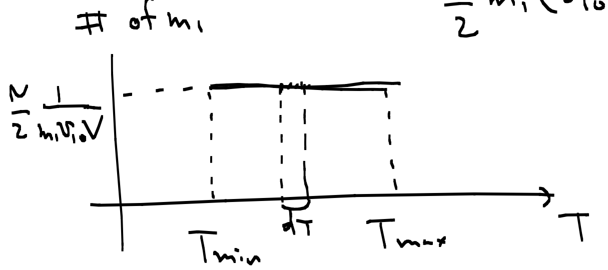
$$dT = m_1 v_{10} V\, d\cos\theta_0$$

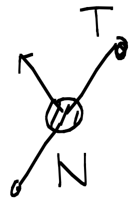
$$dn_T = \frac{N}{2}|d\cos\theta_0| = \frac{N}{2}\frac{dT}{m_1 v_{10} V}$$

of m_1 between T & $T + dT$

$$T = \text{between } T_{min} \ \& \ T_{max}$$

$$T_{min} = \frac{1}{2} m_1 (v_{10} - V)^2 \qquad T_{max} = \frac{1}{2} m_1 (v_{10} + V)^2$$





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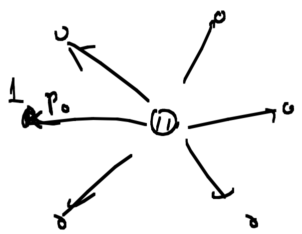
more two-particles

CM frame



internal energy of the rest

$$E_i = E_{1i} + \underbrace{T_1}_{= \frac{p_0^2}{2m_1}} + E_i' + T_i'$$

$$\underbrace{E_i - (E_{1i} + E_i')}_{\equiv \epsilon} = T_1 + T_i'$$

$T_{1,max}$ occurs when T_i' is min.

T_i' is minimum when

$\vec{v_{20}}$, $\vec{v_{30}}$, $\vec{v_{40}}$, $\vdots$ — all particles $m_2, \ldots, m_N$ have velocity in the same direction.

$$\vec{v_{i0}} = v_{i0} \vec{n_0} \qquad i = 2, \ldots, N$$

$$\vec{p_{i0}} = m_i v_{i0} \vec{n_0} \quad (\vec{n_0}^2 = 1)$$

$$0 = m_1 \vec{v}_{10} + \underbrace{m_2 \vec{v}_{20} + \cdots + m_N \vec{v}_{N0}}_{(m_2 v_{20} + \cdots + m_N v_{N0})\vec{n}_0}$$

$\vec{P}_2, \cdots, \vec{P}_N$ $\qquad$ $\vec{P}_2 + \cdots + \vec{P}_N = \text{constant} = \vec{C}$

$m_2 \qquad m_N$

$$T = \frac{P_2^2}{2m_2} + \cdots + \frac{P_N^2}{2m_N} \quad \text{minimize}$$

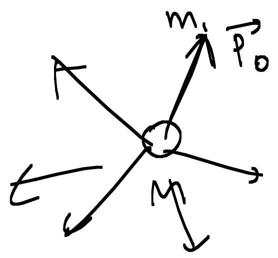
$$T - \vec{\lambda}\cdot\left(\sum_{i=2}^{N} \vec{P}_i - \vec{C}\right)$$

$$= \sum_i \frac{\vec{P}_i^{\,2}}{2m_i} - \vec{\lambda}\cdot\sum_{i=2}^{N} \vec{P}_i + \vec{\lambda}\cdot\vec{C} = T'$$

$$\frac{\partial T'}{\partial \vec{P}_i} = 0 \rightarrow \frac{\vec{P}_i}{m_i} - \vec{\lambda} = 0$$

$$\Rightarrow \vec{P}_i = m_i \vec{\lambda}$$

$$i = 2, \cdots, N$$



$$T_{10} = \epsilon - T_{i'}$$

$$\uparrow$$

$$E_i - E_{1i} - E_{i'}$$

$$T_{10,max} = \epsilon - T_{i',min}$$

$$T_{i',min} = \sum_{i=2}^{N} \frac{m_i^2 \lambda^2}{2m_i} = \frac{1}{2}\lambda^2 (M - m_1)$$

when $\vec{P}_i = m_i \vec{\lambda}$ $\quad i = 2, \cdots, N$

$$\vec{P}_1 + \vec{P}_2 + \cdots \vec{P}_N = 0 \qquad \vec{P}_1 + \overbrace{(m_2 + \cdots + m_N)}^{M-m_1}\vec{\lambda} = 0$$

$$\vec{\lambda} = -\frac{\vec{p}_1}{M-m_1}$$

$$T_{i',\min} = \frac{1}{2}(M-m_1)\frac{p_0^2}{(M-m_1)^2} = \frac{p_0^2}{2(M-m_1)}$$

$$T_{10,\max} = \frac{p_0^2}{2m_1} = \epsilon - \frac{p_0^2}{2(M-m_1)}$$

$$= \epsilon - \frac{m_1}{M-m_1}\underbrace{\frac{p_0^2}{2m_1}}_{T_{10,\max}}$$

$$\therefore T_{10,\max}\left(\overbrace{1+\frac{m_1}{M-m_1}}^{\frac{M}{M-m_1}}\right) = \epsilon$$

$$\therefore T_{10,\max} = \frac{M-m_1}{M}\epsilon \quad //$$

§17. Elastic Collisions

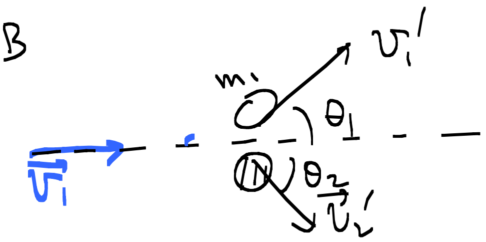
both # & Kinetic Energy are conserved

two-particle system.

LAB

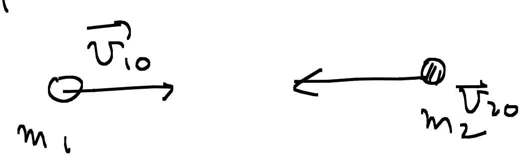
before collision

LAB



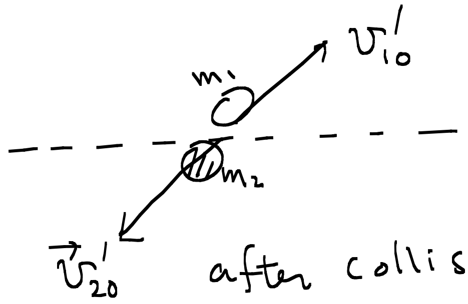
after collision

CM



before collision

CM.



after collision

$$\vec{v}_i = \vec{V} + \vec{v}_{i0} \qquad i=1,2$$

$$\vec{v}_i' = \vec{V} + \vec{v}_{i0}'$$

$$\sum_{i=1}^{2} m_i \vec{v}_{i0} = \sum_{i=1}^{2} m_i \vec{v}_{i0}' = 0$$

$$\frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1+m_2} = \vec{V}$$

$$\vec{v} \equiv \vec{v}_1 - \vec{v}_2$$

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1+m_2)\vec{V}$$

$$\vec{v}_1 - v_2 = \vec{v}$$

LAB
$$\vec{v}_1 = \vec{V} + \frac{m_2}{m_1+m_2}\vec{v}$$
$$\vec{v}_2 = \vec{V} - \frac{m_1}{m_1+m_2}\vec{v}$$

CM

before
$$\vec{v}_{10} = \frac{m_2}{m_1+m_2}\vec{v}$$
$$\vec{v}_{20} = -\frac{m_1}{m_1+m_2}\vec{v}$$

after $\quad m_1\vec{v}_{10}' + m_2\vec{v}_{20}' = 0$

$$\vec{v}_{10}' \propto \vec{n}_0 \longrightarrow \vec{v}_{20}' \propto -\vec{n}_0$$

$$\vec{v}_{10}' = \underbrace{\frac{m_2}{m_1+m_2} v}_{|\vec{v}_{10}'|} \vec{n}_0$$

$$\vec{v}_{20}' = -\frac{m_1}{m_1+m_2} v \vec{n}_0$$

$$\vec{v}_i' = \vec{V} + \vec{v}_{i0}'$$

$$\Rightarrow \vec{v}_1' = \vec{V} + \frac{m_2}{m_1+m_2} v \vec{n}_0$$

$$\vec{v}_2' = \vec{V} - \frac{m_1}{m_1+m_2} v \vec{n}_0$$

$$\vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2}$$

$$\vec{P}_1 = m_1\vec{v}_1' = m_1\vec{V} + \frac{m_1 m_2}{m_1+m_2} v \vec{n}_0$$

$$\vec{P}_2' = m_2\vec{v}_2' = m_2\vec{V} - \frac{m_1 m_2}{m_1+m_2} v \vec{n}_0$$

$$M\vec{V} = \vec{P} = \vec{P}_1 + \vec{P}_2$$

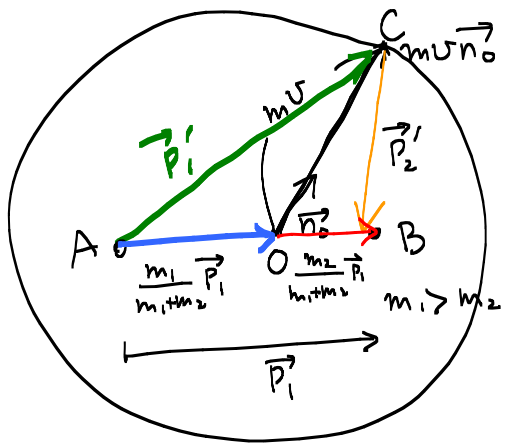
$$\vec{V} = \frac{\vec{P}_1 + \vec{P}_2}{m_1+m_2}$$

Now,
$\vec{v}_2 = 0$ (target is at rest.)

$\vec{P}_2 = 0$

$$\vec{p}_1' = \frac{m_1}{m_1+m_2}\vec{p}_1 + \overset{\text{reduced mass}}{m} v \vec{n}_0$$

$$\vec{p}_2' = \frac{m_2}{m_1+m_2}\vec{p}_1 - m v \vec{n}_0$$



CAB

$$T_b = \frac{p_1^2}{2m_1} + \cancel{\frac{\vec{p}_2^{\,2}}{2m_2}} \to 0 \qquad \vec{v}_2 \equiv 0$$

$$T_a = \frac{p_1'^2}{2m_1} + \frac{p_2'^2}{2m_2}$$

$$\begin{pmatrix} \vec{p}_1' = \frac{m_1}{m_1+m_2}\vec{p}_1 + m v \vec{n}_0 \\ \vec{p}_2' = \frac{m_2}{m_1+m_2}\vec{p}_1 - m v \vec{n}_0 \end{pmatrix}$$

$$= \frac{1}{2m_1}\left(\left(\frac{m_1}{m_1+m_2}\right)^2 p_1^2 + (mv)^2 + \cancel{2\frac{m_1 m v}{m_1+m_2}\vec{p}_1\cdot\vec{n}_0}\right)$$

$$+ \frac{1}{2m_2}\left(\left(\frac{m_2}{m_1+m_2}\right)^2 p_1^2 + (mv)^2 - \cancel{2\frac{m_2 m v}{m_1+m_2}\vec{p}_1\cdot\vec{n}_0}\right)$$

$$= \frac{p_1^2}{2(m_1+m_2)} + \frac{1}{2}\underbrace{\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}_{\frac{1}{m}} m^2 v^2$$

$$= \frac{p_1^2}{2m_1}$$

$$\frac{1}{2} m v^2 = \frac{p_1^2}{2}\left(\frac{1}{m_1} - \frac{1}{m_1+m_2}\right)$$

$$\cancel{\frac{1}{2}}\frac{m_1 \cancel{m_2}}{\cancel{m_1+m_2}} v^2 = \frac{p_1^2}{\cancel{2}} \frac{\cancel{m_2}}{m_1 \cancel{(m_1+m_2)}}$$

$$v^2 = \frac{p_1^2}{m_1^2} \longrightarrow v = v_1 = \frac{p_1}{m_1}$$

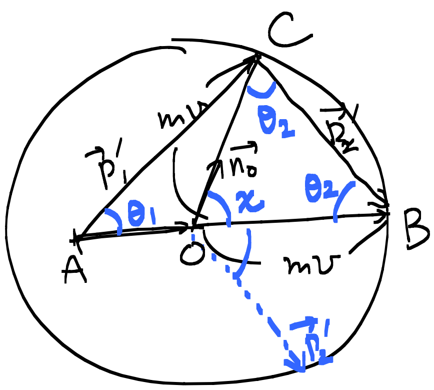
$$m v = \frac{m_1 m_2}{m_1+m_2} v_1 = \frac{m_2}{m_1+m_2} p_1$$

$$\vec{AO} = \frac{m_1}{m_1+m_2} \vec{p_1}$$

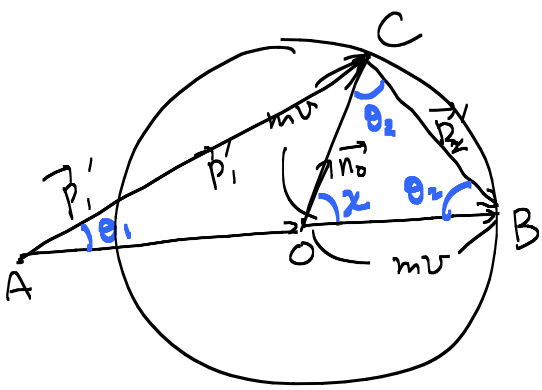
if $m_1 > m_2 \to |mv| < |\vec{AO}|$

$$\vec{OB} = \frac{m_2}{m_1+m_2} \vec{p_1} \to |\vec{OB}| = mv$$

$\vec{v_2}=0 \to$ B should be on circle

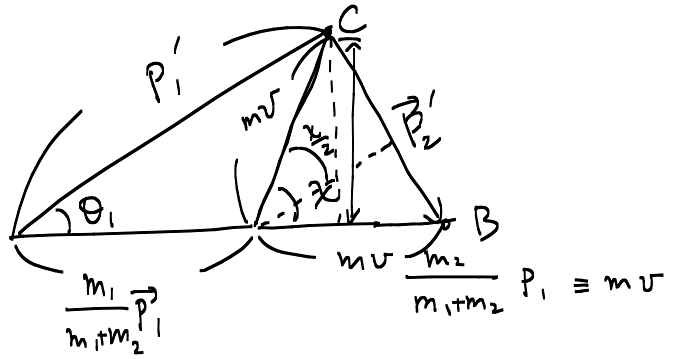


$m_2 > m_1$



$m_2 < m_1$

$$x + 2\theta_2 = \pi \to \theta_2 = \frac{1}{2}(\pi - x)$$



$$p_1' \sin\theta_1 = mv \sin x$$

$$\div)\quad p_1' \cos\theta_1 = \frac{m_1 p_1}{m_1+m_2} + mv\cos x$$

$$\tan\theta_1 = \frac{\frac{m_2}{m_1+m_2}\vec{p}_1 \sin x}{\frac{m_1 \vec{p}_1}{m_1+m_2} + \frac{m_2}{m_1+m_2}\vec{p}_1 \cos x}$$

$$\boxed{\tan\theta_1 = \frac{m_2 \sin x}{m_1 + m_2 \cos x}}$$

$mv =$

$$(p_1' \sin\theta_1)^2 = (mv \sin x)^2$$

$$+)\quad (p_1' \cos\theta_1)^2 = \left(\underbrace{\frac{m_1 p_1}{m_1+m_2}}_{\frac{m_1}{m_2}mv} + mv\cos x\right)^2$$

$\frac{m_2}{m_1+m_2}p_1 = mv$

$$p_1'^2 = (mv)^2\left[\sin^2 x + \left(\frac{m_1}{m_2} + \cos x\right)^2\right]$$

$$= (mv)^2\left[1 + \frac{m_1^2}{m_2^2} + 2\frac{m_1}{m_2}\cos x\right]$$

$$p_1' = (mv)\sqrt{\left[1 + \frac{m_1^2}{m_2^2} + 2\frac{m_1}{m_2}\cos x\right]}$$

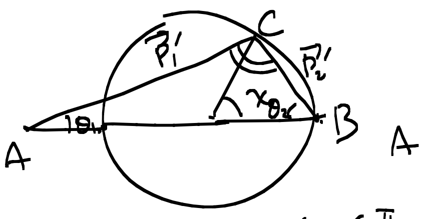
$$v_1' = \frac{(mv)}{m_1}\sqrt{\left[1 + \frac{m_1^2}{m_2^2} + 2\frac{m_1}{m_2}\cos x\right]}$$

//

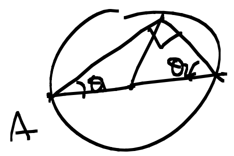
$$\frac{m}{m_1} = \frac{m_2}{m_1+m_2}$$

$$p_2' = 2mv\sin\frac{\chi}{2} = 2\frac{m_1m_2}{m_1+m_2}v\sin\frac{\chi}{2}$$

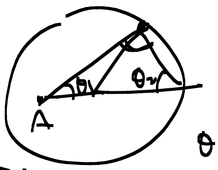
$$v_2' = 2\frac{mv}{m_2}\sin\frac{\chi}{2} = 2\frac{m_1}{m_1+m_2}v\sin\frac{\chi}{2}$$



$m_1 > m_2$ $\theta_1+\theta_2 < \frac{\pi}{2}$



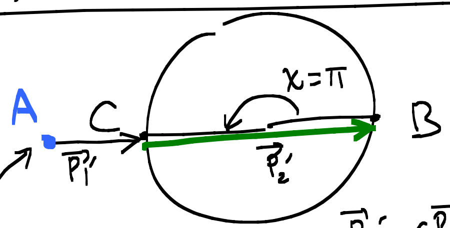
$\theta_1+\theta_2 = \frac{\pi}{2}$



$\theta_1+\theta_2 > \frac{\pi}{2}$

$m_2 > m_1$

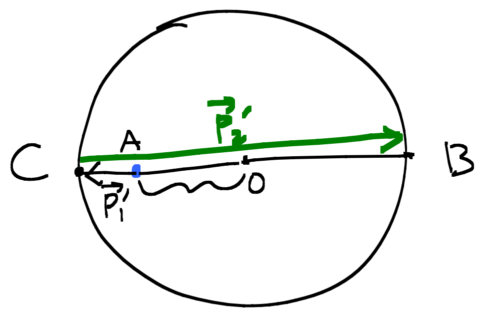
$\underline{\chi = \pi}$



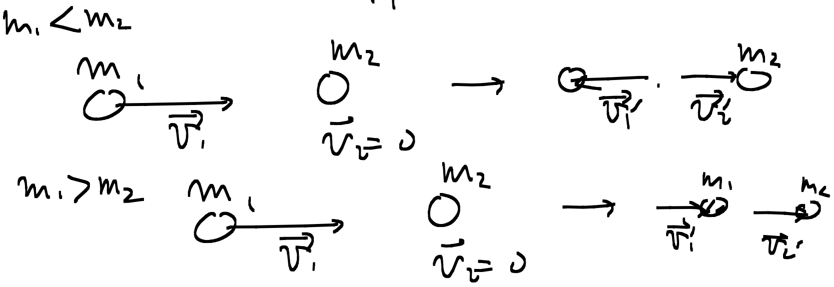
$\vec{p}_1' \propto \vec{p}_2'$

$m_1 > m_2$

$\chi = \pi$ $m_1 < m_2$



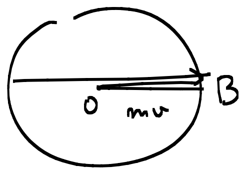
$\vec{p}_1' \propto -\vec{p}_2'$



$$m_1 \vec{v}_1 = m_1 \vec{v}_1{}' + m_2 \vec{v}_2{}'$$

$$\vec{v} = \vec{v}_1$$

$$m v = \frac{m_1 m_2}{m_1+m_2} v_1 = \frac{m_2}{m_1+m_2} p_1$$

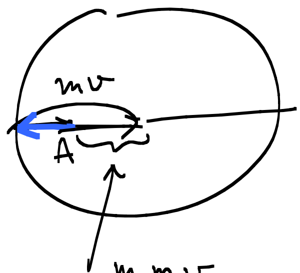


$$p_2{}' = m_2 v_2{}' = 2 m v$$

$$\cancel{m_2} v_2{}' = 2 \frac{m_1 \cancel{m_2}}{m_1+m_2} v$$

$$v_2{}' = \frac{2 m_1}{m_1+m_2} v$$

$\frac{m_1}{m_1+m_2} \vec{p}_1$



$m_1 < m_2$

$\frac{m_1 m_2}{m_1+m_2}$ $\frac{m_1 m_1 v}{m_1+m_2}$

$$|p_1{}'| = m v - \frac{m_1^2}{m_1+m_2} v = m_1 \frac{m_2-m_1}{m_1+m_2} v$$

$$m_1 \vec{v}_1{}' = - m_1 \frac{m_2-m_1}{m_1+m_2} \vec{v}$$

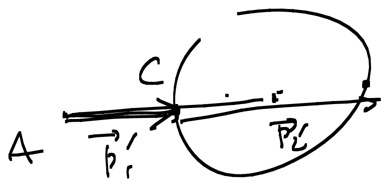
$$\shortparallel$$

$$\vec{p}_1{}'$$

$$\Rightarrow \vec{v}_1{}' = \frac{m_1-m_2}{m_1+m_2} \vec{v}$$

$$\vec{v}_2{}' = \frac{2 m_1}{m_1+m_2} \vec{v}$$

$m_1 > m_2$



$$|\vec{p}_1{}'| = m_1 \frac{|m_2-m_1|}{m_1+m_2} v = \frac{m_1(m_1-m_2)}{m_1+m_2} v$$

$$\vec{p}_1{}' = m_1 \vec{v}_1{}' = m_1 \frac{m_1 - m_2}{m_1 + m_2} \vec{v}$$

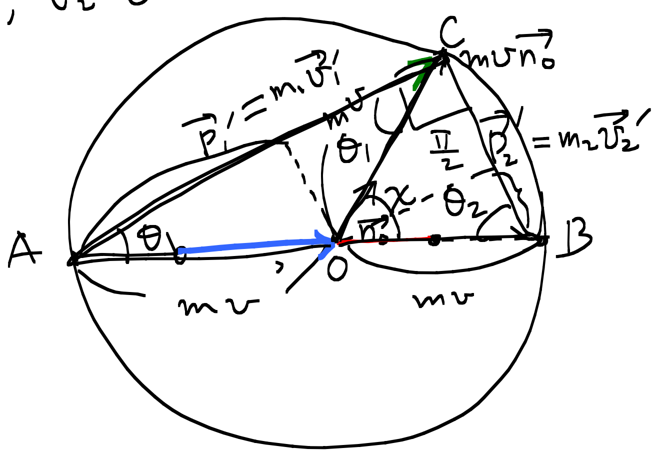
$$\vec{v}_1{}' = \frac{m_1 - m_2}{m_1 + m_2} \vec{v}$$

$$\vec{v}_2{}' = \frac{2m_1}{m_1 + m_2} \vec{v}$$

$$\boxed{\begin{aligned} \vec{v}_1{}' &= \frac{m_1 - m_2}{m_1 + m_2} \vec{v} \\ \vec{v}_2{}' &= \frac{2m_1}{m_1 + m_2} \vec{v} \end{aligned}}$$ for all m_1, m_2

$\boxed{\chi = \pi}$

$m_1 = m_2$, $\vec{v}_2 = 0$



$$\theta_1 + \theta_2 = \frac{\pi}{2} \rightarrow \theta_2 = \frac{\pi}{2} - \theta_1 = \frac{\pi - \chi}{2}$$

$$\chi = 2\theta_1 \rightarrow \theta_1 = \frac{\chi}{2}$$

$$m_1 v_1{}' = 2 m v \cos\theta_1 \qquad m = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1}{2}$$

$$\cancel{m_1} v_1{}' = \cancel{m_1} v \cos\frac{\chi}{2}$$

$$v_1{}' = v \cos\frac{\chi}{2}$$

$$m_2 v_2{}' = 2 \underset{\frac{m_1}{2}}{m} v \cos\theta_2$$

$$\rightarrow v_2{}' = v \cos\frac{\pi - \chi}{2} = v \sin\frac{\chi}{2}$$

HW. Probs. on page 47, 44, 45

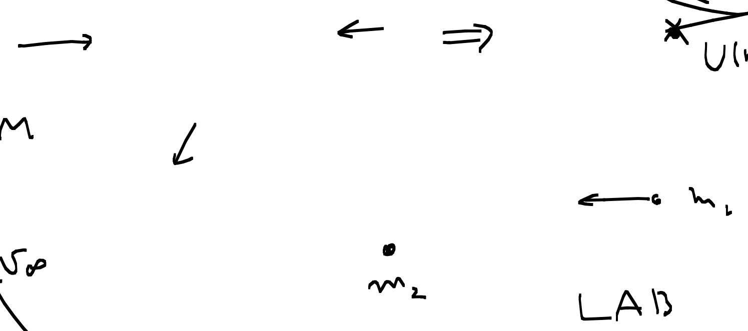
§ 18. scattering

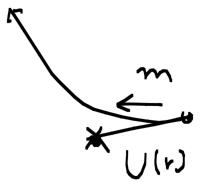
2 - particles $\vec{r}_1, \vec{r}_2 \to U(|\vec{r}_1 - \vec{r}_2|)$

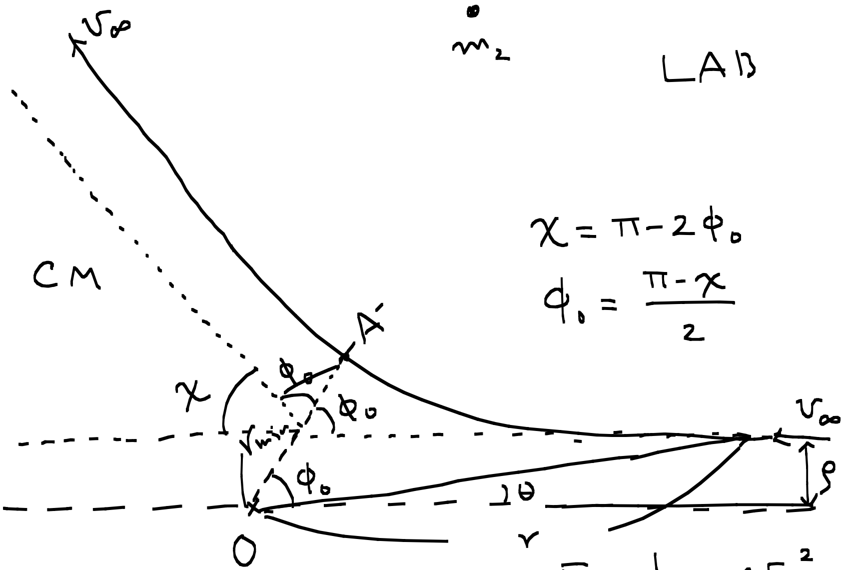
CM frame : $\vec{r} = \vec{r}_1 - \vec{r}_2$

$$L = \frac{1}{2} m \dot{\vec{r}}^2 - U(|\vec{r}|) \ : \text{1-body}$$

$$m \leftarrow \frac{m_1 m_2}{m_1 + m_2}.$$







$$\chi = \pi - 2\phi_0$$

$$\phi_0 = \frac{\pi - \chi}{2}$$

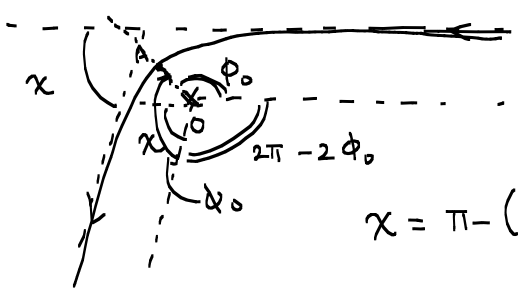
$$E = \frac{1}{2} m v_\infty^2$$

$U(r \to \infty) = 0$ $\quad M = |\vec{M}| = |\vec{r} \times \vec{p}|$

$$= r\, m v_\infty \sin\theta$$

$$M = m v_\infty \rho$$

ρ : impact parameter



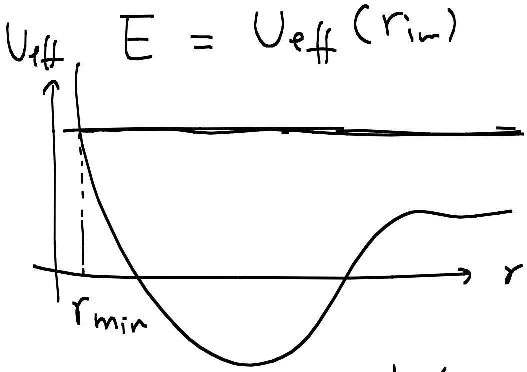
$$\chi = \pi - (2\pi - 2\phi_0) = 2\phi_0 - \pi$$

$$\chi = |\pi - 2\phi_0|$$

$$\boxed{\phi_0 = \int_{r_{min}}^{\infty} \frac{M \, dr/r^2}{\sqrt{2m(E-U(r)) - \frac{M^2}{r^2}}}}$$

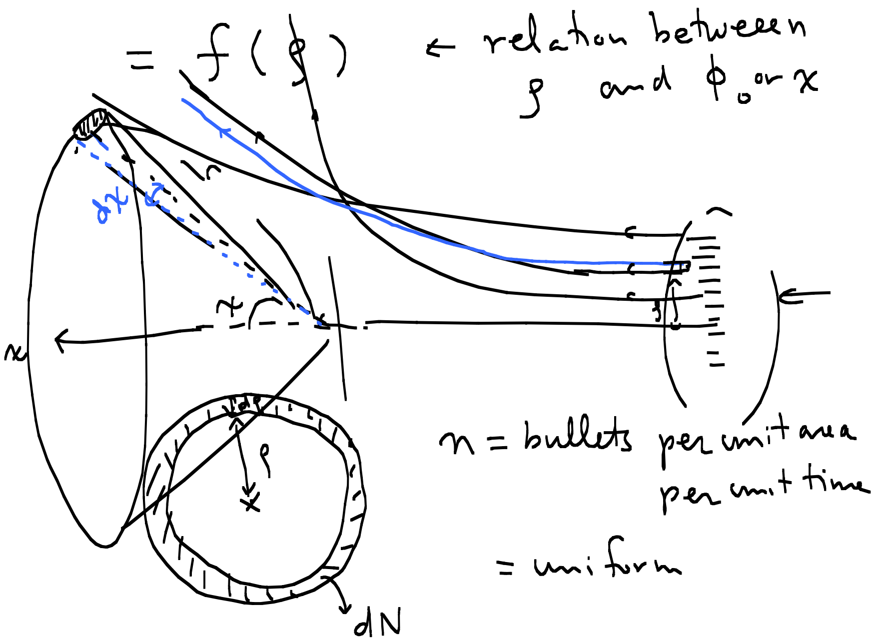
$$\phi_0 = \int_{r_{min}}^{\infty} \frac{m \rho v_\infty \, dr/r^2}{\sqrt{\underbrace{m^2 v_\infty^2 - 2mU - \frac{m^2\rho^2 v_\infty^2}{r^2}}_{=0 \text{ at } r=r_{min}}}}$$

$$E = U_{eff}(r_{min})$$



$$\underbrace{\phi_0 = \int_{r_{min}}^{\infty} \frac{\rho \, dr/r^2}{\sqrt{1 - \frac{\rho^2}{r^2} - \frac{2U}{m v_\infty^2}}} = \frac{\pi - \chi}{2}}$$

$= f(\rho)$ ← relation between ρ and ϕ_0 or χ



n = bullets per unit area per unit time

= uniform

$$dN = n\, dA = n \cdot 2\pi \rho\, d\rho \rightarrow$$

after scattering ; angle between χ and $\chi + d\chi$

$$d\sigma = \frac{dN}{n} = 2\pi \rho\, d\rho$$

$$\frac{d\sigma}{d\chi}(\chi) = ?$$ differential cross section

$$[d\sigma] = \frac{1/[s]}{1/[m^2 \cdot s]} = [m^2]$$

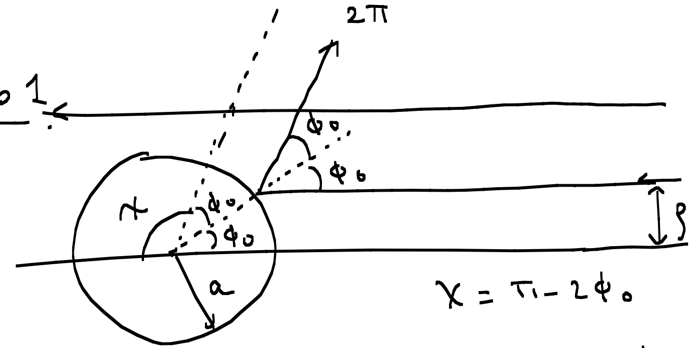
$$d\sigma = 2\pi\rho\, d\rho = 2\pi\rho \frac{d\rho}{d\chi} d\chi$$

$$\therefore \boxed{\frac{d\sigma}{d\chi}(\chi) = 2\pi \rho \left| \frac{d\rho}{d\chi} \right|}$$

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{2\pi \sin\chi\, d\chi} = \frac{\rho}{\sin\chi}\left|\frac{d\rho}{d\chi}\right|$$

$$d\Omega = \sin\chi\, d\chi \underbrace{d\phi}_{2\pi} = 2\pi \sin\chi\, d\chi$$

· Prob 1



$\chi = \pi - 2\phi_0$

$\rho > a$; $\chi = 0$ $\qquad \rho = a \sin\phi_0$

$0 < \rho < a$; $\chi = \pi - 2\phi_0 \rightarrow \phi_0 = \frac{\pi - \chi}{2}$

$$\boxed{\rho = a \cos\frac{\chi}{2}}$$

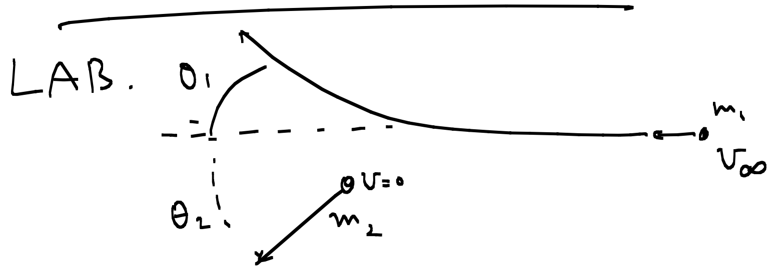
$$\frac{d\rho}{d\chi} = -\frac{a}{2} \sin\frac{\chi}{2}$$

$$\frac{d\sigma}{d\chi} = 2\pi a \cos\frac{\chi}{2}\,\frac{a}{2}\sin\frac{\chi}{2}$$

$$= \pi a^2 \sin\frac{\chi}{2}\cos\frac{\chi}{2} = \frac{\pi a^2}{2}\sin\chi$$

$$\frac{d\sigma}{d\Omega} = \frac{a^2}{4} \longrightarrow \sigma = \int \frac{d\sigma}{d\Omega}\,d\Omega = \frac{a^2}{4}4\pi$$

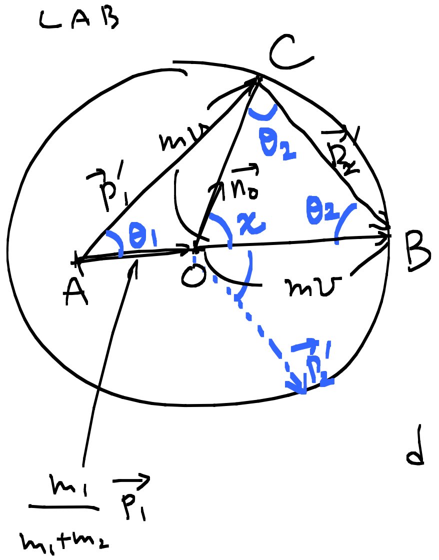
$$= \underline{\pi a^2}$$



$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi} = \frac{\sin\chi}{\frac{m_1}{m_2} + \cos\chi}$$

γ

LAB CM



$$\theta_2 = \frac{\pi - \chi}{2}$$

$$d\sigma = d\sigma_1 = \frac{dN}{n}$$

LAB :
$$\frac{d\sigma_1}{d\theta_1} = \frac{d\sigma}{d\chi}\,\frac{d\chi}{d\theta_1}$$

$$\frac{d\sigma_1}{d\Omega_1} = \frac{1}{2\pi\sin\theta_1}\underbrace{\frac{d\sigma_1}{d\theta_1}}_{\frac{d\sigma}{d\chi}\frac{d\chi}{d\theta_1}}$$

$$= \frac{\sin\chi}{\sin\theta_1}\,\frac{d\chi}{d\theta_1}\,\frac{1}{2\pi\sin\chi}\,\frac{d\sigma}{d\chi}$$

$$\left(\frac{d\sigma_1}{d\Omega_1}\right)_{LAB} = \frac{\sin\chi}{\sin\theta_1}\frac{d\chi}{d\theta_1}\underbrace{\left(\frac{d\sigma}{d\Omega}\right)_{CM}}$$

$$\tan\theta_1 = \frac{\sin\chi}{\gamma+\cos\chi} \qquad \gamma = \frac{m_1}{m_2}$$

$$\gamma \underbrace{\tan\theta_1}_{t_1} + \tan\theta_1\cos\chi = \sqrt{1-\cos^2\chi}$$

$$t_1\,(\gamma+\cos\chi)$$

$$t_1^2\,(\gamma^2+2\gamma\cos\chi+\cos^2\chi) = 1-\cos^2\chi$$

$$\underbrace{(t_1^2+1)}_{\frac{1}{\cos^2\theta_1}}\cos^2\chi + 2\gamma t_1^2\cos\chi + \gamma^2 t_1^2 - 1 = 0$$

$$\underbrace{\cos^2\chi + 2\gamma\sin^2\theta_1\cos\chi} + \gamma^2\sin^2\theta_1 - \cos^2\theta_1 = 0$$

$$\left(\cos\chi + \gamma\sin^2\theta_1\right)^2 - \gamma^2\sin^4\theta_1 = 0$$

$$+\,\gamma^2\sin^2\theta_1 - \cos^2\theta_1$$

$$\gamma^2\sin^2\theta_1\underbrace{(1-\sin^2\theta_1)}_{\cos^2\theta_1}$$

$$\cos^2\theta_1\,(\gamma^2\sin^2\theta_1-1)$$

$$\therefore \cos\chi = -\gamma\sin^2\theta_1 + \cos\theta_1\sqrt{1-\gamma^2\sin^2\theta_1}$$

$$\frac{d\cos\chi}{d\theta_1} = -\sin\chi\frac{d\chi}{d\theta_1}$$

$$= -2\gamma\sin\theta_1\cos\theta_1 - \sin\theta_1\sqrt{}$$

$$+\cos\theta_1\,\frac{1}{2}\,\frac{-2\gamma^2\sin\theta_1\cos\theta_1}{\sqrt{1-\gamma^2\sin^2\theta_1}}$$

$$= -2\gamma\sin\theta_1\cos\theta_1$$

$$-\sin\theta_1\left[\sqrt{} + \frac{\gamma^2\cos^2\theta_1}{\sqrt{1-\gamma^2\sin^2\theta_1}}\right]$$

$$\frac{1-\gamma^2\sin^2\theta_1+\gamma^2\cos^2\theta_1}{\sqrt{}}$$

$$= -2\gamma \sin\theta_1 \cos\theta_1 - \sin\theta_1 \frac{1+\gamma^2\cos 2\theta_1}{\sqrt{1-\gamma^2\sin^2\theta_1}}$$

$$= -\sin x \frac{dx}{d\theta_1}$$

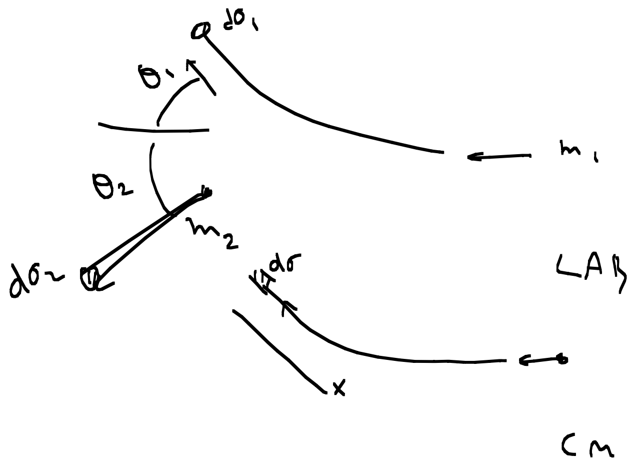
$$\left(\frac{d\sigma_1}{d\Omega_1}\right)_{LAB} = \underbrace{\frac{\sin x}{\sin\theta_1}\,\frac{dx}{d\theta_1}}\;\underbrace{\left(\frac{d\sigma}{d\Omega}\right)_{CM}}_{\uparrow\ \frac{a^2}{4}}$$

$$\downarrow$$

$$2\gamma\cos\theta_1 + \frac{1+\gamma^2\cos 2\theta_1}{\sqrt{}}$$

<u># of m_2 between θ_2 & $\theta_2 + d\theta_2$</u>

$$\theta_2 = \frac{\pi - x}{2}$$



$$\therefore \frac{d\sigma_2}{d\theta_2} = \frac{d\sigma}{dx}\;\frac{dx}{d\theta_2} \qquad 2\pi\sin x\frac{d\sigma}{d\Omega}$$

$$\frac{d\sigma_2}{d\Omega_2} = \frac{1}{2\pi\sin\theta_2}\,\frac{d\sigma_2}{d\theta_2} = \frac{1}{2\pi\sin\theta_2}\,\overline{\frac{d\sigma}{dx}}\,\frac{dx}{d\theta_2}$$

$x = \pi - 2\theta_2$

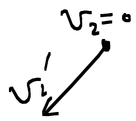
$$= \left|\frac{\sin x}{\sin\theta_2}\,\frac{dx}{d\theta_2}\,\frac{d\sigma}{d\Omega}\right|$$

$$\left.\begin{aligned}&\sin x = \sin 2\theta_2 = 2\sin\theta_2\cos\theta_2\\ &\frac{dx}{d\theta_2} = -2\end{aligned}\right\}$$

$$= 4\left|\cos\theta_2\right|\frac{a^2}{4} = a^2\left|\cos\theta_2\right|$$

[Prob 2]

$$v_2' = \frac{2m_1 v}{m_1+m_2}\sin\frac{\chi}{2}$$



$\leftarrow v_1$

after collision

$$E_2 = \frac{1}{2} m_2 v_2'^2 = \frac{4 m_1^2 v^2 \sin^2\frac{\chi}{2}}{(m_1+m_2)^2}\,\frac{m_2}{2}$$

energy loss of m_1 = ϵ

$$\epsilon = \underbrace{\frac{2 m_1^2 m_2}{(m_1+m_2)^2} v^2}_{\epsilon_{max}} \sin^2\frac{\chi}{2}$$

$$\boxed{\epsilon = \epsilon_{max}\sin^2\frac{\chi}{2}}$$

$$\frac{d\sigma}{d\epsilon} = \frac{d\sigma}{d\Omega}\frac{d\Omega}{d\epsilon} = \underbrace{\frac{d\sigma}{d\Omega}}_{=\frac{a^2}{4}} 2\pi \cancel{\sin\chi}\,\underbrace{\frac{d\chi}{d\epsilon}}_{\frac{2}{\epsilon_{max}\cancel{\sin\chi}}}$$

$$(d\Omega = 2\pi \sin\chi\, d\chi)$$

$$\frac{d\epsilon}{d\chi} = \epsilon_{max}\,\cancel{2}\sin\frac{\chi}{2}\cos\frac{\chi}{2}\cdot\frac{1}{\cancel{2}} = \frac{\epsilon_{max}}{2}\sin\chi$$

$$\therefore \boxed{\frac{d\sigma}{d\epsilon} = \frac{\pi a^2}{\epsilon_{max}}}$$

Prob 3. $U \sim \frac{1}{r^n}$

$$U(\alpha r) = \alpha^k U(r) \rightarrow k=-n$$

$$\frac{v'}{v} = \left(\frac{l'}{l}\right)^{\frac{k}{2}} \qquad \frac{E'}{E} = \left(\frac{l'}{l}\right)^{k}$$

$$\frac{l'}{l} = \left(\frac{v'}{v}\right)^{\frac{2}{k}} \rightarrow \rho \sim (v_\infty)^{\frac{2}{k} = -\frac{2}{n}}$$

$$\rho = v_\infty^{-\frac{2}{n}} f(\chi)$$

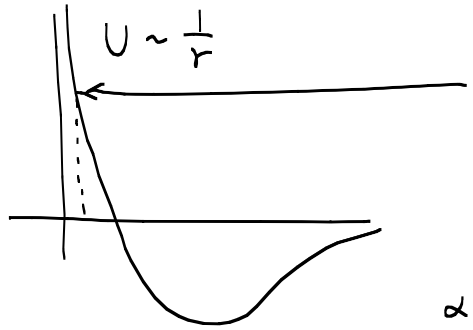
$$\frac{d\sigma}{d\Omega} = \frac{\rho}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| = \left(v_\infty^{-\frac{2}{n}}\right)^2 g(\chi)$$

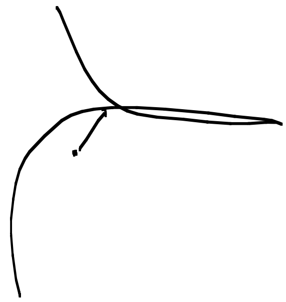
$$\sim v_\infty^{-\frac{4}{n}}$$

<u>Prob 4</u>. $U = -\dfrac{\alpha}{r^2}$

cross section of "fall" to the center $r=0$

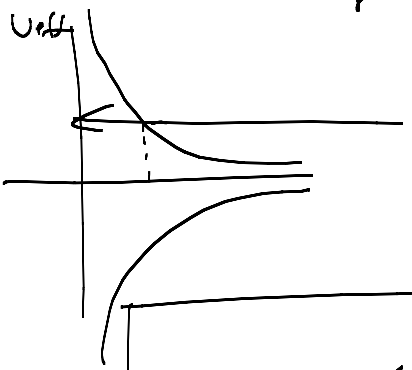
$$U_{eff} = U(r) + \frac{M^2}{2m}\frac{1}{r^2}$$





$$\alpha > \frac{M^2}{2m}$$

$$U_{eff} = \frac{1}{r^2}\underbrace{\left(\frac{M^2}{2m} - \alpha\right)}_{\text{negative}}$$



$$\alpha > \frac{(m\rho v_\infty)^2}{2m} = \frac{m v_\infty^2}{2}\rho^2$$

$$\boxed{0 \le \rho \le \sqrt{\frac{2\alpha}{m v_\infty^2}}} = \rho_{max}$$

$$\frac{d\sigma}{d\rho} = 2\pi\rho$$

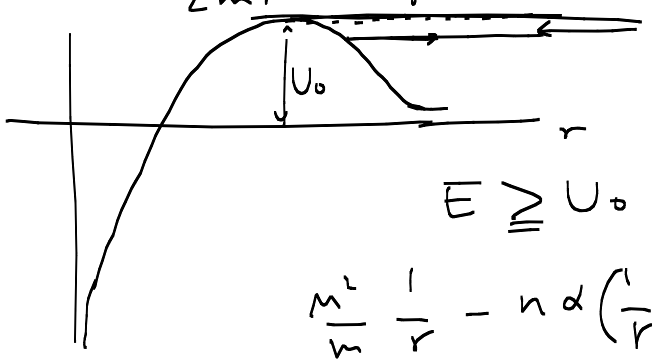
$$\sigma = \int_0^{\rho_{max}} \frac{d\sigma}{d\rho}\, d\rho = \pi\rho^2\Big|_0^{\rho_{max}}$$

$$= \pi\rho_{max}^2$$

$$\underline{\underline{\sigma = \pi \frac{2\alpha}{m v_\infty^2}}}$$

Prm 5. $U = -\dfrac{\alpha}{r^n} \quad (n > 2,\ \alpha > 0)$

$$U_{eff} = \frac{M^2}{2mr^2} - \frac{\alpha}{r^n} \leftarrow E$$



$$E \geqq U_0$$

$$\frac{M^2}{m}\frac{1}{r} - n\alpha\left(\frac{1}{r}\right)^{n-1} = 0$$

$$\left(\frac{1}{r}\right) = \left(\frac{M^2}{mn\alpha}\right)^{\frac{1}{n-2}}$$

$$U_0 = \frac{M^2}{2m}\left(\frac{M^2}{mn\alpha}\right)^{\frac{2}{n-2}} - \alpha\left(\frac{M^2}{mn\alpha}\right)^{\frac{n}{n-2}}$$

$$\frac{2}{n-2} = \underbrace{\frac{2-n}{n-2}}_{=-1} + \frac{n}{n-2}$$

$$= \frac{\cancel{M^2}}{2\cancel{m}}\underbrace{\left(\frac{M^2}{mn\alpha}\right)^{-1+\frac{n}{n-2}}}_{\frac{\cancel{m}n\alpha}{\cancel{M^2}}\cdot\left(\frac{M^2}{mn\alpha}\right)^{\frac{n}{n-2}}} - \alpha\left(\frac{M^2}{mn\alpha}\right)^{\frac{n}{n-2}}$$

$$U_0 = \left(\frac{(m\rho v_\infty)^2}{mn\alpha}\right)^{\frac{n}{n-2}} \cdot \frac{n-2}{2} \cdot \alpha$$

$$\leqq \frac{1}{2} m v_\infty^2$$

$$0 \leqq \rho \leqq \rho_{max}$$

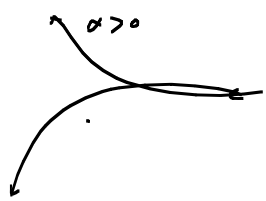
$$\left(\rho_{max}^2 \frac{m v_\infty^2}{n\alpha} \right)^{\frac{n}{n-2}} = \frac{m}{n-2} \frac{v_\infty^2}{\alpha}$$

$$\rho_{max} = \sqrt{\frac{n\alpha}{m v_\infty^2}} \left(\frac{m}{n-2} \frac{v_\infty^2}{\alpha} \right)^{\frac{n-2}{2n}}$$

$$\sigma = \pi \rho_{max}^2$$
$$= \pi \frac{n\alpha}{m v_\infty^2} \left(\frac{m}{n-2} \frac{v_\infty^2}{\alpha} \right)^{\frac{n-2}{n}} //$$

§19. Rutherfurd.

$$U(r) = \frac{\alpha}{r}$$



Note Title 2015-04-03

Kepler's problem

$$U(r) = +\frac{\alpha}{r}$$

$$\phi = \int \frac{M \quad dr/r^2}{\sqrt{2m\left[E - \frac{1}{2} m r^2 \left(\frac{M}{m r^2}\right)^2 - \underbrace{\frac{\alpha}{r}}_{U}\right]}}$$

$$\frac{1}{r} \equiv u \qquad du = -\frac{dr}{r^2}$$

$$\phi = \int \frac{(-M)\, du}{\sqrt{2m}\sqrt{E - \frac{M^2}{2m} u^2 + \alpha u}}$$

$$E - \frac{M^2}{2m} u^2 + \alpha u = E - \frac{M^2}{2m}\Big(\underbrace{u^2 - \frac{2m\alpha}{M^2} u}_{\left(u - \frac{m\alpha}{M^2}\right)^2 - \frac{m^2\alpha^2}{M^4}}\Big)$$

$$\sqrt{} = \sqrt{E + \frac{m\alpha^2}{2M^2} - \frac{M^2}{2m}\left(u - \frac{m\alpha}{M^2}\right)^2}$$

$$= \sqrt{E + \frac{m\alpha^2}{2M^2}}\ \sqrt{1 - \underbrace{\frac{M^2/2m}{E + \frac{m\alpha^2}{2M^2}}\left(u - \frac{m\alpha}{M^2}\right)^2}_{\equiv v^2}}$$

$$v = \sqrt{\frac{M^2/2m}{E + \frac{m\alpha^2}{2M^2}}}\left(u - \frac{m\alpha}{M^2}\right)$$

$$dv \equiv \sqrt{\frac{M^2}{2m\left(E+\frac{m\alpha^2}{2M^2}\right)}}\; du$$

$$\phi = -\frac{\cancel{M}}{\cancel{\sqrt{2m}}}\,\frac{1}{\cancel{\sqrt{E+\frac{m\alpha^2}{2M^2}}}}\int \frac{\cancel{\sqrt{2m\left(E+\frac{m\alpha^2}{2M^2}\right)/M^2}}\;dv}{\sqrt{1-v^2}}$$

$$= -\int \frac{dv}{\sqrt{1-v^2}} \qquad \underline{v \equiv \sin\theta}$$

$$dv = \cos\theta\, d\theta$$

$$\sqrt{1-v^2} = \cos\theta$$

$$\phi = -\int \frac{\cancel{\cos}\theta\, d\theta}{\cancel{\cos}\theta} = -\theta + \phi_0$$

$$\phi - \phi_0 = -\theta = -\sin^{-1} v$$

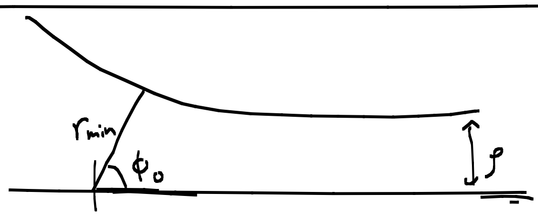
$$v = \sqrt{\frac{M^2/2m}{E+\frac{m\alpha^2}{2M^2}}}\left(\frac{1}{r} - \frac{m\alpha}{M^2}\right)$$

$$v = -\sin(\phi - \phi_0)$$

$$\frac{1}{r} = \frac{m\alpha}{M^2} - \sqrt{\frac{E+\frac{m\alpha^2}{2M^2}}{M^2/2m}}\,\sin(\phi-\phi_0)$$

$\alpha \to -\alpha$

$$\boxed{\frac{1}{r} = -\frac{m\alpha}{M^2} + \sqrt{\frac{E+\frac{m\alpha^2}{2M^2}}{M^2/2m}}\,\cos(\phi-\theta_0)}$$



$r=\infty \quad \phi=0$

$$\cos\theta_0 = \frac{m\alpha/M^2}{\sqrt{\quad}}$$

"

$$\frac{1}{r_{min}} \Rightarrow \frac{1}{r} = \max \to \cos(\phi-\theta_0)=1$$

$$\phi_0=\theta_0$$

$$\therefore \cos\phi_0 = \frac{m\alpha/M^2}{\sqrt{\quad}}$$

"

$$\phi_0 = \cos^{-1}\left(\frac{m\alpha/M^2}{\sqrt{\frac{E+\frac{m\alpha^2}{2M^2}}{M^2/2m}}}\right)$$

$M = m\rho v_\infty$

$E = \frac{1}{2}mv_\infty^2$

$$\phi_0 = \cos^{-1}\left(\frac{\alpha/m\rho v_\infty^2}{\sqrt{1+\frac{\alpha^2}{m^2 v_\infty^4\rho^2}}}\right)$$

$$\cos^2\phi_0 = \frac{(\alpha/m\rho v_\infty^2)^2}{1+\left(\frac{\alpha}{mv_\infty^2\rho}\right)^2}$$

$$\sin^2\phi_0 = 1-\cos^2\phi_0 = \frac{1}{1+\left(\frac{\alpha}{mv_\infty^2\rho}\right)^2}$$

$$\tan^2\phi_0 = \left(\frac{m\rho v_\infty^2}{\alpha}\right)^2$$

"

$$\rho = \frac{\alpha}{mv_\infty^2}\tan\phi_0$$

$$\phi_0 = \frac{\pi-\chi}{2}$$

$$\rho = \underbrace{\frac{\alpha}{m v_\infty^2}}_{a} \cot\frac{\chi}{2}$$

$$\frac{d\sigma}{d\chi}(\chi) = 2\pi\rho \left|\frac{d\rho}{d\chi}\right|$$

$$\frac{d\rho}{d\chi} = \frac{\alpha}{m v_\infty^2}\,\frac{1}{2}\,\frac{1}{\sin^2\frac{\chi}{2}}$$

$$\therefore\ \frac{d\sigma}{d\chi} = 2\pi a^2 \cot\frac{\chi}{2}\,\frac{1}{2}\,\frac{1}{\sin^2\frac{\chi}{2}}$$

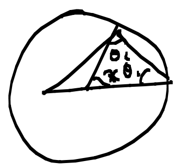
$$\frac{d\sigma}{d\chi} = \pi a^2 \frac{\cos\frac{\chi}{2}}{\sin^3\frac{\chi}{2}}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi\sin\chi}\frac{d\sigma}{d\chi} = \frac{a^2}{2}\frac{\cancel{\cos\frac{\chi}{2}}}{\underbrace{\sin\chi}_{2\sin\frac{\chi}{2}\cancel{\cos\frac{\chi}{2}}}\sin^3\frac{\chi}{2}}$$

$$= \frac{a^2}{4}\frac{1}{\sin^4\frac{\chi}{2}}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{4}\frac{\alpha^2}{m^2 v_\infty^4}\frac{1}{\sin^4\frac{\chi}{2}}}$$

CM



$$\chi = \pi - 2\theta_2$$

LAB.

$$\frac{d\sigma_2}{d\theta_2} = \frac{d\sigma}{d\chi}\underbrace{\left|\frac{d\chi}{d\theta_2}\right|}_{2}$$

$$\theta_2 = \frac{\pi-\chi}{2}$$

$$\frac{d\sigma_2}{d\theta_2} = 2\pi a^2 \frac{\cos\frac{\chi}{2}}{\sin^3\frac{\chi}{2}}$$

$$\frac{d\sigma_2}{d\Omega_2} = 2\cancel{\pi} a^2 \frac{\cos\frac{\chi}{2}}{\sin^3\frac{\chi}{2}} \cdot \frac{1}{2\cancel{\pi}\sin\theta_2}$$

$$= a^2 \frac{1}{\sin^3\frac{\pi-2\theta_2}{2}} = \frac{a^2}{\cos^3\theta_2}$$

LAB. for m_1

$$\tan\theta_1 = \frac{\sin\chi}{\gamma + \cos\chi}$$

$$\frac{d\sigma_1}{d\theta_1} = \underbrace{\frac{d\sigma}{d\chi}}_{\pi a^2 \frac{\cos\frac{\chi}{2}}{\sin^3\frac{\chi}{2}}} \frac{d\chi}{d\theta_1}$$

If. $m_2 \gg m_1 \quad \gamma = \frac{m_1}{m_2} \ll 1 \quad \theta_1 \approx \chi$

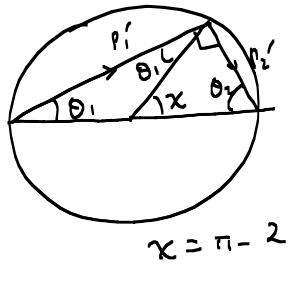
$$\therefore \quad \frac{d\sigma_1}{d\theta_1} \approx \pi a^2 \frac{\cos\frac{\theta_1}{2}}{\sin^3\frac{\theta_1}{2}}$$

$$\frac{d\sigma_1}{d\Omega_1} = \frac{1}{2\pi\sin\theta_1}\frac{d\sigma_1}{d\theta_1} = \frac{a^2}{4}\frac{1}{\sin^4\frac{\theta_1}{2}}$$

$$a = \frac{\alpha}{m v_\infty^2} = \frac{\alpha/2}{E_1}$$

If $m_1 = m_2 \longrightarrow m = \frac{m_1}{2}$

$\chi = 2\theta_1$



$\chi = \pi - 2\theta_2$

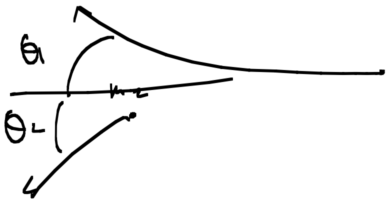
$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\frac{\alpha^2}{m^2 v_\infty^4}\frac{1}{\sin^4\frac{\chi}{2}}$$

$$\frac{d\sigma_1}{d\theta_1} = \underbrace{\frac{d\sigma}{d\chi}}_{= 2\pi\sin\chi\frac{d\sigma}{d\Omega}} \underbrace{\frac{d\chi}{d\theta_1}}_{2} = 2\pi\frac{\alpha^2}{m^2 v_\infty^4}\frac{\cos\theta_1}{\sin^3\theta_1}$$

$$\frac{d\sigma_2}{d\theta_2} = \underbrace{\frac{d\sigma}{d\chi}}_{2\pi \sin\chi \frac{d\sigma}{d\Omega}} \frac{d\chi}{d\theta_2} = 2\pi \frac{\alpha^2}{m^2 v_\infty^4} \frac{\sin\theta_2}{\cos^3\theta_2}$$

$$\boxed{\begin{aligned} \frac{d\sigma_1}{d\Omega_1} &= \frac{\alpha^2}{m^2 v_\infty^4} \frac{\cos\theta_1}{\sin^4\theta_1} \\ \frac{d\sigma_2}{d\Omega_2} &= \frac{\alpha^2}{m^2 v_\infty^4} \frac{1}{\cos^3\theta_2} \end{aligned}}$$

$m_1 = m_2$, indistinguishable



$$\left.\frac{d\sigma}{d\Omega}\right|_{LAB} = \frac{\alpha^2}{m^2 v_\infty^4} \left(\frac{1}{\sin^4\theta} + \frac{1}{\cos^4\theta} \right) \cos\theta$$

In general, CM

$$\frac{d\sigma}{2\pi \sin\chi\, d\chi} = \frac{1}{4} \frac{\alpha^2}{m^2 v_\infty^4} \frac{1}{\sin^4 \frac{\chi}{2}}$$

α

$$\epsilon = \frac{1}{2} m_2 v_2'^2 = \underbrace{2 \frac{m_1^2 m_2}{(m_1+m_2)^2} v^2}_{\beta} \sin^2 \frac{\chi}{2}$$

$$d\epsilon = 4 \frac{m_1^2 m_2}{(m_1+m_2)^2} v^2 \underbrace{\sin\frac{\chi}{2} \cdot \frac{1}{2} \cos\frac{\chi}{2}}_{\frac{\sin\chi}{4}} d\chi$$

$$\sin\chi\, d\chi = \frac{1}{\alpha} d\epsilon \qquad \sin^2\frac{\chi}{2} = \frac{\epsilon}{2\beta}$$

$$\frac{\beta}{2\pi} \frac{d\sigma}{d\epsilon} = \frac{1}{\cancel{4}} \frac{\alpha^2}{m^2 v_\infty^4} \frac{\cancel{4}\beta^2}{\epsilon^2}$$

$$\frac{d\sigma}{d\epsilon} = 2\pi \frac{\alpha^2}{v_\infty^4} \left(\frac{m_1 + \cancel{m_2}}{m_1 m_2}\right)^2 \frac{1}{\epsilon^2} \frac{m_1^2 \cancel{m_2} v_\infty^2}{(m_1 + \cancel{m_2})^2}$$

$$\left(m = \frac{m_1 m_2}{m_1 + m_2} \right)$$

$$\boxed{\frac{d\sigma}{d\epsilon} = \frac{2\pi\alpha^2}{v_\infty^2 m_2} \frac{1}{\epsilon^2}}$$

"

Prob 1. $U = \frac{\alpha}{r^2}$

$$\phi = \int \frac{M \, dr/r^2}{\sqrt{2m\left[E - \frac{M^2 + 2m\alpha}{2mr^2}\right]}} \qquad M^2 + 2m\alpha \equiv \tilde{M}^2$$

$$\tilde{M}^2 = M^2 + 2m\alpha$$

$$\phi = \int \frac{M \, dr/r^2}{\sqrt{2mE - \frac{\tilde{M}^2}{r^2}}} \qquad dr/r^2 = -du$$

$$\frac{1}{r} \equiv u \qquad du = -\frac{1}{r^2} dr$$

$$= -\int \frac{M \, du}{\sqrt{2mE}\sqrt{1 - \frac{\tilde{M}^2}{2mE} u^2}}$$

$$\frac{\tilde{M}^2}{2mE} u^2 \equiv v^2$$

$$dv = \frac{\tilde{M}}{\sqrt{2mE}} du$$

$$= -\frac{M}{\tilde{M}} \int \frac{dv}{\sqrt{1 - v^2}}$$

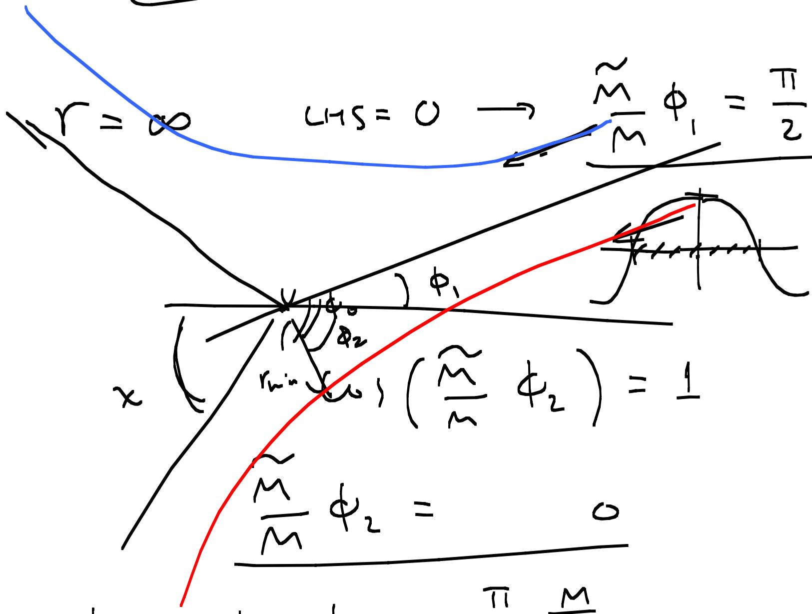
$$v = \cos\beta \qquad dv = -\sin\beta \, d\beta$$

$$= \frac{M}{\tilde{M}} \int \frac{\cancel{\sin}\beta \, d\beta}{\cancel{\sin}\beta}$$

$$\phi = \frac{M}{\tilde{M}} \beta \rightarrow \beta = \frac{\tilde{M}}{M} \phi$$

$$v = \frac{\tilde{M}}{\sqrt{2mE}} \cdot \frac{1}{r} = \cos\left(\frac{\tilde{M}}{M} \phi\right)$$

$$\boxed{\frac{1}{r} = \frac{\sqrt{2mE}}{\tilde{M}} \cos\left(\frac{\tilde{M}}{M}\phi\right)}$$



$$\text{LHS} = 0 \longrightarrow \frac{\tilde{M}}{M}\phi_1 = \frac{\pi}{2}$$

$$\cos\left(\frac{\tilde{M}}{M}\phi_2\right) = 1$$

$$\frac{\tilde{M}}{M}\phi_2 = 0$$

$$\phi_0 = \phi_2 + \phi_1 = \frac{\pi}{2}\frac{M}{\tilde{M}}$$

$$\chi = \pi - 2\phi_0 = \pi - \pi\frac{M}{\tilde{M}} = \pi\left(1 - \frac{M}{\tilde{M}}\right)$$

$$\tilde{M}^2 = M^2 + 2m\alpha$$

$$\tilde{M} = \sqrt{M^2 + 2m\alpha}$$

$$\frac{M}{\tilde{M}} = \frac{1}{\frac{\tilde{M}}{M}} = \frac{1}{\sqrt{1 + \frac{2m\alpha}{M^2}}} = \frac{1}{\sqrt{1 + \frac{2\alpha}{m v_\infty^2 \rho^2}}}$$

$$M = m\rho v_\infty$$

$$\boxed{\chi = \pi\left(1 - \frac{1}{\sqrt{1 + \frac{2\alpha}{m v_\infty^2 \rho^2}}}\right)}$$

$$\frac{d\sigma}{d\chi}(\chi) = 2\pi\rho\left|\frac{d\rho}{d\chi}\right|$$

$$\left(\left(\frac{\chi}{\pi} - 1\right)^{-2} - 1\right)^{-1} = \left(\frac{2\alpha}{m v_\infty^2 \rho^2}\right)^{-1}$$

$$= \frac{\rho^2 m v_\infty^2}{2\alpha}$$

$$\rho = \sqrt{\frac{2\alpha}{m v_\infty^2}} \; \frac{\pi - \chi}{\sqrt{2\pi\chi - \chi^2}}$$

$$\frac{d\rho}{d\chi} = \sqrt{\;''\;} \left[-\frac{1}{\sqrt{\;\;}} - \frac{(\pi - \chi)^2}{\sqrt{\;\;}^{\,3}} \right]$$

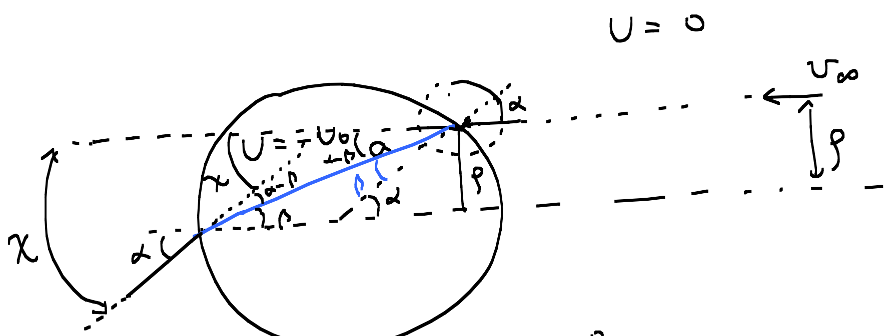
$$\left| \frac{d\rho}{d\chi} \right| = \sqrt{\;''\;} \; \frac{\cancel{2\pi\chi} - \cancel{\chi^2} + (\pi^2 - \cancel{2\pi\chi} + \cancel{\chi^2})}{\sqrt{\;\;}^{\,3}}$$

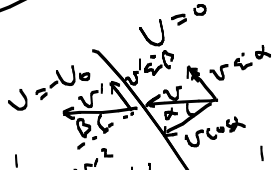
$$= \sqrt{\;''\;} \; \frac{\pi^2}{\left(\sqrt{2\pi\chi - \chi^2}\right)^3}$$

$$\frac{d\sigma}{d\chi}(\chi) = 2\pi\rho \left| \frac{d\rho}{d\chi} \right| = 4\pi^3 \frac{\alpha}{m v_\infty^2} \frac{(\pi - \chi)}{(2\pi\chi - \chi^2)^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi \sin\chi} \frac{d\sigma}{d\chi} = \frac{2\pi^2}{\sin\chi} \frac{\alpha}{m v_\infty^2} \frac{\pi - \chi}{\chi^2 (2\pi - \chi)^2}$$

Prob 2





$$\frac{1}{2} m v'^2 - U_0 = \frac{1}{2} m v^2$$

$$v'^2 = v^2 + \frac{2U_0}{m}$$

$$v' \sin\beta = v \sin\alpha$$

$$\frac{\sin\alpha}{\sin\beta} = \frac{v'}{v} = n = \frac{\sqrt{v^2 + \frac{2U_0}{m}}}{v}$$

$$= \sqrt{1 + \frac{2U_0}{m v_\infty^2}}$$

$$\chi = 2(\alpha - \beta) \longrightarrow \beta = \alpha - \frac{\chi}{2}$$

$$\frac{\sin\left(\alpha - \frac{\chi}{2}\right)}{\sin\alpha} = \frac{1}{\sqrt{}}$$

$$\shortparallel$$

$$\frac{\sin\alpha \cos\frac{\chi}{2} - \cos\alpha \sin\frac{\chi}{2}}{\sin\alpha}$$

$$\shortparallel$$

$$\cos\frac{\chi}{2} - \cot\alpha \sin\frac{\chi}{2} = \frac{1}{\sqrt{}}$$

$$\rho = a \sin\alpha$$

$$\cot^2\alpha = \frac{1 - \sin^2\alpha}{\sin^2\alpha} = \frac{1 - \left(\frac{\rho}{a}\right)^2}{\left(\frac{\rho}{a}\right)^2}$$

$$\cot\alpha = \frac{\sqrt{}}{\frac{\rho}{a}} = \sqrt{\left(\frac{a}{\rho}\right)^2 - 1}$$

$$\cos\frac{\chi}{2} - \sqrt{\left(\frac{a}{\rho}\right)^2 - 1}\ \sin\frac{\chi}{2} = \frac{1}{\sqrt{}} = \frac{1}{n}$$

$$\cos\frac{\chi}{2} - \frac{1}{n} = \sqrt{\left(\frac{a}{\rho}\right)^2 - 1}\ \sin\frac{\chi}{2}$$

$$\cos^2\frac{\chi}{2} + \frac{1}{n^2} - \frac{2\cos\frac{\chi}{2}}{n} = \left(\left(\frac{a}{\rho}\right)^2 - 1\right)\sin^2\frac{\chi}{2}$$

$$\left(\frac{a}{\rho}\right)^2 = \frac{1}{\sin^2\frac{\chi}{2}}\left(\underbrace{1 + \frac{1}{n^2} - \frac{2\cos\frac{\chi}{2}}{n}}_{\frac{n^2 - 2n\cos\frac{\chi}{2} + 1}{n^2}}\right)$$

$$\rho^2 = \frac{n^2 a^2 \sin^2\frac{\chi}{2}}{n^2 - 2n\cos\frac{\chi}{2} + 1}$$

$$\frac{d\sigma}{d\chi}(\chi) = 2\pi\rho\left|\frac{d\rho}{d\chi}\right|$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi\sin\chi}\frac{d\sigma}{d\chi}$$

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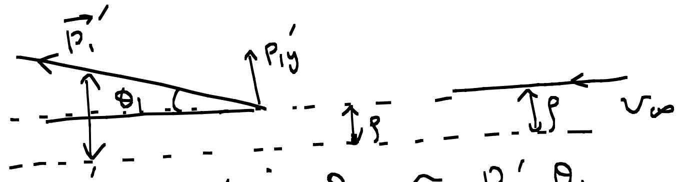
§20. small-angle scattering

LAB



$$p_{1y} = p_1'\sin\theta_1 \approx p_1'\theta_1$$

$$p_1' \cong p_1 = m_1 v_\infty$$

$$\therefore \quad \theta_1 \approx \frac{p_{1y}'}{m_1 v_\infty}$$

$$\dot{p}_y = F_y \longrightarrow \Delta p_y = p_{1y}' = \int_{-\infty}^{\infty} F_y\,dt$$

$$F_y = -\frac{\partial U(r)}{\partial y} = -U'\frac{\partial r}{\partial y} = -U'\frac{y}{r}$$

$$r = \sqrt{x^2+y^2}$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{}} = \frac{y}{r}$$

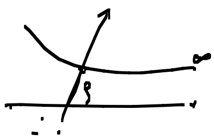
$$F_y = -U'\frac{y}{r} \approx -U'\frac{\rho}{r}$$

$$p_{1y}' = -\rho\int U'\frac{1}{r}\,dt = -\frac{\rho}{v_\infty}\int U'\frac{dx}{r}$$

$$\frac{dx}{dt} = v_\infty$$

$$r^2 = x^2+y^2 \approx x^2+\rho^2 \qquad x = \sqrt{r^2-\rho^2}$$

$$dx = \frac{r}{\sqrt{r^2-\rho^2}}\,dr \rightarrow \frac{dx}{r} = \frac{1}{\sqrt{r^2-\rho^2}}\,dr$$



$$\boxed{\theta_1 = -\frac{2\rho}{m_1 v_\infty^2}\int_\rho^\infty U' \frac{1}{\sqrt{r^2-\rho^2}}\,dr}$$

$$\frac{d\sigma}{d\theta_1} = 2\pi\rho \left|\frac{d\rho}{d\theta_1}\right|$$

$$\frac{d\sigma}{d\Omega_1} \simeq \frac{\rho}{\theta_1}\left|\frac{d\rho}{d\theta_1}\right|$$

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{\rho/r^2\,dr}{\sqrt{1-\frac{\rho^2}{r^2}-\frac{2U}{mv_\infty^2}}}$$

$$= -\frac{\partial}{\partial\rho}\int \sqrt{1-\frac{\rho^2}{r^2}-\frac{2U}{mv_\infty^2}}\,dr$$

if U is small

$$\sqrt{1-\frac{\rho^2}{r^2}}\ \sqrt{1-\frac{2U}{mv_\infty^2\left(1-\frac{\rho^2}{r^2}\right)}}$$

$$\simeq \sqrt{1-\frac{\rho^2}{r^2}}\left(1-\frac{U}{mv_\infty^2\left(1-\frac{\rho^2}{r^2}\right)}+\cdots\right)$$

$$\phi_0 \cong -\frac{\partial}{\partial\rho}\int\sqrt{1-\frac{\rho^2}{r^2}}\,dr = \underbrace{\int_\rho^\infty \frac{\rho\,dr}{r^2\sqrt{1-\frac{\rho^2}{r^2}}}}_{\frac{\pi}{2}}$$

$$+\frac{\partial}{\partial\rho}\underbrace{\int \frac{U/mv_\infty^2}{\sqrt{1-\frac{\rho^2}{r^2}}}\,dr}_{-\frac{1}{mv_\infty^2}\int U\sqrt{\ }} + \cdots$$

$$r = \frac{\rho}{\sin\theta} \qquad dr = -\frac{\rho\cos\theta}{\sin^2\theta}\,d\theta$$

$$\int_{\frac{\pi}{2}}^{0} \frac{-\cancel{\rho^2}\cancel{\cos\theta}/\cancel{\sin^2\theta}\,d\theta}{\cancel{\rho^2}/\cancel{\sin^2\theta}\ \cancel{\cos\theta}} = \frac{\pi}{2}$$

$$\frac{1}{mv_\infty^2}\underbrace{\int U \frac{1}{\sqrt{\left(\frac{r^2-\rho^2}{r^2}\right)}}\, dr}_{} \qquad = \frac{r}{\sqrt{r^2-\rho^2}}$$

$$= \int U \frac{r}{\sqrt{r^2-\rho^2}}\, dr$$

$$= \cancel{U \sqrt{r^2-\rho^2}\Big|_{\rho}^{\infty}} - \int U' \sqrt{r^2-\rho^2}\, dr$$

$$\frac{\partial}{\partial\rho}\int U'\sqrt{r^2-\rho^2}\, dr = -\int \frac{U'\rho}{\sqrt{r^2-\rho^2}}\, dr$$

$$\therefore\ \phi_0 = \frac{\pi}{2} + \frac{\rho}{mv_\infty^2}\int \frac{U'}{\sqrt{r^2-\rho^2}}\, dr$$

$$\theta_1 = \pi - 2\phi_0 = -\frac{2\rho}{mv_\infty^2}\int \ 〃$$

(cf) $$\theta_1 = -\frac{2\rho}{m_1 v_\infty^2}\int U' \frac{1}{\sqrt{r^2-\rho^2}}\, dr$$

<u>Prob2</u>. $U = \dfrac{\alpha}{r^n}\quad (n>0)$

$$\theta_1 = -\frac{2\rho}{mv_\infty^2}\int_\rho^\infty \frac{-n\alpha}{r^{n+1}}\frac{1}{\sqrt{r^2-\rho^2}}\, dr$$

$$r = \frac{\rho}{\sin\theta} \qquad dr = -\frac{\rho\cos\theta}{\sin^2\theta}\, d\theta$$

$$r^2-\rho^2 = \rho^2\left(\frac{1}{\sin^2\theta}-1\right) = \frac{\cos^2\theta}{\sin^2\theta}\rho^2$$

$$= -\frac{\overset{(-\rho)}{2\rho(-n\alpha)}}{mv_\infty^2}\int_{\frac{\pi}{2}}^{0} \frac{\sin^{n+1}\theta}{\rho^{n+1}}\frac{\cos\theta}{\sin^2\theta}\frac{\sin\theta\, d\theta}{\rho\cos\theta}$$

$$= \frac{2\cancel{n}\alpha}{mv_\infty^2\rho^n}\underbrace{\int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta}_{\frac{\sqrt{\pi}}{\cancel{2}}\frac{\Gamma\left(\frac{1+n}{2}\right)}{\frac{\cancel{n}}{\cancel{2}}\Gamma\left(\frac{n}{2}\right)}}$$

$$= \frac{2\alpha\sqrt{\pi}}{mv_\infty^2\rho^n}\frac{\Gamma\left(\frac{1+n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \ //$$

$$\Gamma(1+x) = x\,\Gamma(x)$$

$$\Gamma\left(1+\frac{n}{2}\right) = \frac{n}{2}\,\Gamma\left(\frac{n}{2}\right)$$

$$\theta_1 = \frac{c}{\rho^n} \rightarrow \rho = \left(\frac{c}{\theta_1}\right)^{1/n}$$

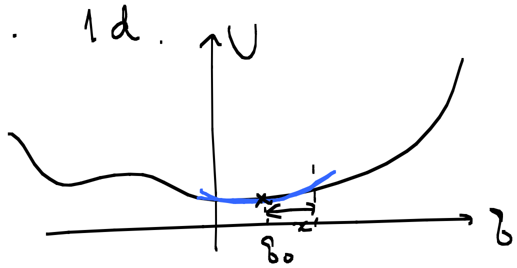
$$1 = -nc\,\underbrace{\frac{1}{\rho^{n+1}}}_{\left(\frac{\theta_1}{c}\right)^{\frac{n+1}{n}}}\,\frac{d\rho}{d\theta_1}$$

$$\left|\frac{d\rho}{d\theta_1}\right| = \frac{1}{nc}\left(\frac{\theta_1}{c}\right)^{-\frac{n+1}{n}}$$

$$\frac{d\sigma}{d\Omega_1} = \frac{1}{2\pi\,\theta_1}\left|\frac{d\rho}{d\theta_1}\right|\left(\frac{c}{\theta_1}\right)^{\frac{1}{n}}$$

6/12 2pm – 5pm : Final exam.
(open book, closed note)

§21. 1d.



$q - q_0 = x$

$$U(q) = \underbrace{U(q_0)}_{0} + \cancel{U'(q_0)}^{\,0}(q-q_0) + \frac{1}{2}\underbrace{U''(q_0)}(q-q_0)^2 \ldots$$

$$U = \frac{1}{2}kx^2$$

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \qquad k = m\omega^2$$

$$m\ddot{x} + kx = 0 \longrightarrow \ddot{x} + \omega^2 x = 0$$

$$x = C_1 \cos\omega t + C_2 \sin\omega t$$

$$= a\cos(\omega t + \alpha)$$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 a^2$$

$$x = \mathrm{re}\left[A e^{i\omega t}\right] \qquad A = a e^{i\alpha}$$

HW. Prob 1–6 on P.60, 61

§22. Forced osc. $F(t)$

$$m\ddot{x} = -kx + F(t)$$

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m}$$

inhomogeneous d.e.

$$x = \underset{\substack{\uparrow \\ \text{homogeneous d.e.} \\ a\cos(\omega t+\alpha)}}{x_0} + x_1 \leftarrow \text{special solution}$$

$$F(t) = f\cos(\gamma t+\beta) \text{ 의 경우}$$

$$x(t) = b\cos(\gamma t+\beta)$$

$$\left(-b\gamma^2 + \omega^2 b\right)\cancel{\cos(\gamma t+\beta)} = \frac{f}{m}\cancel{\cos}$$

$$b = \frac{f}{m(\omega^2-\gamma^2)}$$

$$x = a\cos(\omega t+\alpha) + \frac{f}{m(\omega^2-\gamma^2)}\cos(\gamma t+\beta)$$

---

resonance occurs when $\gamma = \omega$

$$x = \underline{a'\cos(\omega t+\alpha)} + \underbrace{\frac{f}{m(\omega^2-\gamma^2)}\left(\cos(\gamma t+\beta) - \cos(\omega t+\beta)\right)}$$

$$\lim_{\gamma\to\omega} \frac{\cos(\omega t+\beta) - \cos(\gamma t+\beta)}{\omega-\gamma} = \frac{-t\sin(\omega t+\beta)}{1}$$

$$x = a'\cos(\omega t+\alpha) + \frac{t}{2m\omega}\sin(\omega t+\beta)$$

near resonance

$$\gamma = \omega + \epsilon \quad (\epsilon \ll 1)$$

$$x = Ae^{i\omega t} + Be^{i\overset{\omega+\epsilon}{\overset{\shortparallel}{\gamma}} t}$$

$$A = ae^{i\alpha} \qquad B = \underbrace{\frac{f}{m(\omega^2-\gamma^2)}}_{b} e^{i\beta}$$

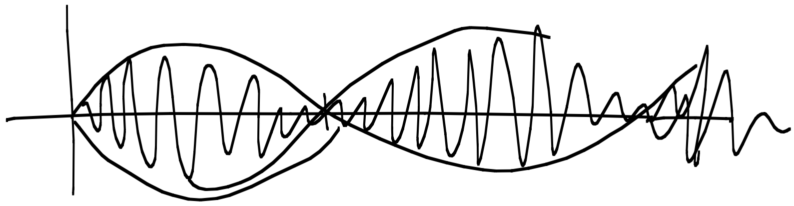
$$= Ce^{i\omega t}, \quad C(t) = A + Be^{i\epsilon t}$$

$$= |C|e^{i\delta}$$

$$|C|^2 = |A|^2 + |B|^2 + \underbrace{A^* B\, e^{i\epsilon t}}_{ab\,e^{i(\epsilon t-\alpha+\beta)}} + \underbrace{A B^*\, e^{-i\epsilon t}}_{ab\,e^{-i(\epsilon t-\alpha+\beta)}}$$

$$= a^2 + b^2 + 2ab\cos(\epsilon t - \alpha + \beta)$$

$$\frac{2\pi}{\epsilon} \gg \frac{2\pi}{\omega}$$



$$\ddot{x} + \omega^2 x = \frac{F}{m}$$

$$\dot{x} + i\omega x = \xi = \mathrm{Re}\,\xi + i\,\mathrm{Im}\,\xi \qquad x = \frac{\mathrm{Im}\,\xi}{\omega}$$

$$\dot{\xi} = \ddot{x} + i\omega\dot{x}$$

$$-)\; i\omega\xi = i\omega\dot{x} - \omega^2 x$$

$$\dot{\xi} - i\omega\xi = \ddot{x} + \omega^2 x = \frac{F}{m}$$

$$\xi = A(t)\,e^{i\omega t}$$

$$\dot{\xi} = (\dot{A} + i\omega A)\,e^{i\omega t}$$

$$-i\omega\xi = -i\omega A\,e^{i\omega t}$$

$$\dot{A}\,e^{i\omega t} = \frac{F}{m}$$

$$\dot{A} = \frac{F(t)\,e^{-i\omega t}}{m}$$

$$A(t) = A_0 + \frac{1}{m}\int_0^t F(t)\,e^{-i\omega t}\,dt$$

$$\xi(t) = \xi_0 e^{i\omega t} + \frac{1}{m} e^{i\omega t}\int_0^t F(t')\,e^{-i\omega t'}\,dt'$$

$$(\xi(0) = \xi_0 = A_0)$$

$t = -\infty \qquad \xi = 0$

$$\xi(t) = \frac{1}{m} e^{i\omega t} \int_{-\infty}^{t} F(t') e^{-i\omega t'} dt'$$

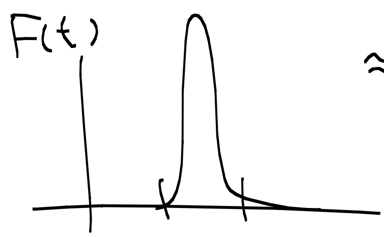
$$|\xi(\infty)|^2 = \frac{1}{m^2} \underbrace{|e^{i\omega t}|}_{1} \left| \int_{-\infty}^{\infty} F(t') e^{-i\omega t'} dt' \right|^2$$

$$= \dot{x}(\infty)^2 + \omega^2 x(\infty)^2$$

$$= \underbrace{\frac{1}{2} m (\quad)}_{E(\infty)} \cdot \frac{2}{m}$$

$$\therefore E(\infty) \frac{2}{m} = \frac{1}{m^2} \underbrace{\left| \int_{-\infty}^{\infty} F(t) e^{-i\omega t} dt \right|^2}_{2\pi \tilde{F}(\omega)}$$

$$\boxed{E(\infty) = \frac{4\pi^2}{2m} \left| \tilde{F}(\omega) \right|^2}$$



$$\approx \frac{1}{2m} \left| \int_{-\infty}^{\infty} F(t)\, dt \right|^2$$

H.W. Prob 1–4 on p. 64, 65

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§ 23. more than 1 d.o.f

$$T = \frac{1}{2} \sum_{i,k} a_{ik}(\vec{q}) \dot{q}_i \dot{q}_k$$

$$q_i = q_{i0} + x_i + \cdots$$

$$a_{ik}(\vec{q}) = a_{ik}(\vec{q}_0) + \frac{\partial^2 a_{ik}}{\partial q_j \partial q_\ell} \bigg|_{\vec{q} = \vec{q}_0} x_j x_\ell$$

$$= \frac{1}{2} \sum_{i,k} \underbrace{a_{ik}(\vec{q}_0)}_{\equiv m_{ik}} \dot{x}_i \dot{x}_k + O(x^4) \dots$$

$$U = U(\vec{q}) = \underset{\overset{|||}{0}}{U(\vec{q}_0)} + \frac{1}{2} \sum_{ik} \underbrace{\frac{\partial^2 U}{\partial q_i \partial q_k}\bigg|_{\vec{q}=\vec{q}_0}}_{\equiv k_{ik} = k_{ki}} x_i x_k$$

$$T = \frac{1}{2} \sum m_{ik} \dot{x}_i \dot{x}_k$$

$$U = \frac{1}{2} \sum k_{ik} x_i x_k$$

$$L = T - U = \frac{1}{2} \sum \left(m_{ik} \dot{x}_i \dot{x}_k - k_{ik} x_i x_k \right)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_j} - \frac{\partial L}{\partial x_j} = 0$$

$$\frac{\partial L}{\partial x_j} = -\frac{1}{2} \sum_{i,k} \left(k_{ik} \delta_{ij} x_k + k_{ik} x_i \delta_{kj} \right)$$

$$= -\frac{1}{2} \left(\sum_i \underbrace{k_{ji}}_{k_{ij}} x_i + \sum_i k_{ij} x_i \right)$$

$$= - \sum_i k_{ij} x_i$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_j} = + \sum_i m_{ij} \ddot{x}_i$$

$$\sum_i \underset{\overset{||}{m_{ji}}}{m_{ij}} \ddot{x}_i + \sum_i \underbrace{k_{ij}}_{k_{ji}} x_i = 0$$

$$= (m \ddot{\vec{x}})_j + (k \vec{x})_j = 0$$

$$\boxed{m \ddot{\vec{x}} + k \vec{x} = 0} \quad \begin{pmatrix} & k & \end{pmatrix} \begin{pmatrix} \vdots \\ \vec{x} \\ \vdots \end{pmatrix}$$

Assume

$$\vec{x} = \begin{pmatrix} x_1(t) \\ \vdots \\ x_s(t) \end{pmatrix} = \begin{pmatrix} A_1 e^{i\omega t} \\ A_2 e^{i\omega t} \\ \vdots \\ A_s e^{i\omega t} \end{pmatrix}$$

$$\longrightarrow \quad \vec{x} = \vec{A}\, e^{i\omega t}$$

$$\ddot{\vec{x}} = -\omega^2 \vec{A}\, e^{i\omega t}$$

$$(k - \omega^2 m)\, \vec{A}\, e^{i\omega t} = 0$$

$$(k - \omega^2 m)\, \vec{A} = 0$$

$$\begin{array}{cc} & s \\ s & \begin{pmatrix} - & - & \cdot & - \\ - & - & & - \\ - & & - & - \\ - & & - & - \end{pmatrix} \end{array} \begin{pmatrix} A_1 \\ \vdots \\ A_s \end{pmatrix}$$

$$\vec{A} \neq 0 \longrightarrow \boxed{\det(k - \omega^2 m) = 0}$$

characteristic eq.

polynomial of order s of ω^2

$\longrightarrow \omega^2 = s$ solutions $= \omega_\alpha^2$

$\omega = \omega_\alpha, \quad \alpha = 1, \cdots, s$

$$\underbrace{(k - \omega_\alpha^2 m)}_{M}\, \vec{A}^{(\alpha)} = 0$$

$$\det M = 0 \qquad M = \begin{array}{c} \quad\quad j \\ \begin{pmatrix} \cdot & \cdot & \oplus & \cdot & - \\ - & - & \bullet & - & \cdot \\ - & - & | & - & \cdot \\ - & - & | & - & \cdot \end{pmatrix} \end{array}$$

$$\det M = M_{1j}\Delta_{1j} + M_{2j}\Delta_{2j} + \cdots + M_{sj}\Delta_{sj}$$

$$= 0$$

$$= \sum_{l} M_{lj} \Delta_{lj}$$

$$= \sum_{l} M_{jl} \Delta_{lj} = 0$$

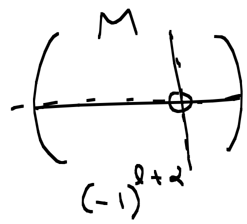
$$\left(M \vec{A}^{(\alpha)}\right)_j = 0$$

$$\sum_{l} M_{jl} \left(\vec{A}^{(\alpha)}\right)_l = 0$$

$$\left(\vec{A}^{(\alpha)}\right)_l = \Delta_{lj}$$

$$j = 1 - s, \quad \alpha = 1 \cdots s$$

$$\boxed{\vec{A}^{(\alpha)}_{\ \ l} \propto \Delta_{l\alpha}}$$



$\omega = \omega_\alpha$

$$\vec{X}^{(\alpha)}(t) = \vec{\Delta}_\alpha \underbrace{C_\alpha e^{i\omega_\alpha t}}_{\equiv \Theta_\alpha(t)}$$

$$\left(\vec{\Delta}_\alpha\right)_l \equiv \Delta_{l\alpha}$$

$$\vec{X}^{(\alpha)} = \vec{\Delta}_\alpha \Theta_\alpha(t)$$

$$\boxed{X^{(\alpha)}_{\ \ k}(t) = \Delta_{k\alpha} \Theta_\alpha}$$

$$\mathrm{re}\left(C_\alpha e^{i\omega_\alpha t}\right)$$

$$\| $$

$$|C_\alpha| \cos(\omega_\alpha t + \varphi)$$

$$\boxed{\ddot{\Theta}_\alpha + \omega_\alpha^2 \Theta_\alpha = 0}$$

normal mode

$$L = \sum_{\alpha=1}^{s} \frac{1}{2} m_\alpha \dot{\Theta}_\alpha^2 - \frac{1}{2} k_\alpha \Theta_\alpha^2$$

$$\tilde{k}_\alpha = \tilde{m}_\alpha \omega_\alpha^2$$

$$\sqrt{m_\alpha}\, \Theta_\alpha = Q_\alpha$$

$$L = \frac{1}{2}\sum_\alpha \left(\dot{Q}_\alpha^2 - \omega_\alpha^2 Q_\alpha^2\right)$$

$$\Updownarrow$$

$$L = \frac{1}{2}\sum_{ik} m_{ik}\dot{x}_i \dot{x}_k - \frac{1}{2}\sum k_{ik} x_i x_k$$

$$\boxed{x_k = \sum_\alpha \frac{\Delta_{k\alpha}}{\sqrt{m_\alpha}} Q_\alpha}$$

with ext. force F_k

$$\boxed{m_{ji}\ddot{x}_i + k_{ji}x_i = F_j}$$

$$L = \boxed{\frac{1}{2}\sum m_{ji}\dot{x}_i\dot{x}_j - \frac{1}{2}\sum k_{ji}x_i x_j} + \sum_j F_j x_j$$

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j} - \frac{\partial L}{\partial x_j}\right)$$

$$= \frac{1}{2}\sum_\alpha\left(\dot{Q}_\alpha^2 - \omega_\alpha^2 Q_\alpha^2\right)$$

$$+ \underbrace{\sum_j F_j \sum_\alpha \frac{\Delta_{j\alpha}}{\sqrt{m_\alpha}} Q_\alpha}_{\sum_\alpha \Big(\underbrace{\frac{\sum_j F_j \Delta_{j\alpha}}{\sqrt{m_\alpha}}}_{\equiv f_\alpha(t)}\Big) Q_\alpha}$$

$$= \;\; \cdots \; + \sum_\alpha f_\alpha(t) Q_\alpha$$

$$\underline{\ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = f_\alpha(t)}$$

Prob 1. $L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}\left(\omega_0^2(x^2+y^2) - 2\alpha xy\right)$

$= \frac{1}{2}\sum m_{ij}\dot{x}_i\dot{x}_j - \frac{1}{2}\sum k_{ij}x_i x_j$

$m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad k = \begin{pmatrix} \omega_0^2 & -\alpha \\ -\alpha & \omega_0^2 \end{pmatrix}$

$k - \omega^2 m = \begin{pmatrix} \omega_0^2 - \omega^2 & -\alpha \\ -\alpha & \omega_0^2 - \omega^2 \end{pmatrix}$

$|\ \ | = (\omega_0^2 - \omega^2)^2 - \alpha^2 = 0$

$\omega^2 - \omega_0^2 = \pm\alpha$

$\omega^2 = \omega_0^2 \pm \alpha = \omega_1^2, \omega_2^2$

$\omega_1 = \sqrt{\omega_0^2 + \alpha}, \quad \omega_2 = \sqrt{\omega_0^2 - \alpha}$

$A_\ell^{(a)} = \Delta_{\ell a} C_a \qquad a = 1, 2$

$M_{(1)} = \begin{pmatrix} -\alpha & -\alpha \\ -\alpha & -\alpha \end{pmatrix} = -\alpha\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$A^{(1)}_1 = \underset{\Delta_{11}}{\overset{\shortparallel}{-\alpha}} \qquad A^{(1)}_2 = -\alpha \cdot (-1)^3 = \underset{\Delta_{21}}{\overset{\shortparallel}{\alpha}}$

$\therefore \vec{A}^{(1)} = C_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\boxed{\vec{X}^{(1)} = C_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\omega_1 t}}$$

$M_{(2)} = M(\omega = \omega_2) = \begin{pmatrix} \alpha & -\alpha \\ -\alpha & \alpha \end{pmatrix}$

$= \alpha\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$A^{(2)}_1 = \underset{\Delta_{12}}{+\alpha}\, C_2 \qquad C_2$

$A^{(2)}_2 = \alpha\, C_2 \qquad C_2$

$$\boxed{\begin{matrix} \Delta_{12} = \alpha & \Delta_{21} = \alpha \\ \Delta_{22} = \alpha & \Delta_{11} = -\alpha \end{matrix}}$$

$$\vec{X}^{(2)} = C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{i\omega_2 t}$$

$$X_k = \sum_a \Delta_{ka} Q_a$$

$$X_1 = -\frac{\alpha}{\sqrt{m_1}} Q_1 + \frac{\alpha}{\sqrt{m_2}} Q_2$$

$$X_2 = \frac{\alpha}{\sqrt{m_1}} Q_1 + \frac{\alpha}{\sqrt{m_2}} Q_2$$

$$X_k = \sum_\alpha \frac{\Delta_{k\alpha}}{\sqrt{m_\alpha}} Q_\alpha$$

$$T = \sum_{a=1}^{s} \frac{1}{2} \dot{Q}_a^2 = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2)$$

$$= \frac{1}{2}\left(\underbrace{2\left(\frac{\alpha}{\sqrt{m_1}}\right)^2}_{1} \dot{Q}_1^2 + \underbrace{2\left(\frac{\alpha}{\sqrt{m_2}}\right)^2}_{1} \dot{Q}_2^2\right)$$

$\Rightarrow$

$$\therefore \frac{\alpha}{\sqrt{m_1}} = +\frac{1}{\sqrt{2}} \quad \frac{\alpha}{\sqrt{m_2}} = \frac{1}{\sqrt{2}}$$

$$x_1 = \frac{1}{\sqrt{2}}(-Q_1 + Q_2)$$

$$x_2 = \frac{1}{\sqrt{2}}(Q + Q_2)$$

H.W. Prob 2-3, on p.170

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§25. Damped Osc.

$$F_r = -\alpha \dot{x}$$

$$m\ddot{x} = -kx - \alpha\dot{x} \qquad \frac{k}{m} = \omega_0^2$$

$$\frac{\alpha}{m} = 2\lambda$$

$$\ddot{x} + 2\lambda \dot{x} + \omega_0^2 x = 0$$

linear

$$x = A e^{rt}$$

$$(r^2 + 2\lambda r + \omega_0^2) A e^{rt} = 0$$

$$\boxed{r = -\lambda \pm \sqrt{\lambda^2 - \omega_0^2}}$$

① $\lambda < \omega_0$ (under damped)

$$\lambda^2 - \omega_0^2 < 0 \rightarrow \sqrt{\lambda^2 - \omega_0^2} = \sqrt{\omega_0^2 - \lambda^2} \cdot i$$

$$x = \mathrm{Re}\left\{ \underset{a e^{i\alpha}}{A}\, e^{-\lambda t \pm i \underbrace{\sqrt{\omega_0^2 - \lambda^2}}_{\omega} t} \right\}$$

$$= \underbrace{a e^{-\lambda t}}_{A} \cos(\omega t + \alpha)$$

$$E = \frac{1}{2} m \omega_0^2 A^2 = E_0 e^{-2\lambda t}$$

② $\lambda > \omega_0$: overdamped

$$e^{-(\lambda \pm \sqrt{\lambda^2 - \omega_0^2})t}$$

$$x = C_1 e^{-(\lambda - \sqrt{\lambda^2 - \omega_0^2})t} + C_2 e^{-(\lambda + \sqrt{\lambda^2 - \omega_0^2})t}$$

③ $\lambda = \omega_0$: critical damping.

$$x = C_1 e^{-\lambda t} + C_2 t e^{-\lambda t}$$

§26. forced osc.

$$F(t) = f \cos \gamma t$$

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = \frac{f}{m}\cos\gamma t$$

$$= \frac{f}{m}\,\mathrm{Re}\left[e^{i\gamma t}\right]$$

$$x = \mathrm{Re}\left(B e^{i\gamma t}\right)$$

$$\left(-\gamma^2 + 2i\lambda\gamma + \omega_0^2\right) B e^{i\gamma t} = \frac{f}{m} e^{i\gamma t}$$

$$\therefore \quad B = \frac{f/m}{-\gamma^2 + 2i\lambda\gamma + \omega_0^2}$$

$$= b\, e^{i\delta} \rightarrow e^{-i\delta} = \frac{b\left(\omega_0^2 - \gamma^2 + 2i\lambda\gamma\right)}{f/m}$$

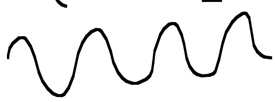
$$e^{-i\delta} = \cos\delta - i\sin\delta$$

$$b = \frac{(f/m)}{\sqrt{\left(\omega_0^2 - \gamma^2\right)^2 + 4\lambda^2\gamma^2}}$$

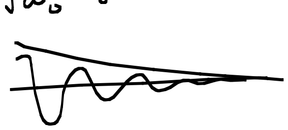
$$\frac{\sin\delta}{\cos\delta} = \tan\delta = -\frac{2\lambda\gamma}{\omega_0^2 - \gamma^2} = \frac{2\lambda\gamma}{\gamma^2 - \omega_0^2}$$

$$x_1 = \mathrm{Re}\left[B e^{i\gamma t}\right] = \mathrm{Re}\left(b e^{i\delta} e^{i\gamma t}\right)$$

$$= b\cos(\gamma t + \delta)$$

$$=$$ 

$$x_0 = a e^{-\gamma t}\cos(\omega t + \alpha) \qquad \omega = \sqrt{\omega_0^2 - \gamma^2}$$

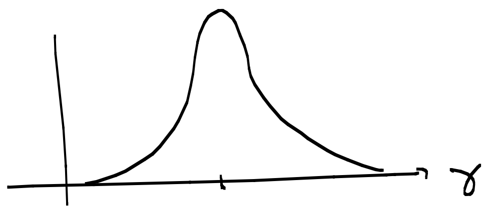


$$\boxed{x = x_0 + x_1}$$

transient sol. $x_0 \rightarrow 0$

$$b = \frac{(f/m)}{\sqrt{\left(\omega_0^2 - \gamma^2\right)^2 + 4\lambda^2\gamma^2}}$$

$b(\gamma)$



$\gamma^2 \equiv x \quad f(x) = 4\lambda^2 x + (\omega_0^2 - x)^2$

$$f'(x) = 4\lambda^2 - 2(\omega_0^2 - x) = 0$$

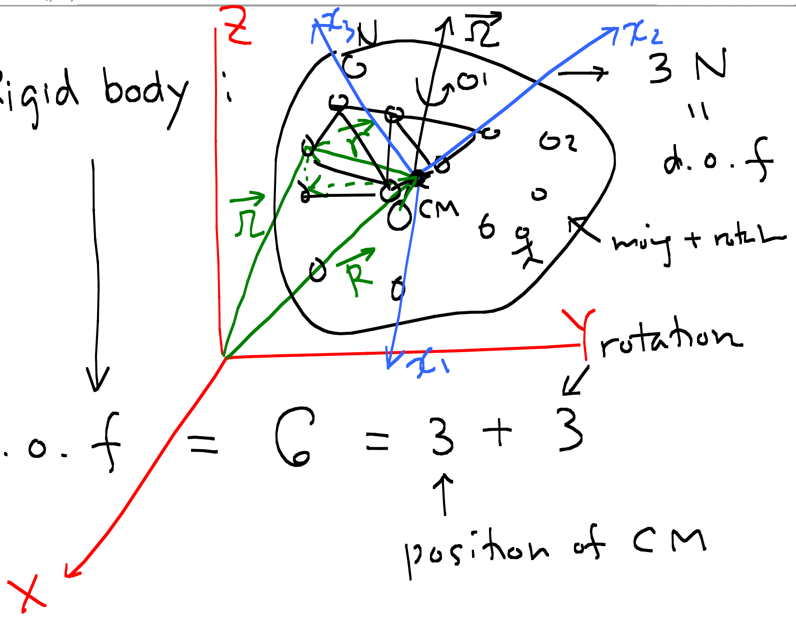
$$2\lambda^2 = \omega_0^2 - x$$

$$\gamma = \sqrt{\omega_0^2 - 2\lambda^2}$$

$$f((\sqrt{})^2) = 4\lambda^2(\omega_0^2 - 2\lambda^2) + 4\lambda^4$$
$$= 4\lambda^2(\omega_0^2 - \lambda^2)$$

$$b_{max} = \frac{f/m}{2\lambda\sqrt{\omega_0^2 - \lambda^2}}$$

Rigid body :



d.o.f = 6 = 3 + 3

↑ position of CM

Coordinate system

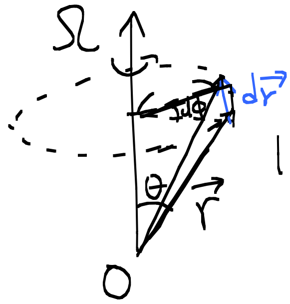
① Fixed XYZ

② Moving (Body-fixed) $x_1 x_2 x_3$

$$\vec{r} = \vec{R} + \vec{r}$$

during dt

$$d\vec{r} = d\vec{R} + d\vec{\phi} \times \vec{r}$$



$d\vec{r} \perp \vec{\Omega}$ & $\vec{r}$

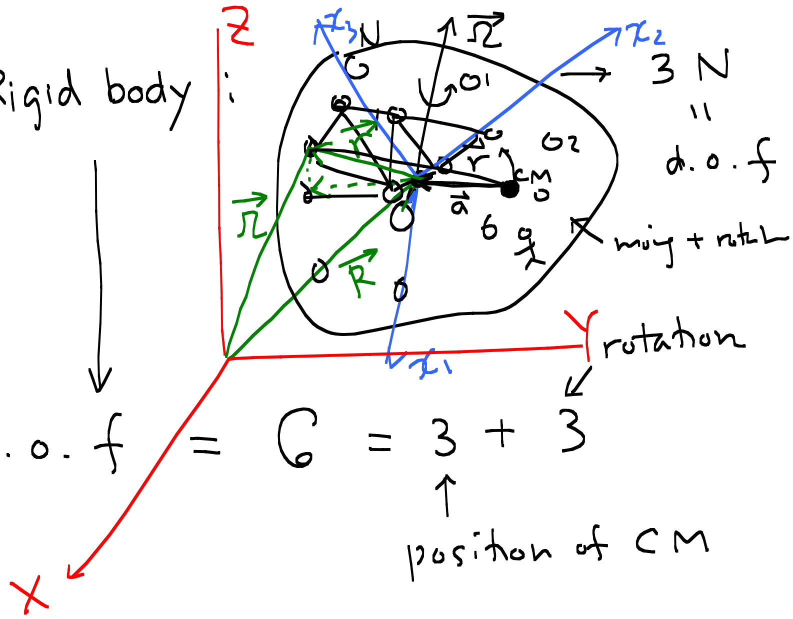
$|d\vec{r}| = d\phi\, r \sin\theta$

$$d\vec{r} = d\vec{\phi} \times \vec{r}$$

$$\vec{v}_{\text{Fixed}} = \vec{V} + \frac{d\vec{\phi}}{dt} \times \vec{r} = \vec{V} + \vec{\Omega} \times \vec{r}$$

If $O \neq CM$

Rigid body :



$\rightarrow$ 3 N

$\shortparallel$

d.o.f

$\nwarrow$ moving + rotation

rotation

$$\text{d.o.f} = 6 = 3 + 3$$

$\uparrow$ position of CM

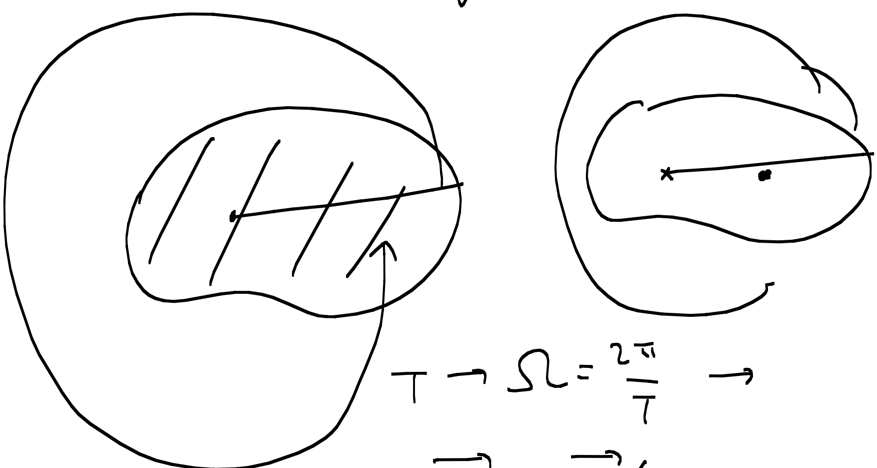
$$\vec{r} = \vec{a} + \vec{r}'$$

$\vec{r}$: position from $O = CM$; $\vec{a}$: OO'; $\vec{r}'$: position from O'

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{V} + \vec{\Omega} \times \vec{r}$$

$$= \underbrace{\vec{V} + \vec{\Omega} \times \vec{a}}_{\vec{V}'} + \vec{\Omega} \times \vec{r}'$$

$$\vec{v} = \vec{V}' + \vec{\Omega} \times \vec{r}'$$

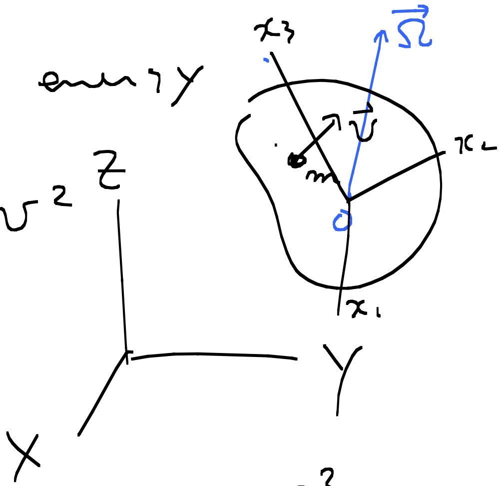


$T \rightarrow \Omega = \frac{2\pi}{T} \rightarrow$

$\vec{\Omega} = \vec{\Omega}'$ as far as rot. axes are parallel

$$\vec{v} = \underline{\underline{\vec{V}'}} + \vec{\Omega}' \times \vec{r}'$$

§32. Kinetic energy



$$T = \sum_m \frac{1}{2} m v^2$$

$$= \sum_m \frac{1}{2} m \left(\vec{V} + \vec{\Omega} \times \vec{r}\right)^2$$

$$= \underbrace{\sum_m \frac{1}{2} m V^2} + \cancel{\sum_m m \vec{V} \cdot (\vec{\Omega} \times \vec{r})}$$
$$+ \sum_m \frac{1}{2} m (\vec{\Omega} \times \vec{r})^2$$

$$= \frac{1}{2} \underbrace{\left(\sum_m m\right)}_{\mu \leftarrow \text{total mass}} V^2$$

$$\vec{V} \cdot \left(\sum_m m(\vec{\Omega} \times \vec{r})\right) = \vec{V} \cdot \left(\vec{\Omega} \times \underbrace{\sum_m m\vec{r}}_{\vec{r}_{CM} = 0}\right)$$

$$= 0$$

$$(\vec{\Omega} \times \vec{r})^2 = |\vec{\Omega} \times \vec{r}|^2$$
$$= |\vec{\Omega}|^2 |\vec{r}|^2 \sin^2\theta$$
$$= |\vec{\Omega}|^2 |\vec{r}|^2 (1 - \cos^2\theta)$$
$$= |\vec{\Omega}|^2 |\vec{r}|^2 - (\underbrace{|\vec{\Omega}||\vec{r}|\cos\theta}_{\vec{\Omega} \cdot \vec{r}})^2$$

$$T_{rot} = \frac{1}{2} \sum_m m \left[|\Omega|^2 |\vec{r}|^2 - (\vec{\Omega} \cdot \vec{r})^2 \right]$$

$$\vec{\Omega} = (\Omega_1, \Omega_2, \Omega_3) \quad \text{w.r.t. } x_1\text{-}x_2\text{-}x_3$$

$$r^2 = x_1^2 + x_2^2 + x_3^2$$

$$= \frac{1}{2} \sum_m m \left[\underbrace{\left(\sum_{i=1}^{3} \Omega_i^2\right) \sum_{i=1}^{3} x_i^2} - \left(\sum_i \Omega_i x_i\right)^2 \right]$$

$$\left(\sum_i \Omega_i x_i\right)^2 = \left(\sum_{i=1}^{3} \Omega_i x_i\right)\left(\sum_{j=1}^{3} \Omega_j x_j\right)$$

$$= \sum_{i,j=1}^{3} \Omega_i \Omega_j x_i x_j$$

$$\sum_{i,j} \Omega_i \Omega_j \delta_{ij} \left(\sum_{l=1}^{3} x_l^2\right)$$

$$T = \frac{1}{2} \sum_m \sum_{i,j} m \Omega_i \Omega_j \left(\Big(\underbrace{\sum_{l=1}^{3} x_l^2}_{x^2}\Big) \delta_{ij} - x_i x_j \right)$$

$$= \frac{1}{2} \sum_{i,j} \Big(\underbrace{\sum_m m (x^2 \delta_{ij} - x_i x_j)}_{I_{ij}} \Big) \Omega_i \Omega_j$$

$$= \frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \Omega_j$$

I_{ij} : inertia tensor; 3×3 matrix

I_{ii} : moment of inertia

$$T = \frac{1}{2} \mu V^2 + \frac{1}{2} \vec{\Omega}^T I \vec{\Omega}$$

$$L = T - U$$

$$I_{11} = \sum_m m\left(\sum_{l=1}^{3} x_l^2 - x_1^2\right)$$

$$I_{11} = \sum_m m\left(x_2^2 + x_3^2\right)$$

$$I_{22} = \sum_m m\left(x_1^2 + x_3^2\right)$$

$$I_{33} = \sum_m m\left(x_1^2 + x_2^2\right)$$

$i \neq j \longrightarrow$ symmetric 3×3 matrix

$$I_{ij} = -\sum_m m x_i x_j \equiv I_{ji}$$

Continuous case

$$I_{ij} = \int \rho(\vec{x})\left(x^2 \delta_{ij} - x_i x_j\right) dV$$

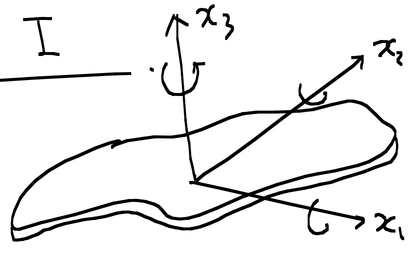
if $I_{ij} = 0$ for $i \neq j$,

or if I is diagonal matrix

we call x_1, x_2, x_3 as principal axes.

If not, we can always diagonalize I

<u>computation of I</u>



① <u>flat objects</u>:

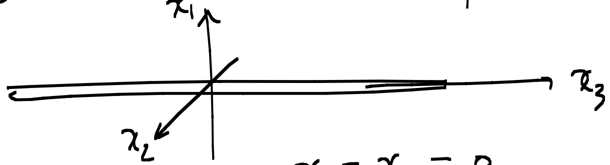
$x_3 = 0 \qquad I_{13} = I_{23} = 0$

$$I_1 = \sum m (x_2^2 + \cancel{x_3^2}) = \sum m x_2^2$$

$$I_2 = \sum m (x_1^2 + \cancel{x_3^2}) = \sum m x_1^2$$

$$I_3 = \sum m (x_1^2 + x_2^2) = I_1 + I_2$$

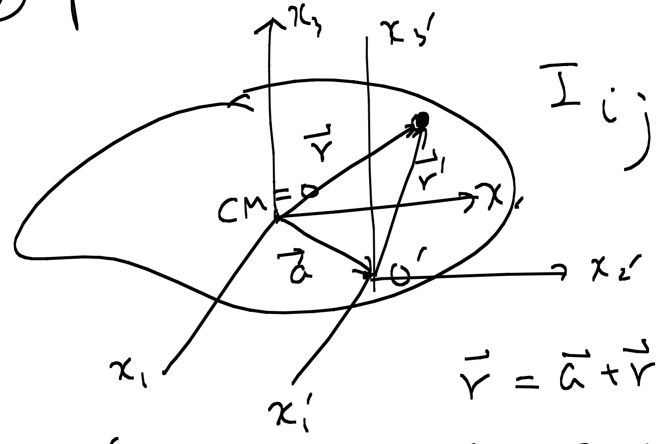
② rotator = 1 dim object



$$x_1 = x_2 = 0$$

$$\left.\begin{aligned} I_1 &= \sum m x_3^2 \\ I_2 &= \sum m x_3^2 \\ I_3 &= 0 \end{aligned}\right\} \quad I_1 = I_2,\ I_3 = 0$$

③ parallel axis theorem



I_{ij}

$$\vec r = \vec a + \vec r'$$

$$x_i' = x_i - a_i \leftarrow x_i = a_i + x_i'$$

$$I_{ij} \equiv \sum m \left(\left(\sum_l x_l^2\right) \delta_{ij} - x_i x_j \right)$$

$$I'_{ij} \equiv \sum m \left(\left(\sum_l x_l'^2\right) \delta_{ij} - x_i' x_j' \right)$$

$$= \sum m \left(\left[\sum_l (x_l - a_l)^2\right] \delta_{ij} - (x_i - a_i)(x_j - a_j) \right)$$

$$\therefore = \sum m \left(\sum_l x_l^2 \delta_{ij} - x_i x_j \right) + \sum m \left(\sum_l a_l^2 \delta_{ij} - a_i a_j \right)$$

$$\sum_\ell (x_\ell - a_\ell)^2 = \sum_\ell \left(x_\ell^2 + a_\ell^2 - \cancel{2x_\ell a_\ell}\right)$$

$$\sum m\, \delta_{ij} \sum_\ell (x_\ell a_\ell)$$

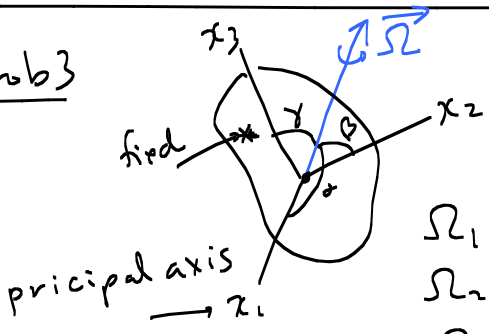
$$= \delta_{ij} \sum_\ell a_\ell \underbrace{\sum m x_\ell}_{(\sum m) x_{cm} = 0} = 0$$

$$\sum_{ij} \sum m\, x_i a_j = \sum_{ij} a_j \underbrace{\left(\sum m x_i\right)}_{(\sum m)\cdot x_{icm}}$$

$$\sum_{ij} \sum m\, x_j a_i = \sum_{ij} a_i \left(\sum m x_j^{0}\right) = (\sum m)\, x_{jcm} = 0$$

$$\therefore \boxed{I'_{ij} = I_{ij} + \mu\left(a^2\delta_{ij} - a_i a_j\right)}$$

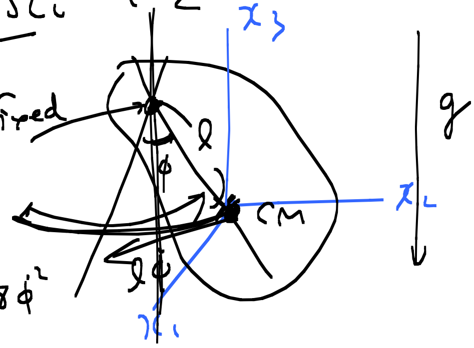
Prob3



$$\Omega = \dot\phi$$

$$\Omega_1 = \Omega\cos\alpha$$
$$\Omega_2 = \Omega\cos\beta$$
$$\Omega_3 = \Omega\cos\gamma$$

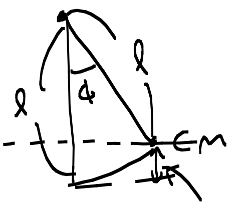
$$T = \frac{1}{2} \sum \underline{I_i \Omega_i^2} + \frac{1}{2}\mu \ell^2 \dot\phi^2$$



$$I_1 \cos^2\alpha\, \dot\phi^2 + I_2 \cos^2\beta\, \dot\phi^2 + I_3 \cos^2\gamma\, \dot\phi^2$$

$$U(\phi) = \mu g l \underbrace{(1-\cos\phi)}_{\approx \frac{1}{2}\phi^2}$$

$\phi \ll 1$



$h = l(1-\cos\phi)$

$$L = T - U$$

$$= \frac{1}{2}\mu l^2 \dot{\phi}^2 + \frac{1}{2}\left(I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right)\dot{\phi}^2$$

$$- \frac{\mu g l}{2}\phi^2$$

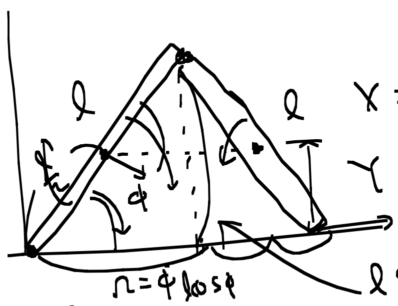
$$= \frac{1}{2} m \dot{\phi}^2 - \frac{1}{2} k \phi^2$$

$$\phi = \phi_0 \cos(\omega t + \alpha)$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{\mu g l/2}{\mu l^2 + (I_1\cos^2\alpha + \cdots)}} \;//$$

Prob 4



$X = \frac{3}{2} l \cos\phi$

$Y = \frac{l}{2}\sin\phi$

$$T = T^{(1)}_{trans} + T^{(1)}_{rot}$$

$$= \frac{1}{2}\mu\left(\frac{l}{2}\dot{\phi}\right)^2 + \frac{1}{2} I \dot{\phi}^2, \quad I = \frac{1}{12}\mu l^2$$

$$\frac{1}{2}\boxed{\mu l^2 \underbrace{\left(\frac{1}{4}+\frac{1}{12}\right)}_{\frac{1}{3}}}\dot{\phi}^2, \quad \mu l^2 \cdot \frac{1}{3} = I_{zz}$$

$$v^2 = \dot{X}^2 + \dot{Y}^2 = \left(-\frac{3}{2} l \sin\phi\, \dot\phi\right)^2 + \left(\frac{l}{2}\cos\phi\, \dot\phi\right)^2$$

$$T^{(2)}_{trans} = \frac{\mu}{2}\ \frac{1}{4}\left(9\sin^2\phi + \cos^2\phi\right) l^2 \dot\phi^2$$

$$T^{(2)}_{rot} = \frac{1}{2}\left(\frac{1}{12}\mu l^2\right)\dot\phi^2$$

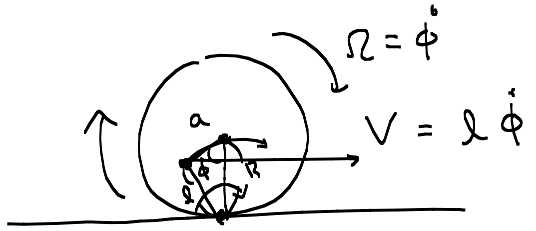
$$T = \frac{1}{2}\mu l^2 \dot\phi^2 \left[\frac{1}{3} + \frac{1}{12} + \frac{\overbrace{9\sin^2\phi + \cos^2\phi}^{8\sin^2\phi + 1}}{4}\right]$$

$$\left[\underbrace{\frac{4+1+3}{12}}_{\frac{2}{3}} + 2\sin^2\phi\right]$$

$$= \frac{1}{\cancel{2}}\ \frac{\cancel{2}}{3}\mu l^2 \dot\phi^2 \left(1 + 3\sin^2\phi\right)$$

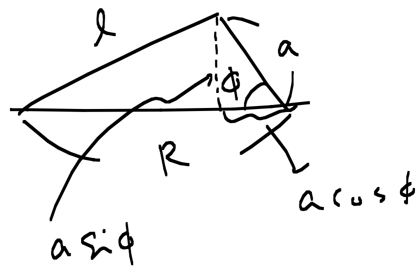
//

Prob 5



$$T = T_{trans} + \underbrace{T_{rot}}_{\frac{1}{2} I \dot\phi^2}$$

$$= \frac{1}{2}\mu V^2$$

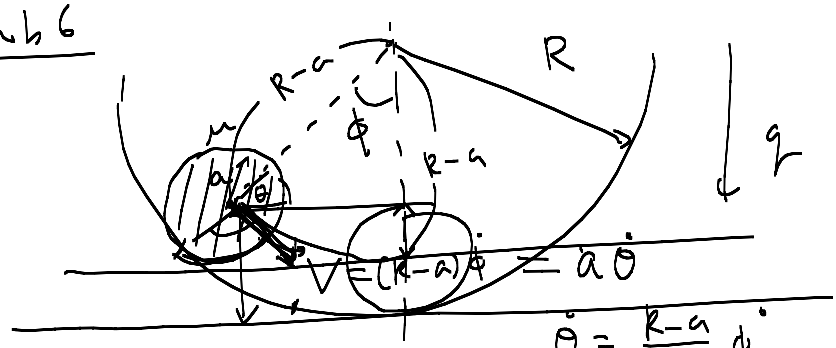


$$l^2 = a^2\sin^2\phi + (R - a\cos\phi)^2$$

$$= R^2 + a^2 - 2Ra\cos\phi$$

$$T = \frac{1}{2}\mu\left(R^2 + a^2 - 2Ra\cos\phi\right)\dot\phi^2 + \frac{1}{2}I\dot\phi^2$$

Prob 6



$$L = T - V$$

$$T = T_{trans} + T_{rot}$$

$$= \frac{1}{2}\mu\left((R-a)\dot\phi\right)^2 + \frac{1}{2}\underset{\boxed{\frac{1}{2}\mu a^2}}{I}\dot\theta^2 \left(\frac{R-a}{a}\dot\phi\right)^2$$

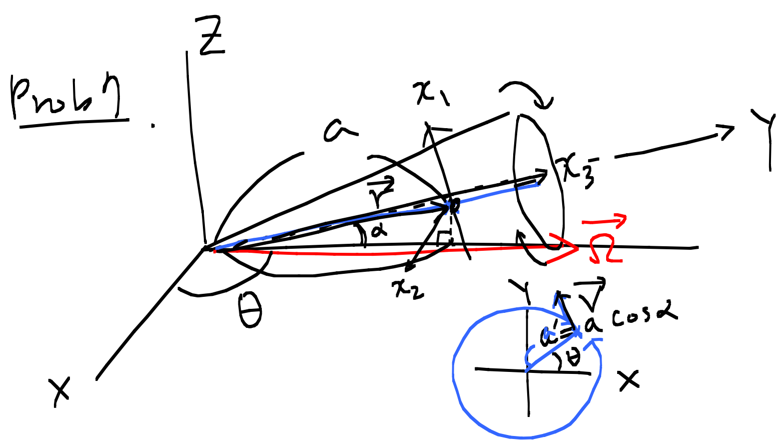
$$= \frac{1}{2}\mu\left((R-a)^2 + \frac{(R-a)^2}{2}\right)\dot\phi^2$$

$$= \frac{1}{2}\underbrace{\frac{3}{2}\mu(R-a)^2}_{m}\dot\phi^2$$

$$V = \underbrace{\mu g(R-a)}_{k}\Big(\underbrace{1-\cos\phi}_{\frac{1}{2}\phi^2}\Big)$$

$$\omega = \sqrt{\frac{\cancel{\mu} g(R-\cancel{a})}{\frac{3}{2}\cancel{\mu}(R-a)^{\cancel{2}}}} = \sqrt{\frac{2g}{3(R-a)}}$$

Prob 7.



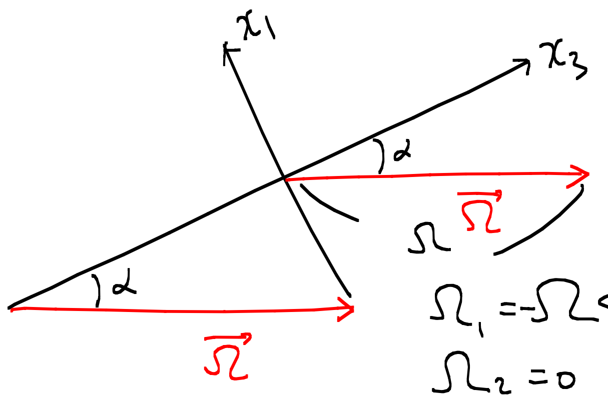
$$\vec{V} = \vec{\Omega} \times \vec{r}$$

$$V = \Omega a \sin\alpha = a'\dot{\theta} = a\cos\alpha\,\dot{\theta}$$

$$\Omega = \dot{\theta}\,\frac{\cos\alpha}{\sin\alpha} = \underline{\underline{\dot{\theta}\cot\alpha}}$$

$$T = T_{trans} + T_{rot}$$

$$= \frac{1}{2}\mu V^2 + \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$



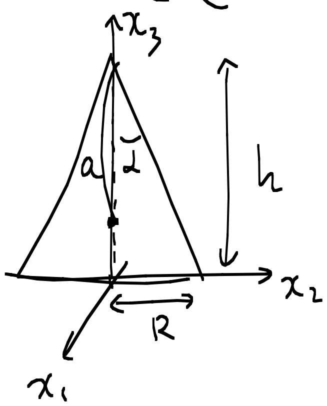
$\Omega_1 = -\Omega \sin\alpha$

$\Omega_2 = 0$

$\Omega_3 = \Omega\cos\alpha$

$$= \frac{1}{2}\mu a^2 \cos^2\alpha\,\dot{\theta}^2$$

$$+ \frac{1}{2}\left(I_1 \sin^2\alpha + I_3 \cos^2\alpha\right)\cot^2\alpha\,\dot{\theta}^2$$



$$I_1 = I_2 = \frac{3}{20}\mu\left(R^2 + \frac{1}{4}h^2\right)$$

$$a = \frac{3}{4}h \rightarrow h = \frac{4}{3}a$$

$$I_3 = \frac{3}{10}\mu R^2$$

$$\frac{R}{h} = \tan\alpha \qquad R = \frac{4}{3}a\tan\alpha$$

$$I_1 = \frac{3}{20}\mu\left(\frac{16}{9}\tan^2\alpha + \underbrace{\frac{1}{4}\cdot\left(\frac{4}{3}\right)^2}_{\frac{4}{9}}\right)a^2$$

$$I_1 = \frac{3}{20}\cdot\frac{4}{9}\mu\left(4\tan^2\alpha + 1\right)a^2$$

$$I_3 = \frac{3}{10}\mu\left(\frac{4}{3}a\tan\alpha\right)^2$$

$$= \underbrace{\frac{3}{10}\cdot\frac{16}{9}}_{\frac{8}{15}}\mu a^2\tan^2\alpha$$

$$+\frac{1}{2}\left(\frac{1}{15}(4\tan^2\alpha+1)\sin^2\alpha + \frac{8}{15}\underbrace{\tan^2\alpha\cos^2\alpha}_{\sin^2\alpha}\right)\cot^2\alpha\,\dot\theta^2\,\mu a^2$$

$$= \frac{1}{30}\underbrace{\left(4\tan^2\alpha + \underbrace{1+8}_{9}\right)\cos^2\alpha}_{4\sin^2\alpha + 9\cos^2\alpha = 4+5\cos^2\alpha}\mu a^2\dot\theta^2$$

$$= \frac{1}{30}\left(4+5\cos^2\alpha\right)\mu a^2\dot\theta^2$$

//

$$T_{trns} = +\frac{1}{2}\mu\cos^2\alpha\, a^2\dot\theta^2$$

$$\left(\frac{4}{30} + \frac{5}{30}\cos^2\alpha + \frac{\cos^2\alpha}{2}\right)\mu a^2\dot\theta^2$$

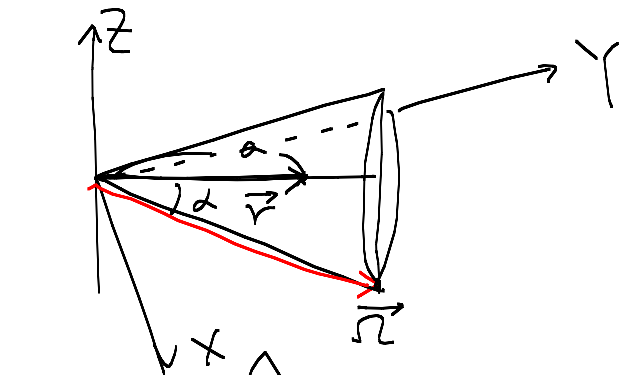
$$\frac{4}{30} + \frac{20}{30}\cos^2\alpha = \frac{4}{30}\left(1+5\cos^2\alpha\right)$$

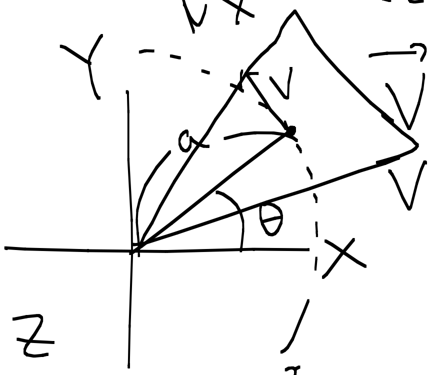
$$= \frac{4}{30}\left(1+5\cos^2\alpha\right)\mu \underbrace{a^2}_{\left(\frac{3}{4}h\right)^2}\dot\theta^2$$

$$= \overbrace{\frac{\cancel{4}}{3\cancel{0}_{10}}\cdot\frac{\cancel{9}\,3}{\cancel{16}\,4}}^{\frac{3}{40}}\left(1+5\cos^2\alpha\right)\mu h^2\dot\theta^2$$

//

Prob A





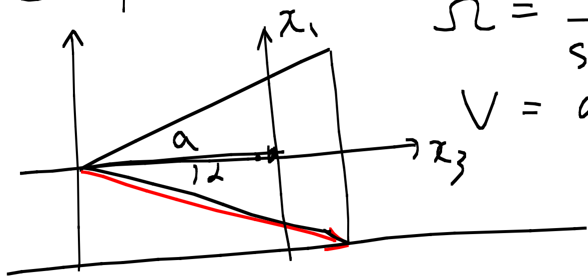
$$\vec{V} = \vec{\Omega} \times \vec{r}$$

$$V = \Omega a \sin\alpha$$

$$= a\dot{\theta}$$

$$\Omega = \frac{\dot{\theta}}{\sin\alpha}$$

$$V = a\dot{\theta}$$



$$T = \underbrace{\frac{1}{2}\mu V^2}_{\frac{1}{2}\mu a^2\dot{\theta}^2} + \frac{1}{2}\left(I_1\Omega_1^2 + I_3\Omega_3^2\right)$$

$$= \frac{1}{2}\left(\frac{1}{15}(4\tan^2\alpha+1)\cancel{\sin^{2}\alpha} + \frac{8}{15}\underbrace{\tan^2\alpha\cos^2\alpha}_{\cancel{\sin^2\alpha}}\right)\underbrace{\Omega^2}_{\frac{\dot{\theta}^2}{\cancel{\sin^2\alpha}}}\mu a^2$$

$$\frac{1}{30}\left(9 + 4\tan^2\alpha\right)$$

$$= \left(\frac{1}{2} + \frac{9+4\tan^2\alpha}{30}\right)\mu a^2\dot{\theta}^2$$

$$= \frac{4(6+\tan^2\alpha)}{30}\mu a^2\dot{\theta}^2$$

$$6 + \tan^2\alpha = 6 + \frac{\sin^2\alpha}{\cos^2\alpha} = \frac{6\cos^2\alpha + \sin^2\alpha}{\cos^2\alpha}$$

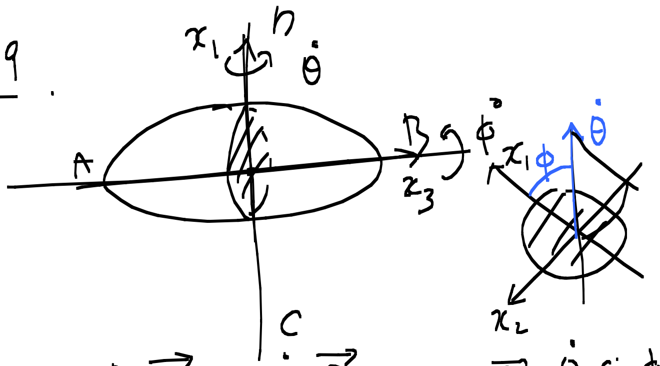
$$= \frac{5\cos^2\alpha + 1}{\cos^2\alpha} = 5 + \sec^2\alpha$$

$$= \frac{2}{15}\left(5+\sec^2\alpha\right)\mu \underbrace{a^2}_{\frac{9}{16}h^2}\dot\theta^2$$

$$\frac{\cancel{2}^{\,1}}{5\,\cancel{15}}\cdot\frac{\cancel{9}^{\,3}}{\cancel{16}\,8} = \frac{3}{40}$$

$$= \frac{3}{40}\left(5+\sec^2\alpha\right)\mu h^2\dot\theta^2 \;//$$

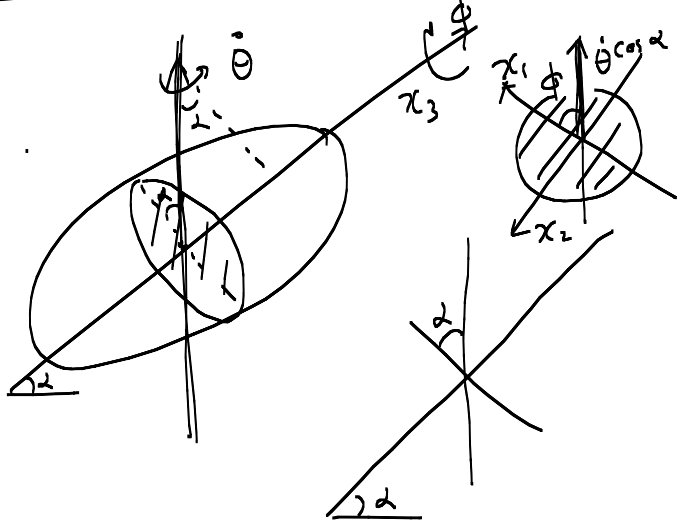
Prob 9.



$$\vec\Omega = \underbrace{\dot\theta\cos\phi}_{\Omega_1}\,\vec e_1 + \underbrace{\dot\phi}_{\Omega_3}\,\vec e_3 + \underbrace{\vec e_2\,\dot\theta\sin\phi}_{\Omega_2}$$

$$T = \frac{1}{2}\left(I_1\dot\theta^2\cos^2\phi + I_2\dot\theta^2\sin^2\phi + I_3\dot\phi^2\right)$$

Prob 10.

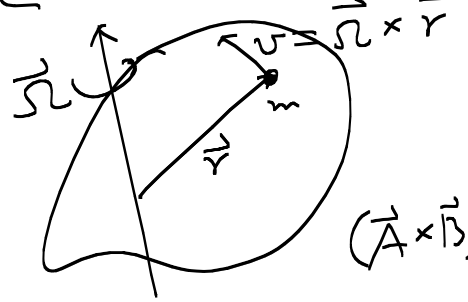


$$\vec\Omega = \vec e_3\left(\dot\phi + \dot\theta\sin\alpha\right) + \vec e_1\left(\dot\theta\cos\alpha\cos\phi\right) + \vec e_2\left(-\dot\theta\cos\alpha\sin\phi\right)$$

$$T = \frac{1}{2}\left[I_3 (\dot\phi + \dot\theta \sin\alpha)^2 + I_1 \dot\theta^2 \cos^2\alpha \cos^2\phi + I_2 \dot\theta^2 \cos^2\alpha \sin^2\phi \right]$$

§33. AM.

$$\vec{M} = \sum m \vec{r} \times \vec{v}$$



$$(\vec{A} \times \vec{B})_i = \varepsilon_{ijk} A_j B_k$$

$$M_i = \left(\sum m \vec{r} \times (\vec{\Omega} \times \vec{r}) \right)_i$$

$$= \sum m \underset{\varepsilon_{123} = 1,}{\varepsilon_{ijk}} x_j \underbrace{(\vec{\Omega} \times \vec{r})_k}_{\varepsilon_{kln} \Omega_l x_n}$$

$$= \sum m \sum_{jkln} \varepsilon_{ijk} \underset{\varepsilon_{lnk}}{\varepsilon_{kln}} \Omega_l x_j x_n$$

$$\sum_k \varepsilon_{ijk} \varepsilon_{lnk} = \delta_{il} \delta_{jn} - \delta_{in} \delta_{jl}$$

$$= \sum m (\underbrace{\delta_{il} \delta_{jn}} - \delta_{in} \delta_{jl}) \Omega_l x_j x_n$$

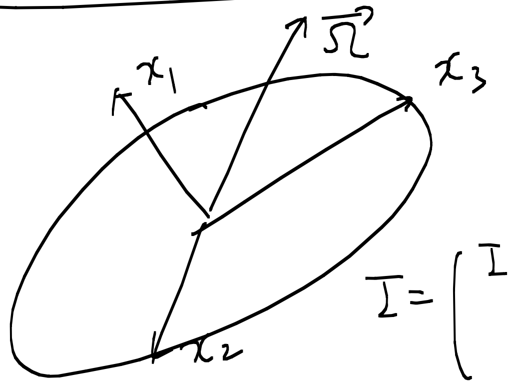
$$= \sum m \; \Omega_i x^2 - \Omega_j x_j x_i$$

$$= \sum m \sum_j \{ x^2 \underbrace{\delta_{ij} \Omega_j}_{\Omega_i} - x_i x_j \Omega_j$$

$$= \underbrace{\sum m(x^2\delta_{ij} - x_i x_j)}_{I_{ij}} \Omega_j$$

$$M_i = I_{ij}\Omega_j$$

$$\boxed{\vec{M} = I \cdot \vec{\Omega}}$$



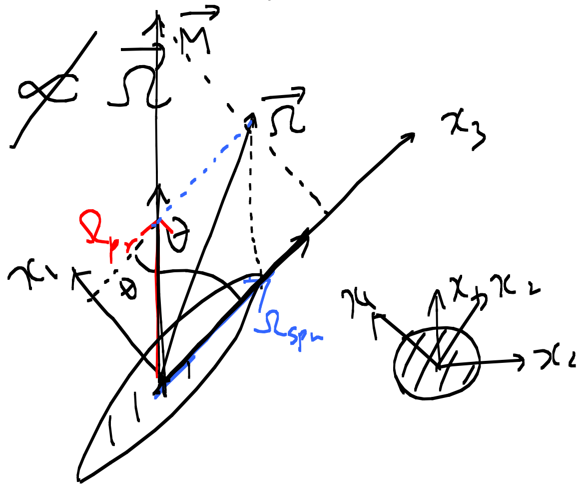
$$I = \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{pmatrix}$$

$$M_1 = I_1\Omega_1$$

$$M_2 = I_2\Omega_2$$

$$M_3 = I_3\Omega_3 \leftarrow$$

$$\vec{M} \not\propto \vec{\Omega}$$



$$M_2 = 0 = I_2\Omega_2 \rightarrow \Omega_2 = 0$$

$$= \Omega_3 = \frac{M_3}{I_3} = \frac{M\cos\theta}{I_3}$$

$$\Omega_1 = \Omega_{pr} \cancel{\sin\theta} = \frac{M_1}{I_1} = \frac{M \cancel{\sin\theta}}{I_1}$$

$$\boxed{\Omega_{pr} = \frac{M}{I_1}}$$

//

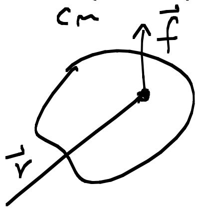
Final on 6/5 (F) lecture time

$$\vec{M} = I \cdot \vec{\Omega}$$

§34. E. of. M. of Rigid body

$6\ (\underbrace{3}_{CM} + \underbrace{3}_{\text{rotation}})$

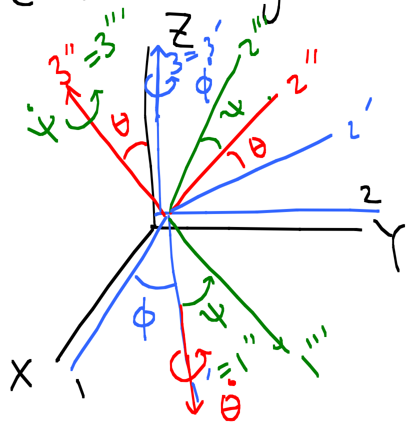
$\vec{P}$: mom. of CM



$$\frac{d\vec{P}}{dt} = \vec{F}$$

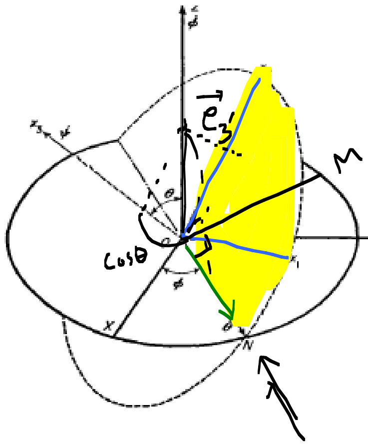
$$\boxed{\frac{d\vec{M}}{dt} = \vec{K}} = \sum \vec{r} \times \vec{f}$$

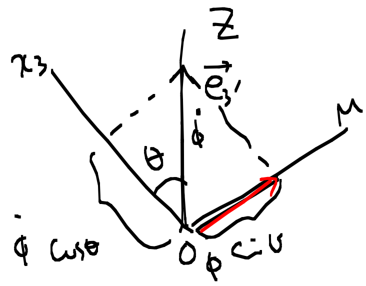
§35. Euler angles

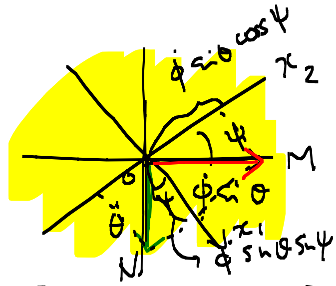


$1''', 2''', 3''' \rightarrow \underbrace{x_1, x_2, x_3}_{\text{Body fixed}}$

(ϕ, θ, ψ) Euler angles







$$\left.\begin{aligned} \vec{\Omega}_1 &= \dot\phi\, \vec{e}_{3'} \\ \vec{\Omega}_2 &= \dot\theta\, \vec{e}_{1'} \\ \vec{\Omega}_3 &= \dot\psi\, \vec{e}_{3''} \end{aligned}\right\} \quad \vec{\Omega} = \vec{\Omega}_1 + \vec{\Omega}_2 + \vec{\Omega}_3$$

$$\vec{\Omega}_1 = \dot\phi\left(\cos\theta\,\vec{e}_3 + \sin\theta\cos\psi\,\vec{e}_2 + \sin\theta\sin\psi\,\vec{e}_1\right)$$

$$\vec{\Omega}_2 = \dot\theta\left(\cos\psi\,\vec{e}_1 - \sin\psi\,\vec{e}_2\right)$$

$$\vec{\Omega}_3 = \dot\psi\,\vec{e}_3$$

$$\vec{\Omega} = \vec{e}_1\big(\overbrace{\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi}^{\Omega_1}\big) + \vec{e}_2\big(\overbrace{\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi}^{\Omega_2}\big) + \vec{e}_3\big(\underbrace{\dot\phi\cos\theta + \dot\psi}_{\Omega_3}\big)$$

$$T_{rot} = \frac{1}{2}\vec{\Omega}\cdot\begin{pmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{pmatrix}\vec{\Omega}$$

$$= \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$

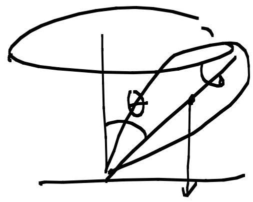
for a sym. top ($I_1 = I_2 \neq I_3$)

$$T_{rot} = \frac{1}{2} I_1 (\Omega_1^2 + \Omega_2^2) + \frac{1}{2} I_3 \Omega_3^2$$

$$= \frac{1}{2} I_1 (\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2} I_3 (\dot\phi\cos\theta + \dot\psi)^2$$

= indep of ψ & ϕ

$$L = \underbrace{T_{rot}} - U(\theta)$$



$$\frac{\partial L}{\partial \psi} = \frac{\partial L}{\partial \phi} = 0 \Rightarrow$$ conservation of two Ang. Mom comp.

Prob 1.

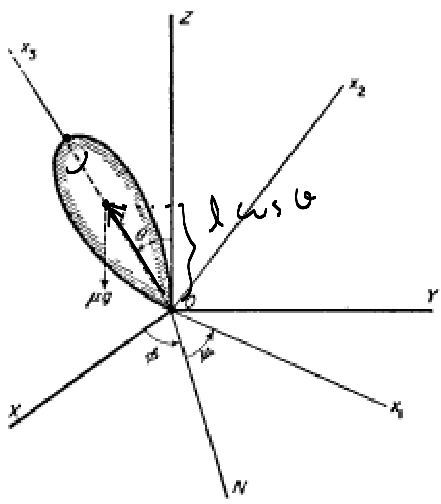
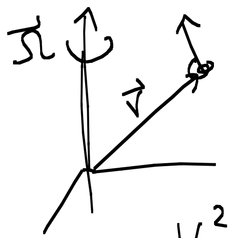


FIG. 48

$$T = T_{CM} + T_{rot}$$

$$= \frac{1}{2}\mu \vec{V}^2 + \frac{1}{2}(I_1\Omega_1^2 + I_1\Omega_2^2 + I_3\Omega_3^2)$$

$$\vec{V} = \vec{\Omega} \times \vec{l} = \vec{\Omega} \times (l\vec{e}_3) = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \Omega_1 & \Omega_2 & \Omega_3 \\ 0 & 0 & l \end{vmatrix}$$



$$= \vec{e}_1(\Omega_2 l) - \vec{e}_2(\Omega_1 l)$$

$$V^2 = \Omega_2^2 l^2 + \Omega_1^2 l^2$$

$$= l^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \right)$$

$$\therefore T = \frac{1}{2} \mu l^2 \left(\underbrace{\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta} \right)$$

$$+ \frac{1}{2} I_1 \left(\underbrace{\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2} \right) + \frac{1}{2} I_3 \left(\dot{\phi}\cos\theta + \dot{\psi} \right)^2$$

$$= \frac{1}{2} \underbrace{\left(I_1 + \mu l^2 \right)}_{I_1'} \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2 \right) + \frac{1}{2} I_3 \left(\dot{\phi}\cos\theta + \dot{\psi} \right)^2$$

$$U = \mu g l \cos\theta$$

$$\boxed{L = \frac{1}{2} \underbrace{\left(I_1 + \mu l^2 \right)}_{I_1'} \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2 \right) + \frac{1}{2} I_3 \left(\dot{\phi}\cos\theta + \dot{\psi} \right)^2 - \mu g l \cos\theta}$$

$$\frac{\partial L}{\partial \psi} = 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}} \right) \rightarrow \frac{\partial L}{\partial \dot{\psi}} = \text{const} = p_\psi$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = \underbrace{I_3 \underbrace{\left(\dot{\phi}\cos\theta + \dot{\psi} \right)}_{\Omega_3}} = M_3$$

$$\vec{M} = I \cdot \vec{\Omega}$$

$$\frac{\partial L}{\partial \phi} = 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}} \right) \rightarrow \frac{\partial L}{\partial \dot{\phi}} = \text{const} = p_\phi$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \underbrace{I_1' \dot{\phi} \sin^2\theta + I_3 \cos\theta \left(\dot{\phi}\cos\theta + \dot{\psi} \right)}$$

$$= \text{constant} \equiv M_z$$

$$I_3 \dot{\phi} \cos\theta + I_3 \dot{\psi} = M_3$$

$$I_3 \dot{\psi} = M_3 - I_3 \dot{\phi} \cos\theta$$

$$M_z = I_1' \dot\phi \sin^2\theta + I_3 \dot\phi \cos^2\theta + I_3 \dot\psi \cos\theta$$

$$= I_1' \dot\phi \sin^2\theta + \cancel{I_3 \dot\phi \cos^2\theta} + \left(M_3 - \cancel{I_3 \dot\phi \cos\theta}\right)\cos\theta$$

$$= I_1' \dot\phi \sin^2\theta + M_3 \cos\theta$$

$$\boxed{\dot\phi = \frac{M_z - M_3\cos\theta}{I_1' \sin^2\theta}}$$

$$I_3(\dot\phi\cos\theta + \dot\psi) = M_3$$

$$\Rightarrow \dot\psi = \frac{M_3}{I_3} - \cos\theta\left(\frac{M_z - M_3\cos\theta}{I_1'\sin^2\theta}\right)$$

$$E = \frac{1}{2} I_1'\left(\left(\frac{M_z - M_3\cos\theta}{I_1'\sin^2\theta}\right)^2 \sin^2\theta + \dot\theta^2\right) + \frac{1}{2} I_3\left(\frac{M_3}{I_3}\right)^2 + \mu g l(\cos\theta - 1) + \mu g l$$

$$E' = E - \frac{1}{2}\frac{M_3^2}{I_3} - \mu g l$$

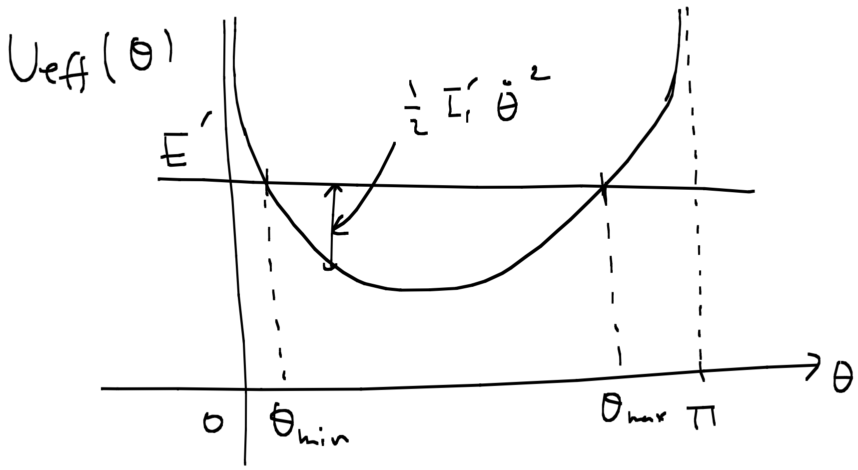
$$= \frac{1}{2} I_1' \dot\theta^2 + U_{eff}(\theta)$$

where

$$U_{eff}(\theta) = \frac{(M_z - M_3\cos\theta)^2}{2 I_1' \sin^2\theta} - \mu g l(1-\cos\theta)$$

1 dim system with coordinate θ

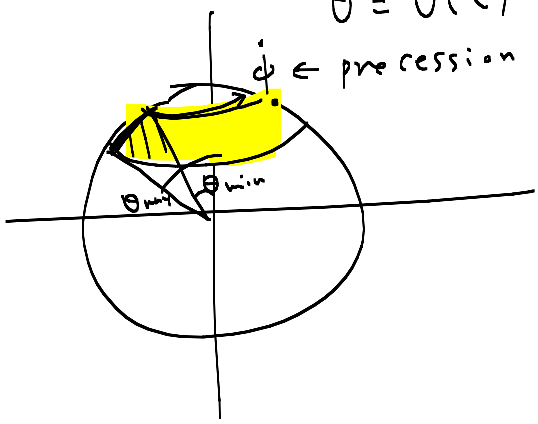
$$0 \leq \theta \leq \pi$$



$$\dot{\theta} = \sqrt{\frac{2}{I_1'}\left(E' - U_{eff}(\theta)\right)} = \frac{d\theta}{dt}$$

$$\int dt = \int \frac{d\theta}{\sqrt{\frac{2}{I_1'}\left(E' - U_{eff}(\theta)\right)}}$$

$\theta = \theta(t)$



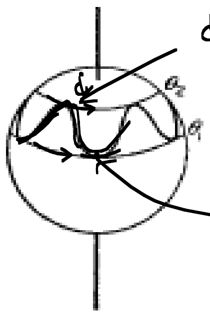
$$\dot{\phi} = \frac{M_z - M_3 \cos\theta}{I_1' \sin^2\theta} > 0$$

$$\frac{M_z}{M_3} > \cos\theta \longrightarrow \dot{\phi} > 0$$

$$< \longrightarrow \dot{\phi} < 0$$

if $M_z > M_3 \rightarrow \frac{M_z}{M_3} > 1 > \cos\theta$

$\rightarrow \dot{\phi} > 0$ always



$\dot{\phi}(\theta = \theta_{min}) = \frac{M_z - M_3 \cos\theta_{min}}{I_1' \sin^2\theta_{min}}$

$\dot{\phi}(\theta = \theta_{min}) = \frac{M_z - M_3 \cos\theta_{max}}{I_1' \sin^2\theta_{max}}$

(a)

if $M_z < M_3 \rightarrow \frac{M_z}{M_3} < 1$

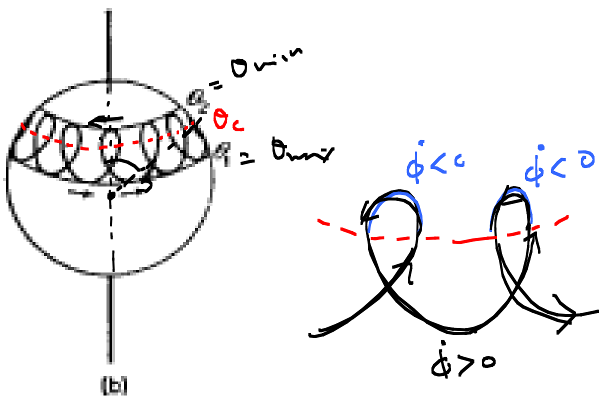
$\rightarrow$ there is θ_c such that

$$\frac{M_z}{M_3} = \cos\theta_c$$

$$\Rightarrow \dot{\phi} = 0 \text{ at } \theta = \theta_c$$

if $\theta_{min} < \theta < \theta_c \rightarrow \dot{\phi} \propto \underbrace{\frac{M_z}{M_3}}_{\cos\theta_c} - \cos\theta < 0$

if $\theta_{max} > \theta > \theta_c \rightarrow \dot{\phi} \propto \frac{M_z}{M_3} - \cos\theta > 0$

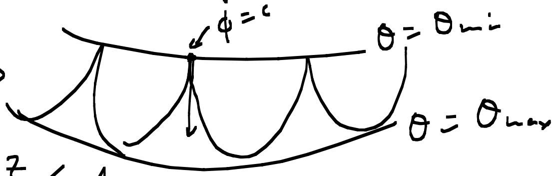


(b)

if $\cos\theta_{min} = \dfrac{M_z}{M_3} \longrightarrow \dot\phi(\theta_{min}) = 0$

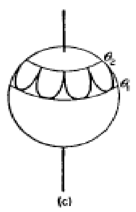
$$\dot\phi = \frac{M_z - M_3\cos\theta}{I_1' \sin^2\theta}$$

$\dot\phi(\theta > \theta_{min}) > 0$

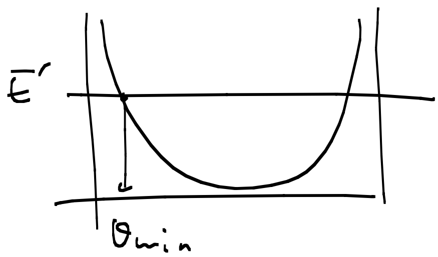


$\cos\theta_{min} = \dfrac{M_z}{M_3} < 1$

$\underline{M_z < M_3}$



$\left(M_z < M_3 \ \& \ \dfrac{M_z}{M_3} = \cos\theta_{min} \right)$



Prob 2.

$$U_{eff}(\theta) = U_{eff}(0) + U'_{eff}(0)\,\theta + \frac{1}{2} U''_{eff}(0)\,\theta^2 + \cdots$$

$$U_{eff}(\theta) = \frac{(M_z - M_3\cos\theta)^2}{2 I_1' \sin^2\theta} - \mu g l (1-\cos\theta)$$

$$= \frac{M_3^2}{2I_1'}\left(\frac{1-\cos\theta}{\sin\theta}\right)^2$$

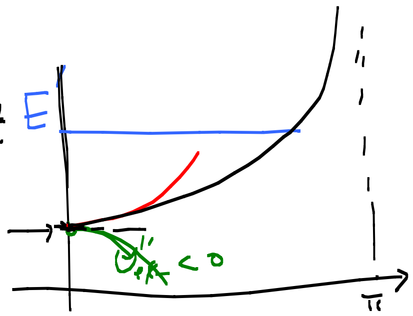
$$\boxed{M_z \equiv M_3}$$

$$\frac{1-\cos\theta}{\sin\theta} = \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \simeq \frac{\theta}{2}$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\approx \frac{M_3^2}{2I_1'}\frac{\theta^2}{4} - \mu g l \frac{\theta^2}{2}$$

$$\underline{U_{eff}(\theta) = \frac{M_3^2}{8 I_1'}\theta^2 - \frac{\mu g l}{2}\theta^2}$$



$\theta_{min} = 0$

$$1 + \frac{\theta^2}{4} = \frac{1}{1 - 2\cdot\frac{\theta^2}{8}} = \frac{1}{\cos^2\frac{\theta}{2}}$$

$$\cos x = 1 - \frac{x^2}{2}$$

$$U'_{eff}(\theta = 0) = 0$$

$$U''_{eff}(\theta=0) = \left(\frac{M_3^2}{8I_1'} - \frac{\mu g l}{2}\right) > 0 \quad (< 0)$$

$$U_{eff}(\theta) \approx \frac{M_3^2}{4I_1'} + \left(\quad\right)\theta^2$$

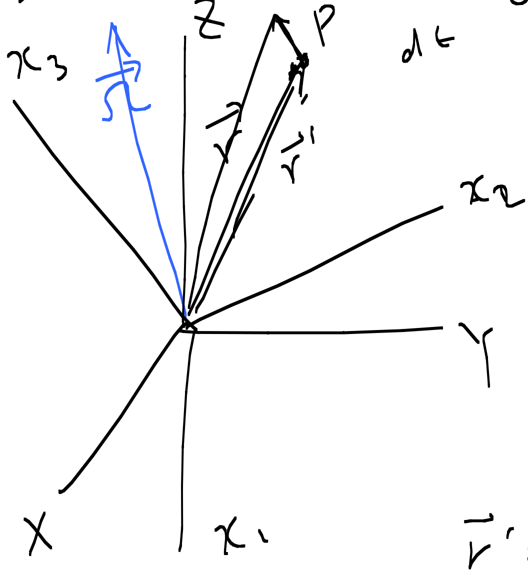
$$\boxed{M_3^2 > 4 I_1' \mu g \ell}$$

stable

Final Exam

- On June 5 (2 – 5 pm)
- 1 problem each from Chapters 3, 4, 6
- Open book (closed note)

§36. Euler's equations



$\vec r = \vec r'$

$d\vec r = d\vec r' - d\vec\theta \times \vec r$

$$\frac{d'}{dt}\vec r'\Big|_{rot} = \left(\frac{d\vec r}{dt}\right)_{Fixed} - \vec\Omega\times\vec r$$

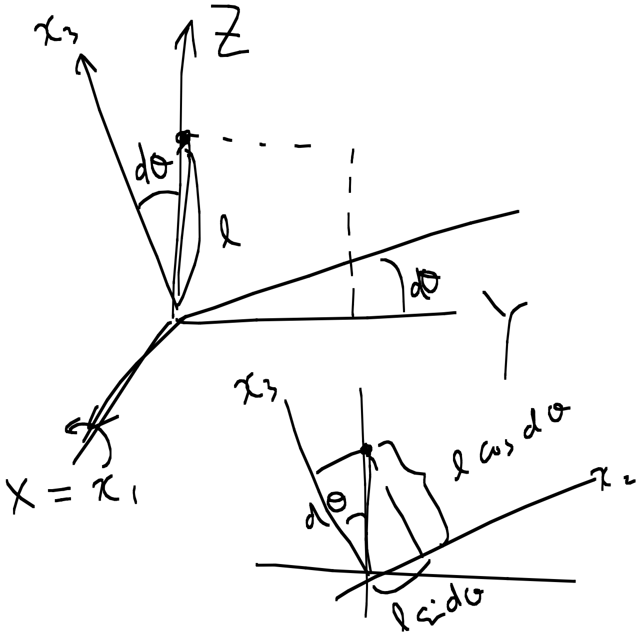
$\vec r' = \vec r$

$$\Rightarrow \boxed{\left(\frac{d}{dt}\right)_{rot}\vec r = \left(\frac{d}{dt}\right)_{Fixed}\vec r - \vec\Omega \times \vec r}$$

$(x_1 - x_2 - x_3)$ $(X - Y - Z)$ ang. vel of $(x_1 - x_2 - x_3)$

valid for any vector, $\vec r$, ..., $\vec A$

$$\left(\frac{d\vec A}{dt}\right)_{rot} = \left(\frac{d}{dt}\vec A\right)_{fixed} - \vec\Omega \times \vec A$$



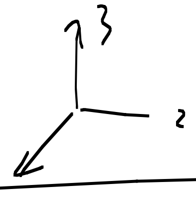
$$dx_1=0,\ dx_2 = l \sin d\theta,\ dx_3 = l\cos(d\theta) - l \approx 0$$

$$dX = dY = dZ = 0 \rightarrow d\vec{r} = 0$$

$$dx_2 = l\, d\theta \Rightarrow \frac{dY}{dt} = 0 = \underbrace{\frac{dx_2}{dt}}\ \underbrace{-l\frac{d\theta}{dt}}_{+(\vec{\Omega}\times\vec{r})_2}$$

$$\vec{\Omega} = \dot{\theta}\, \vec{e}_1$$

$$\vec{\Omega}\times\underset{l\vec{e}_3}{\underset{\shortparallel}{\vec{r}}} = l\,\dot{\theta}\, \underbrace{\vec{e}_1\times\vec{e}_3}_{-\vec{e}_2} = -l\,\dot{\theta}\,\vec{e}_2$$



$$\Rightarrow \boxed{\left(\frac{d\vec{A}}{dt}\right)_{fixed} = \left(\frac{d\vec{A}}{dt}\right)_{rot} + (\vec{\Omega}\times\vec{A})}$$

$$\rightarrow \left(\frac{d\vec{r}}{dt}\right)_{rot} = \left(\frac{d\vec{r}}{dt}\right)_{fixed} - \vec{\Omega}\times\vec{r}$$

$$\overset{\shortparallel}{\vec{v}}_{rot} = \underbrace{\vec{v}_{fixed} - \vec{\Omega}\times\vec{r}}$$

$$\vec{a}_{rot} = \left(\frac{d}{dt}\right)_{rot}\vec{v}_{rot}$$

$$= \left(\frac{d}{dt}\right)_{fixed}\vec{v}_{rot} - \vec{\Omega}\times\vec{v}_{rot}$$

$$= \left(\frac{d}{dt}\right)_{fixed}\left(\vec{v}_{fixed} - \vec{\Omega}\times\vec{r}\right) - \vec{\Omega}\times\left(\vec{v}_{fixed} - \vec{\Omega}\times\vec{r}\right)$$

$$\vec{a}_{fix} = \left(\frac{d}{dt}\right)_{fix} \vec{v}_{fix}$$

$$= \left(\frac{d}{dt}\right)_{rot} \vec{v}_{fix} + \vec{\Omega} \times \vec{v}_{fix}$$

$$= \left(\frac{d}{dt}\right)_{rot} \left(\vec{v}_{rot} + \vec{\Omega} \times \vec{r}\right) + \vec{\Omega} \times \left(\vec{v}_{rot} + \vec{\Omega} \times \vec{r}\right)$$

$$\vec{v}_{fix} = \vec{v}_{rot} + \vec{\Omega} \times \vec{r}$$

$$\left(\frac{d\vec{v}_{fix}}{dt}\right)_{fix} = \underbrace{\left(\frac{d\vec{v}_{rot}}{dt}\right)_{rot}}_{\vec{a}_{rot}} + \dot{\vec{\Omega}} \times \vec{r}_{rot} + \vec{\Omega} \times \underbrace{\left(\frac{d\vec{r}}{dt}\right)_{rot}}_{\vec{v}_{rot}} + \underbrace{\vec{\Omega} \times \vec{v}_{rot}} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

$$\vec{a}_{fixed} = \frac{\vec{F}}{m}$$

$$\Rightarrow \vec{a}_{rot} = \underbrace{\frac{\vec{F}}{m}}_{\text{phys.}} \underbrace{\underbrace{- 2\vec{\Omega} \times \vec{v}}_{\text{Coriolis}} \underbrace{- \dot{\vec{\Omega}} \times \vec{r}}_{\text{transverse}} \underbrace{- \vec{\Omega} \times (\vec{\Omega} \times \vec{r})}_{\text{centrifugal}}}_{\text{pseudo-force}}$$

rot

$$\Rightarrow \left(\frac{d\vec{A}}{dt}\right)_{fixed} = \left(\frac{d\vec{A}}{dt}\right)_{rot} + \left(\vec{\Omega} \times \vec{A}\right)$$

$$\vec{A} = \vec{p}$$

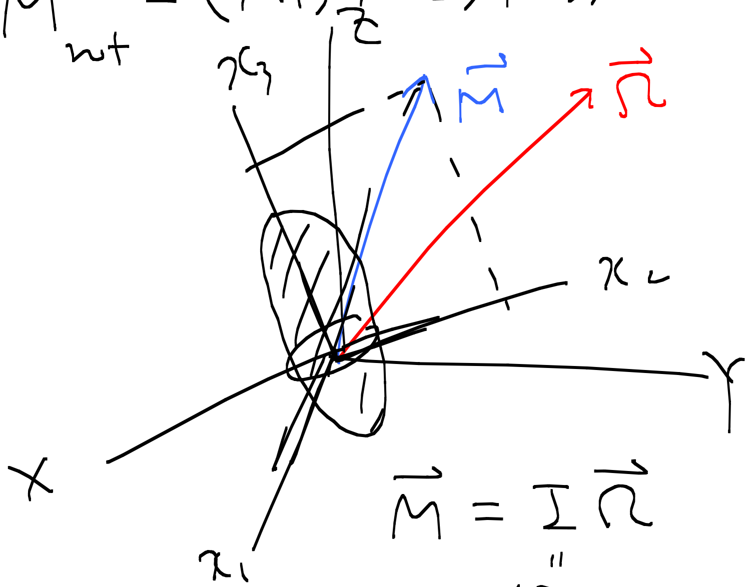
$$\left(\frac{d\vec{p}}{dt}\right)_f = \left(\frac{d\vec{p}}{dt}\right)_{rot} + \vec{\Omega} \times \vec{p}$$

$$\overset{\shortparallel}{\vec{F}} \quad \rightarrow \quad \underline{\left(\frac{d\vec{p}}{dt}\right)_{rot} + \vec{\Omega} \times \vec{p} = \vec{F}}$$

$$\vec{A} = \vec{M}$$

$$\left(\frac{d\vec{M}}{dt}\right)_{Fixed} = \vec{K} = \underline{\underline{\left(\frac{d\vec{M}}{dt}\right)_{rot} + \vec{\Omega} \times \vec{M}}}$$

$$\vec{M}_{rot} = (M_1, M_2, M_3)$$



$$\vec{M} = \overset{\shortparallel}{I}\,\vec{\Omega}$$

$$\begin{pmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{pmatrix}$$

$$\underset{\sim}{M_1} = I_1 \Omega_1, \quad \underset{\sim}{M_2} = I_2 \Omega_2, \quad \underset{\sim}{M_3} = I_3 \Omega_3$$

$$\left(\vec{K} = \left(\frac{d\vec{M}}{dt}\right)_{rot} + \vec{\Omega} \times \vec{M}\right)_{i = x_1, x_2, x_3}$$

$$\left(\frac{d}{dt}\right)_{rot}(I_i\Omega_i) + (\vec{\Omega}\times\vec{M})_i = K_i$$

$$\overset{\shortparallel}{I_i\dot{\Omega}_i} + \overset{\shortparallel}{\varepsilon_{ijk}\Omega_j M_k}$$

$$M_k = (\vec{M})_k = I_k\Omega_k$$

$$\varepsilon_{ijk}\Omega_j I_k\Omega_k$$

$$i=1 \rightarrow \varepsilon_{1jk}\Omega_j I_k\Omega_k$$

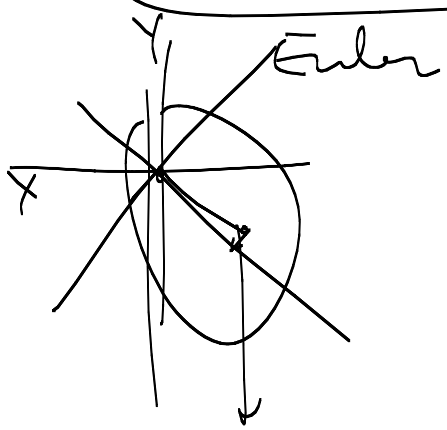
$$= \Omega_2 I_3\Omega_3 - \Omega_3 I_2\Omega_2$$

$$= \Omega_2\Omega_3(I_3 - I_2)$$

$$i=2 \rightarrow \Omega_1\Omega_3(I_1 - I_3)$$

$$i=3 \rightarrow \Omega_1\Omega_2(I_2 - I_1)$$

$$\Rightarrow \boxed{\begin{aligned} I_1\dot{\Omega}_1 + \Omega_2\Omega_3(I_3 - I_2) &= K_1 \\ I_2\dot{\Omega}_2 + \Omega_1\Omega_3(I_1 - I_3) &= K_2 \\ I_3\dot{\Omega}_3 + \Omega_2\Omega_1(I_2 - I_1) &= K_3 \end{aligned}}$$



Complicated

if $\vec{K} = 0$

$$I_1 \dot{\Omega}_1 + \Omega_2 \Omega_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\Omega}_2 + \Omega_1 \Omega_3 (I_1 - I_3) = 0$$

$$I_3 \dot{\Omega}_3 + \Omega_2 \Omega_1 (I_2 - I_1) = 0$$

$I_1 = I_2 \neq I_3$ (symmetric top)

$$\Rightarrow \quad I_3 \dot{\Omega}_3 = 0 \rightarrow \Omega_3 = \text{const.}$$

$$\dot{\Omega}_1 + \Omega_2 \Big(\underbrace{\Omega_3 \frac{I_3 - \overset{= I_2}{I_1}}{I_1}}_{\text{const} \equiv \omega} \Big) = 0$$

$$\dot{\Omega}_2 + \Omega_1 \underbrace{\left(\Omega_3 \frac{I_1 - I_3}{I_2}\right)}_{\text{const} \equiv -\omega} = 0$$

$$\dot{\Omega}_1 + \omega \Omega_2 = 0$$

$$+)\ i(\dot{\Omega}_2 - \omega \Omega_1) = 0$$

$$\frac{d}{dt}(\underbrace{\Omega_1 + i\Omega_2}_{\underset{\Omega}{\parallel}}) + \underbrace{\omega(\Omega_2 - i\Omega_1)}_{-i\omega(\Omega_1 + i\Omega_2)} = 0$$

$$\frac{d\Omega}{dt} - i\omega\Omega = 0 \to \frac{d\Omega}{dt} = \underline{i\omega}\,\Omega$$

$$\Downarrow$$

$$\Omega = \underset{\substack{\parallel \\ A = \text{real}}}{\Omega_0}\, e^{+i\omega t} = \Omega_1 + i\Omega_2$$

$$\boxed{\begin{aligned} \Omega_1 &= A \cos \omega t \\ \Omega_2 &= A \sin \omega t \end{aligned}}$$

$$A = \sqrt{\Omega_1^2 + \Omega_2^2}$$

§37. asymmetric top

$$I_1 < I_2 < I_3$$

$$E = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$

$$= \text{constant}$$

$$\vec{M}^2 = M^2 = I_1^2\Omega_1^2 + I_2^2\Omega_2^2 + I_1\Omega_3^2$$

$$\vec{M} = I\vec{\Omega}$$

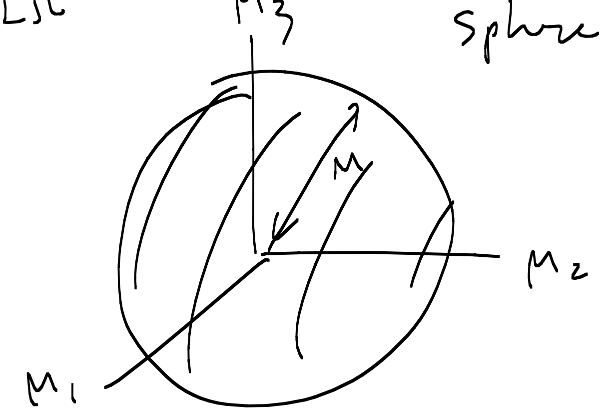
$$E = \frac{1}{2}\left(\frac{M_1^2}{I} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3}\right)$$

$= \text{constant}$

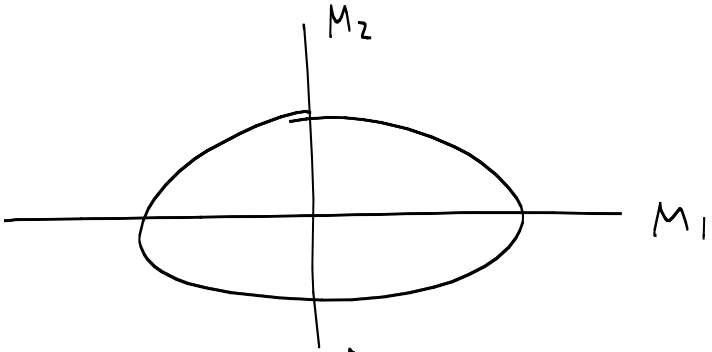
ellipsoid in $\vec{M}$

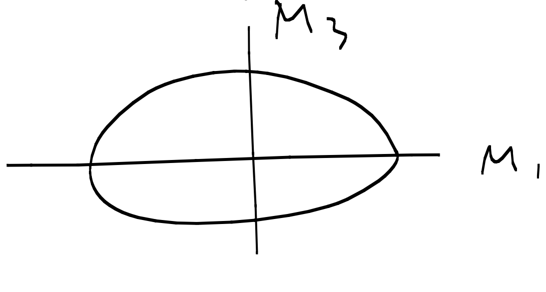
$$\vec{M}^2 = M^2 = \underbrace{M_1^2 + M_2^2 + M_3^2}_{\text{sphere in } \vec{M}}$$

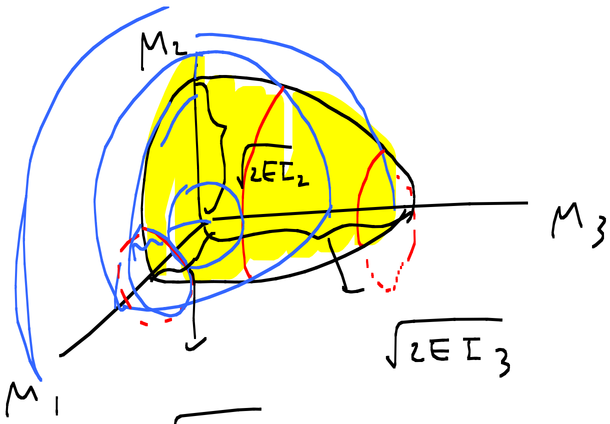
$\vec{M} = I\vec{\Omega}$



$$1 = \frac{1}{2E}\left(\frac{M_1^2}{I} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3}\right)$$







$\sqrt{2EI_1}$ $\qquad$ $I_1 < I_2 < I_3$

if $\sqrt{2EI_1} < M < \sqrt{2EI_3}$

$M \gtrsim \sqrt{2EI_1}$

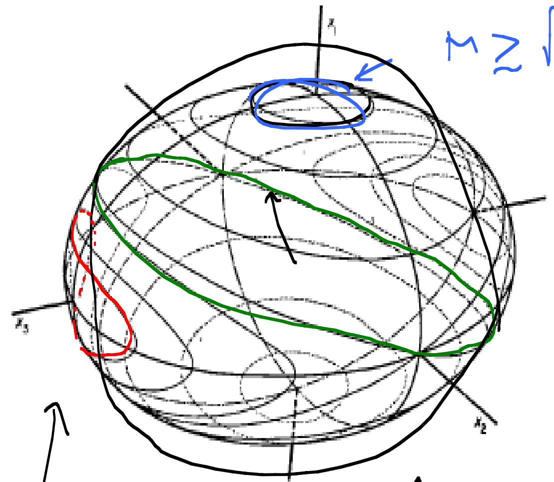


FIG. 51

unstable

$\sqrt{2EI_3} \gtrsim M$

$$2EI_3 = \left(I_1 I_3 \Omega_1^2 + I_2 I_3 \Omega_2^2 + I_3^2 \Omega_3^2\right)$$

$$|\vec{M}|^2 = M^2 = I_1^2 \Omega_1^2 + I_2^2 \Omega_2^2 + I_1^2 \Omega_3^2$$

$$2EI_1 - M^2 = I_2 (I_1 - I_2) \Omega_2^2 + I_3 (I_1 - I_3) \Omega_3^2$$

$$2EI_3 - M^2 = I_2 (I_3 - I_2) \Omega_2^2 + I_1 (I_3 - I_1) \Omega_1^2$$

$$\Omega_3^2 = \frac{2EI_1 - M^2 - I_2(I_1 - I_2)\Omega_2^2}{I_3(I_1 - I_3)}$$

$$\Omega_1 = \sqrt{\frac{2EI_1 - M^2 - I_2(I_1 - I_2)\Omega_2^2}{I_3(I_1 - I_3)}}$$

$$\Omega_3 = \sqrt{\frac{2EI_3 - M^2 - I_2(I_3 - I_2)\Omega_2^2}{I_1(I_3 - I_1)}}$$

$$I_2 \dot{\Omega}_2 + \Omega_1 \Omega_3 (I_1 - I_3) = 0$$

$$\tau = t(\sim)$$

$$s = \Omega_2(\ldots), \quad k = \frac{I_2 - I_1}{}$$

$$\frac{ds}{d\tau} = \sqrt{(1-s^2)(1-k^2 s^2)}$$

$$\Rightarrow \tau = \int d\tau = \underbrace{\int \frac{ds}{\sqrt{(1-s^2)(1-k^2 s^2)}}}_{\text{elliptic}}$$

$\tau = t\sqrt{[(I_3-I_2)(M^2-2EI_1)/I_1I_2I_3]}$,

$s = \Omega_2\sqrt{[I_2(I_3-I_2)/(2EI_3-M^2)]}$,

a positive parameter $k^2 < 1$ by

$k^2 = (I_2-I_1)(2EI_3-M^2)/(I_3-I_1)(M^2-2EI_1)$,

$$= \mathrm{sn}^{-1}(s)$$

$$s = \mathrm{sn}(\tau) = \Omega_2(\ldots)$$

$$\Rightarrow \boxed{\Omega_2 = \frac{1}{\ldots} \mathrm{sn}(t(\ldots))}$$

$$\Omega_1 = \frac{1}{\ldots} \mathrm{cn}(t(\ldots))$$

$$\Omega_3 = \ldots \mathrm{dn}(t(\ldots))$$

sn $\to$ sin when $k \to 0$

(k = elliptic modulus)

§38. $\to$ skip

§39.

$$m\vec{a}'_{\text{rot}} = \vec{F} - m\left(\dot{\vec{\omega}} \times \vec{r}' + 2\vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}')\right)$$

H.W. Prob 2-3 on page 129-130

skip Chap VII